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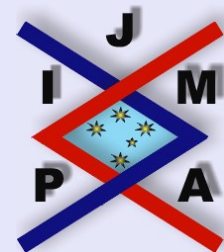
INCLUSION AND NEIGHBORHOOD PROPERTIES OF SOME ANALYTIC AND MULTIVALENT FUNCTIONS

R.K. RAINA AND H.M. SRIVASTAVA

Department of Mathematics
College of Technology and Engineering
Maharana Pratap University of Agriculture and Technology
Udaipur 313001, Rajasthan, India.
EMail: rainark_7@hotmail.com

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada.
EMail: harimsri@math.uvic.ca

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Abstract

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Abstract

By means of a certain extended derivative operator of Ruscheweyh type, the authors introduce and investigate two new subclasses of p -valently analytic functions of complex order. The various results obtained here for each of these function classes include coefficient inequalities and the consequent inclusion relationships involving the neighborhoods of the p -valently analytic functions.

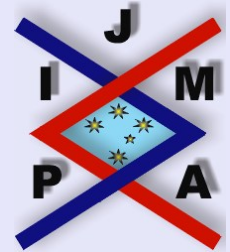
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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ normalized by

$$(1.1) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; n, p \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are analytic and p -valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The Hadamard product (or convolution) of the function $f \in \mathcal{A}_p(n)$ given by (1.1) and the function $g \in \mathcal{A}_p(n)$ given by

$$(1.2) \quad g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geq 0; n, p \in \mathbb{N})$$

is defined (as usual) by

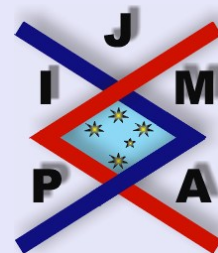
$$(1.3) \quad (f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

We introduce here an extended linear derivative operator of Ruscheweyh type:

$$\mathcal{D}^{\lambda,p} : \mathcal{A}_p \rightarrow \mathcal{A}_p \quad (\mathcal{A}_p := \mathcal{A}_p(1)),$$

which is defined by the following convolution:

$$(1.4) \quad \mathcal{D}^{\lambda,p} f(z) = \frac{z^p}{(1-z)^{\lambda+p}} * f(z) \quad (\lambda > -p; f \in \mathcal{A}_p).$$



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In terms of the binomial coefficients, we can rewrite (1.4) as follows:

$$(1.5) \quad \mathcal{D}^{\lambda,p} f(z) = z^p - \sum_{k=1+p}^{\infty} \binom{\lambda+k-1}{k-p} a_k z^k \quad (\lambda > -p; f \in \mathcal{A}_p).$$

In particular, when $\lambda = n$ ($n \in \mathbb{N}$), it is easily observed from (1.4) and (1.5) that

$$(1.6) \quad \mathcal{D}^{n,p} f(z) = \frac{z^p (z^{n-p} f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p \in \mathbb{N}),$$

so that

$$(1.7) \quad \mathcal{D}^{1,p} f(z) = (1-p)f(z) + z f'(z),$$

$$(1.8) \quad \mathcal{D}^{2,p} f(z) = \frac{(1-p)(2-p)}{2!} f(z) + (2-p)z f'(z) + \frac{z^2}{2!} f''(z),$$

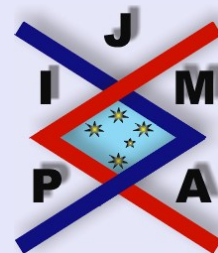
and so on.

By using the operator

$$\mathcal{D}^{\lambda,p} f(z) \quad (\lambda > -p; p \in \mathbb{N})$$

given by (1.5), we now introduce a new subclass $\mathcal{H}_{n,m}^p(\lambda, b)$ of the p -valently analytic function class $\mathcal{A}_p(n)$, which includes functions $f(z)$ satisfying the following inequality:

$$(1.9) \quad \left| \frac{1}{b} \left(\frac{z (\mathcal{D}^{\lambda,p} f(z))^{(m+1)}}{(\mathcal{D}^{\lambda,p} f(z))^{(m)}} - (p-m) \right) \right| < 1$$



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$(z \in \mathbb{U}; p \in \mathbb{N}; m \in \mathbb{N}_0; \lambda \in \mathbb{R}; p > \max(m, -\lambda); b \in \mathbb{C} \setminus \{0\})$.

Next, following the earlier investigations by Goodman [3], Ruscheweyh [5] and Altıntaş *et al.* [2] (see also [1], [4] and [6]), we define the (n, δ) -neighborhood of a function $f(z) \in \mathcal{A}_n(p)$ by (see, for details, [2, p. 1668])

$$(1.10) \quad \mathcal{N}_{n,\delta}(f) := \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \right. \\ \left. \sum_{k=n+p}^{\infty} k |a_k - b_k| \leq \delta \right\}.$$

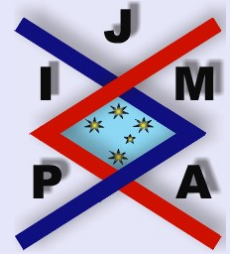
It follows from (1.10) that, if

$$(1.11) \quad h(z) = z^p \quad (p \in \mathbb{N}),$$

then

$$(1.12) \quad \mathcal{N}_{n,\delta}(h) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \right. \\ \left. \sum_{k=n+p}^{\infty} k |b_k| \leq \delta \right\}.$$

Finally, we denote by $\mathcal{L}_{n,m}^p(\lambda, b; \mu)$ the subclass of $\mathcal{A}_p(n)$ consisting of func-



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tions $f(z)$ which satisfy the inequality (1.13) below:

$$(1.13) \quad \left| \frac{1}{b} \left(p(1-\mu) \left(\frac{\mathcal{D}^{\lambda,p} f(z)}{z} \right)^{(m)} + \mu (\mathcal{D}^{\lambda,p} f(z))^{(m+1)} - (p-m) \right) \right| < p-m$$

$(z \in \mathbb{U}; p \in \mathbb{N}; m \in \mathbb{N}_0; \lambda \in \mathbb{R}; p > \max(m, -\lambda); \mu \geq 0; b \in \mathbb{C} \setminus \{0\}).$

The object of the present paper is to investigate the various properties and characteristics of analytic p -valent functions belonging to the subclasses

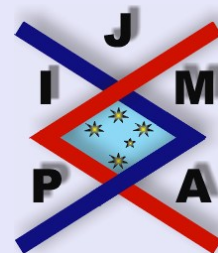
$$\mathcal{H}_{n,m}^p(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^p(\lambda, b; \mu),$$

which we have introduced here. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (n, δ) -neighborhoods of analytic p -valent functions (with negative and missing coefficients) belonging to these subclasses.

Our definitions of the function classes

$$\mathcal{H}_{n,m}^p(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^p(\lambda, b; \mu)$$

are motivated essentially by two earlier investigations [1] and [4], in each of which further details and references to other closely-related subclasses can be found. In particular, in our definition of the function class $\mathcal{L}_{n,m}^p(\lambda, b; \mu)$ involving the inequality (1.13), we have relaxed the parametric constraint $0 \leq \mu \leq 1$, which was imposed earlier by Murugusundaramoorthy and Srivastava [4, p. 3, Equation (1.14)] (see also Remark 3 below).



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2. A Set of Coefficient Bounds

In this section, we prove the following results which yield the coefficient inequalities for functions in the subclasses

$$\mathcal{H}_{n,m}^p(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^p(\lambda, b; \mu).$$

Theorem 1. Let $f(z) \in \mathcal{A}_p(n)$ be given by (1.1). Then $f(z) \in \mathcal{H}_{n,m}^p(\lambda, b)$ if and only if

$$(2.1) \quad \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} (k+|b|-p) a_k \leq |b| \binom{p}{m}.$$

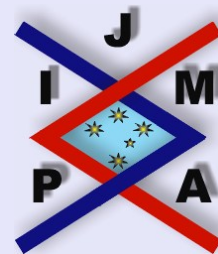
Proof. Let a function $f(z)$ of the form (1.1) belong to the class $\mathcal{H}_{n,m}^p(\lambda, b)$. Then, in view of (1.5), (1.9) yields the following inequality:

$$(2.2) \quad \Re \left(\frac{\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} (p-k) z^{k-p}}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} z^{k-p}} \right) > -|b| \quad (z \in \mathbb{U}).$$

Putting $z = r$ ($0 \leq r < 1$) in (2.2), we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all r ($0 < r < 1$). Thus, by letting $r \rightarrow 1^-$ through *real* values, (2.2) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying (2.1) and setting $|z| = 1$, we find by using (1.5) that

$$\left| \frac{z (\mathcal{D}^{\lambda,p} f(z))^{(m+1)}}{(\mathcal{D}^{\lambda,p} f(z))^{(m)}} - (p-m) \right|$$



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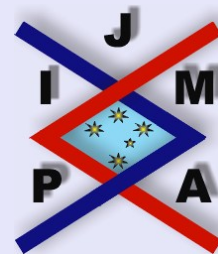


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$$\begin{aligned}
 &= \left| \frac{\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} (p-k) z^{k-m}}{\binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} z^{k-m}} \right| \\
 &\leq \frac{|b| \left[\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} a_k \right]}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} a_k} = |b|.
 \end{aligned}$$

Hence, by the maximum modulus principle, we infer that $f(z) \in \mathcal{H}_{n,m}^p(\lambda, b)$, which completes the proof of Theorem 1. \square

Remark 1. In the special case when

$$(2.3) \quad m = 0, \quad p = 1, \quad \text{and} \quad b = \beta\gamma \quad (0 < \beta \leq 1; \gamma \in \mathbb{C} \setminus \{0\}),$$

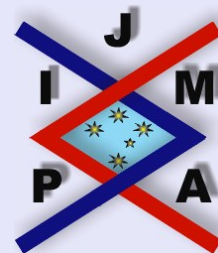
Theorem 1 corresponds to a result given earlier by Murugusundaramoorthy and Srivastava [4, p. 3, Lemma 1].

By using the same arguments as in the proof of Theorem 1, we can establish Theorem 2 below.

Theorem 2. Let $f(z) \in \mathcal{A}_p(n)$ be given by (1.1). Then $f(z) \in \mathcal{L}_{n,m}^p(\lambda, b; \mu)$ if and only if

$$\begin{aligned}
 (2.4) \quad \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k-1}{m} [\mu(k-1) + 1] a_k \\
 \leq (p-m) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right].
 \end{aligned}$$

Remark 2. *Making use of the same parametric substitutions as mentioned above in (2.3), Theorem 2 yields another known result due to Murugusundaramoorthy and Srivastava [4, p. 4, Lemma 2].*



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3. Inclusion Relationships Involving (n, δ) -Neighborhoods

In this section, we establish several inclusion relationships for the function classes

$$\mathcal{H}_{n,m}^p(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^p(\lambda, b; \mu)$$

involving the (n, δ) -neighborhood defined by (1.12).

Theorem 3. *If*

$$(3.1) \quad \delta = \frac{(n+p)|b| \binom{p}{m}}{(n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}} \quad (p > |b|),$$

then

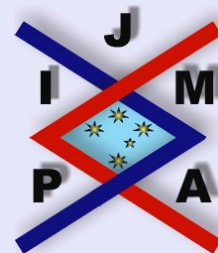
$$(3.2) \quad \mathcal{H}_{n,m}^p(\lambda, b) \subset \mathcal{N}_{n,\delta}(h).$$

Proof. Let $f(z) \in \mathcal{H}_{n,m}^p(\lambda, b)$. Then, in view of the assertion (2.1) of Theorem 1, we have

$$(3.3) \quad (n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m} \sum_{k=n+p}^{\infty} a_k \leq |b| \binom{p}{m}.$$

This yields

$$(3.4) \quad \sum_{k=n+p}^{\infty} a_k \leq \frac{|b| \binom{p}{m}}{(n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}}.$$



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Applying the assertion (2.1) of Theorem 1 again, in conjunction with (3.4), we obtain

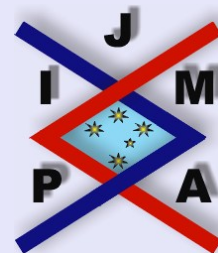
$$\begin{aligned}
 & \binom{\lambda + n + p - 1}{n} \binom{n + p}{m} \sum_{k=n+p}^{\infty} k a_k \\
 & \leq |b| \binom{p}{m} + (p - |b|) \binom{\lambda + n + p - 1}{n} \binom{n + p}{m} \sum_{k=n+p}^{\infty} a_k \\
 & \leq |b| \binom{p}{m} + (p - |b|) \binom{\lambda + n + p - 1}{n} \binom{n + p}{m} \\
 & \quad \cdot \frac{|b| \binom{p}{m}}{(n + |b|) \binom{\lambda + n + p - 1}{n} \binom{n + p}{m}} \\
 & = |b| \binom{p}{m} \binom{n + p}{n + |b|}.
 \end{aligned}$$

Hence

$$(3.5) \quad \sum_{k=n+p}^{\infty} k a_k \leq \frac{|b| (n + p) \binom{p}{m}}{(n + |b|) \binom{\lambda + n + p - 1}{n} \binom{n + p}{m}} =: \delta \quad (p > |b|),$$

which, by virtue of (1.12), establishes the inclusion relation (3.2) of Theorem 3. \square

In an analogous manner, by applying the assertion (2.4) of Theorem 2 instead of the assertion (2.1) of Theorem 1 to functions in the class $\mathcal{L}_{n,m}^p(\lambda, b; \mu)$, we can prove the following inclusion relationship.



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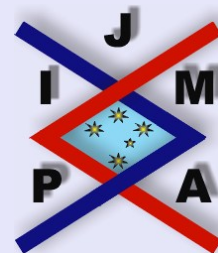
Theorem 4. *If*

$$(3.6) \quad \delta = \frac{(p-m)(n+p) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right]}{[\mu(n+p-1)+1] \binom{\lambda+n+p-1}{n} \binom{n+p}{m}} \quad (\mu > 1),$$

then

$$\mathcal{L}_{n,m}^p(\lambda, b; \mu) \subset \mathcal{N}_{n,\delta}(h).$$

Remark 3. *Applying the parametric substitutions listed in (2.3), Theorems 3 and 4 would yield the known results due to Murugusundaramoorthy and Srivastava [4, p. 4, Theorem 1; p. 5, Theorem 2]. Incidentally, just as we indicated in Section 2 above, the condition $\mu > 1$ is needed in the proof of one of these known results [4, p. 5, Theorem 2]. This implies that the constraint $0 \leq \mu \leq 1$ in [4, p. 3, Equation (1.14)] should be replaced by the less stringent constraint $\mu \geq 0$.*



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4. Further Neighborhood Properties

In this last section, we determine the neighborhood properties for each of the following (slightly modified) function classes:

$$\mathcal{H}_{n,m}^{p,\alpha}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b; \mu).$$

Here the class $\mathcal{H}_{n,m}^{p,\alpha}(\lambda, b)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{H}_{n,m}^p(\lambda, b)$ such that

$$(4.1) \quad \left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < p).$$

Analogously, the class $\mathcal{L}_{n,m}^{p,\alpha}(\lambda, b; \mu)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{L}_{n,m}^p(\lambda, b; \mu)$ satisfying the inequality (4.1).

The proofs of the following results involving the neighborhood properties for the classes

$$\mathcal{H}_{n,m}^{p,\alpha}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b; \mu)$$

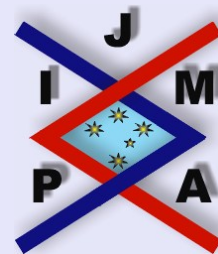
are similar to those given in [1] and [4]. We, therefore, skip their proofs here.

Theorem 5. *Let $g(z) \in \mathcal{H}_{n,m}^p(\lambda, b)$. Suppose also that*

$$(4.2) \quad \alpha = p - \frac{\delta (n + |b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}}{(n+p) \left[(n + |b|) \binom{\lambda+n+p-1}{n+p} \binom{n+p}{m} - |b| \binom{p}{m} \right]}.$$

Then

$$(4.3) \quad \mathcal{N}_{n,\delta}(g) \subset \mathcal{H}_{n,m}^{p,\alpha}(\lambda, b).$$



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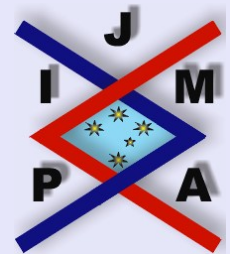
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Theorem 6. Let $g(z) \in \mathcal{L}_{n,m}^p(\lambda, b; \mu)$. Suppose also that

$$(4.4) \quad \alpha = p - \frac{\delta [\mu(n+p-1)+1] \binom{\lambda+n+p-1}{n} \binom{n+p-1}{m}}{(n+p) \left[\mu(n+p-1)+1 \right] \binom{\lambda+n+p-1}{n} \binom{n+p-1}{m} - (p-m) \left\{ \frac{|b-1|}{m!} + \binom{p}{m} \right\}}.$$

Then

$$(4.5) \quad \mathcal{N}_{n,\delta}(g) \subset \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b; \mu).$$



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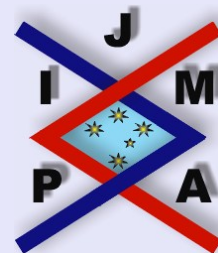
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