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SHARPENING OF JORDAN'S INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. In this paper,the following inequality:

$$\frac{2}{\pi}+\frac{1}{2\pi^5}(\pi^4-16x^4)\le\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^5}(\pi^4-16x^4)$$

is established. An application of this inequality gives an improvement of Yang Le's inequality.

Key words and phrases: Jordan inequality, Yang Le inequality, Upper-lower bound.

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1. INTRODUCTION

The following result is known as Jordan's inequality [1]:

Theorem 1.1.

$$\frac{\sin x}{x}\ge\frac{2}{\pi},\qquad x\in(0,\pi/2]. \tag{1.1}$$

The inequality (1.1) is sharp with equality if and only if $x=\frac{\pi}{2}$.

Jordan's inequality and its refinements have been considered by a number of other authors (see [2], [3]). In [2] Feng Qi obtained new lower and upper bounds for the function $\frac{\sin x}{x}$; his result reads as follows:

Theorem 1.2. *Let* $x\in(0,\pi/2]$, *then*

$$\frac{2}{\pi}+\frac{1}{\pi^3}(\pi^2-4x^2)\le\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^3}(\pi^2-4x^2), \tag{1.2}$$

with equality if and only if $x=\frac{\pi}{2}$.

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In this paper we will consider a new refined form of Jordan's inequality and an application of it on the same problem considered by Zhao [5] – [7]. Our main result is given by the following.

2. MAIN RESULT

In order to prove Theorem 2.2 below, we need the following lemma.

Lemma 2.1 ([8]). *Let* $f, g : [a, b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a, b)*, let* $g' \neq 0$ *on* (a, b)*,if* $\frac{f'}{g'}$ *is decreasing on* (a, b)*, then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)} \quad \text{and} \quad \frac{f(x) - f(a)}{g(x) - g(a)}$$

are also decreasing on (a, b).

Theorem 2.2. *If* $x \in (0, \pi/2]$*, then*

$$\frac{2}{\pi} + \frac{1}{2\pi^5}(\pi^4 - 16x^4) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}(\pi^4 - 16x^4) \tag{2.1}$$

with equality if and only if $x = \frac{\pi}{2}$.

Proof. Let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -16x^4$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^5$, and $x \in (0, \pi/2]$, then we have.

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{64} \cdot \frac{\sin x - x\cos x}{x^5} = \frac{1}{64} \cdot \frac{f_3(x)}{f_4(x)}.$$
$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{5} \cdot \frac{\sin x}{x^3}.$$

It is well-known that $\frac{\sin x}{x^3}$ is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_3'(x)}{f_4'(x)}$ is decreasing on $(0, \frac{\pi}{2})$. By Lemma 2.1,

$$\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)}$$

is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_1'(x)}{f_2'(x)}$is decreasing on $(0, \frac{\pi}{2})$, then

$$h(x) = \frac{f_1(x) - f_1(\frac{\pi}{2})}{f_2(x) - f_2(\frac{\pi}{2})} = \frac{\frac{\sin x}{x} - \frac{\pi}{2}}{\pi^4 - 16x^4}$$

is decreasing on $(0, \frac{\pi}{2})$. By Lemma 2.1.

Furthermore, $\lim_{x \to 0+} h(x) = \frac{\pi-2}{\pi^5}$, $\lim_{x \to \frac{\pi}{2}-} h(x) = \frac{1}{2\pi^5}$. Thus $\frac{\pi-2}{\pi^5}$ and $\frac{1}{2\pi^5}$ are the best constants in (2.1). So the proof is complete □

Note: In a similar manner, we can obtain several interesting inequalities similar to (2.2). For example, let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -4x^2$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^3$, and $x \in (0, \pi/2]$, then (1.2) is obtained. If we let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -8x^3$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^4$, then we have

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^4}(\pi^3 - 8x^3).$$

3. APPLICATIONS

Yang Le's inequality [4] and its generalizations which play an important role in the theory of distribution of values of functions can be stated as follows.

If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A + \cos^2 \lambda B - 2\cos\lambda A \cos\lambda B \cos\lambda\pi \geq \sin^2 \lambda\pi. \tag{3.1}$$

In [5] – [7] some improvements of Yang Le's inequality are obtained. In a similar way, based on the inequality (2.2) we can give the following.

Theorem 3.1. *Let* $A_i > 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^n A_i \leq \pi$, $n \in \mathbb{N}$ *and* $n \neq 1$, $0 \leq \lambda \leq 1$*, then*

$$R(\lambda) \leq \sum_{1\leq i<j\leq n} H_{ij} \leq T(\lambda), \tag{3.2}$$

where

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos\lambda A_i \cos\lambda A_j \cos\lambda\pi,$$
$$R(\lambda) = 4C_n^2 \left(\lambda + \frac{1}{4}\lambda(1-\lambda^4)\right)^2 \cos^2 \frac{\lambda}{2}\pi,$$
$$T(\lambda) = 4C_n^2 \left(\lambda + \frac{\pi-2}{2}\lambda(1-\lambda^4)\right)^2.$$

Proof. Substituting $x = \frac{\lambda}{2}\pi$ in (2.2),we have

$$\sin\frac{\lambda}{2}\pi \geq \lambda + \frac{1}{4}\lambda(1-\lambda^4) \tag{3.3}$$

and

$$\sin\frac{\lambda}{2}\pi \leq \lambda + \frac{\lambda-2}{2}\lambda(1-\lambda^4) \tag{3.4}$$

since

$$\sin^2 \lambda\pi = 4\sin^2 \frac{\lambda}{2}\pi \cos^2 \frac{\lambda}{2}\pi. \tag{3.5}$$

Using the inequality (see [6])

$$\sin^2 \lambda\pi \leq H_{ij} \leq 4\sin^2 \frac{\lambda}{2}\pi \tag{3.6}$$

and the identity (3.5) it follows that

$$4\left(\lambda + \frac{1}{4}\lambda\left(1-\lambda^4\right)\right)^2 \cos^2 \frac{\lambda}{2}\pi \leq H_{ij} \leq 4\left(\lambda + \frac{\pi-2}{2}\lambda(1-\lambda^4)\right)^2 \tag{3.7}$$

let $1 \leq i < j \leq n$.Taking the sum for all the inequalities in (3.7), we obtain (3.2), and the proof of Theorem 3.1 is thus complete. □

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