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A NEW OBSTRUCTION TO MINIMAL ISOMETRIC IMMERSIONS INTO A REAL SPACE FORM

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Abstract

In the theory of minimal submanifolds, the following problem is fundamental: *when does a given Riemannian manifold admit (or does not admit) a minimal isometric immersion into an Euclidean space of arbitrary dimension?* S.S. Chern, in his monograph [6] *Minimal submanifolds in a Riemannian manifold*, remarked that the result of Takahashi (*the Ricci tensor of a minimal submanifold into a Euclidean space is negative semidefinite*) was the only known Riemannian obstruction to minimal isometric immersions in Euclidean spaces. A second obstruction was obtained by B.Y. Chen as an immediate application of his fundamental inequality [1]: *the scalar curvature and the sectional curvature of a minimal submanifold into a Euclidean space satisfies the inequality $\tau \leq k$* . We find a new relation between the Chen invariant, the dimension of the submanifold, the length of the mean curvature vector field and a deviation parameter. This result implies a new obstruction: *the sectional curvature of a minimal submanifold into a Euclidean space also satisfies the inequality $k \leq -\tau$* .

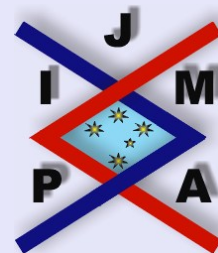
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1. Optimizations on Riemannian Manifolds

Let (N, \tilde{g}) be a Riemannian manifold, (M, g) a Riemannian submanifold, and $f \in \mathcal{F}(N)$. To these ingredients we attach the optimum problem

$$(1.1) \quad \min_{x \in M} f(x).$$

The fundamental properties of such programs are given in the papers [7] – [9]. For the interest of this paper we recall below a result obtained in [7].

Theorem 1.1. *If $x_0 \in M$ is a solution of the problem (1.1), then*

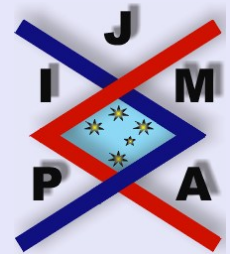
i) $(\text{grad } f)(x_0) \in T_{x_0}^\perp M,$

ii) *the bilinear form*

$$\alpha : T_{x_0} M \times T_{x_0} M \rightarrow R,$$
$$\alpha(X, Y) = \text{Hess } f(X, Y) + \tilde{g}(h(X, Y), (\text{grad } f)(x_0))$$

is positive semidefnite, where h is the second fundamental form of the submanifold M in N .

Remark 1. *The bilinear form α is nothing else but $\text{Hess } f|_M(x_0)$.*



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2. Chen's Inequality

Let (M, g) be a Riemannian manifold of dimension n , and x a point in M . We consider the orthonormal frame $\{e_1, e_2, \dots, e_n\}$ in $T_x M$.

The *scalar curvature* at x is defined by

$$\tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j).$$

We denote

$$\delta_M = \tau - \min(k),$$

where k is the *sectional curvature* at the point x . The invariant δ_M is called the *Chen's invariant* of Riemannian manifold (M, g) .

The Chen's invariant was estimated as the following: “ (M, g) is a Riemannian submanifold in a real space form $\widetilde{M}(c)$, varying with c and the length of the mean curvature vector field of M in $\widetilde{M}(c)$.”

Theorem 2.1. Consider $(\widetilde{M}(c), \widetilde{g})$ a real space form of dimension m , $M \subset \widetilde{M}(c)$ a Riemannian submanifold of dimension $n \geq 3$. The Chen's invariant of M satisfies

$$\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1)c \right\},$$

where H is the mean curvature vector field of submanifold M in $\widetilde{M}(c)$. Equality is attained at a point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_x M$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ in



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which the Weingarten operators take the following form

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & h_{33}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn}^{n+1} \end{pmatrix},$$

with $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$ and

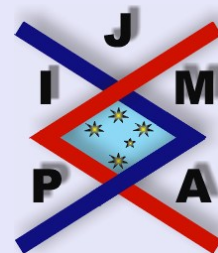
$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad r \in \overline{n+2, m}.$$

Corollary 2.2. *If the Riemannian manifold (M, g) , of dimension $n \geq 3$, admits a minimal isometric immersion into a real space form $\widetilde{M}(c)$, then*

$$k \geq \tau - \frac{(n-2)(n+1)c}{2}.$$

The aim of this paper is threefold:

- to formulate a new theorem regarding the relation between δ_M , the dimension n , the length of the mean curvature vector field, and a deviation parameter a ;



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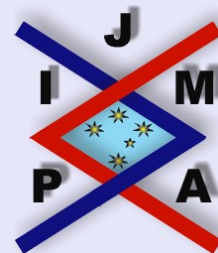
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- to prove this new theorem using the technique of Riemannian programming;
- to obtain a new obstruction, $k \leq -\tau + \frac{(n^2-n+2)c}{2}$, for minimal isometric immersions in real space forms.



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3. A New Obstruction To Minimal Isometric Immersions Into A Real Space Form

Let (M, g) be a Riemannian manifold of dimension n , and a a real number. We define the following invariants

$$\delta_M^a = \begin{cases} \tau - a \min k, & \text{for } a \geq 0, \\ \tau - a \max k, & \text{for } a < 0, \end{cases}$$

where τ is the scalar curvature, and k is the sectional curvature.

With these ingredients we obtain

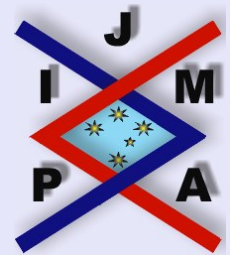
Theorem 3.1. *For any real number $a \in [-1, 1]$, the invariant δ_M^a of a Riemannian submanifold (M, g) , of dimension $n \geq 3$, into a real space form $\widetilde{M}(c)$, of dimension m , verifies the inequality*

$$\delta_M^a \leq \frac{(n^2 - n - 2a)c}{2} + \frac{n(a+1) - 3a - 1}{n(a+1) - 2a} \frac{n^2 \|H\|^2}{2},$$

where H is the mean curvature vector field of submanifold M in $\widetilde{M}(c)$.

If $a \in (-1, 1)$, equality is attained at the point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_x M$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ in which the Weingarten operators take the form

$$A_r = \begin{pmatrix} h_{11}^r & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^r & 0 & \cdots & 0 \\ 0 & 0 & h_{33}^r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn}^r \end{pmatrix},$$



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with $(a + 1)h_{11}^r = (a + 1)h_{22}^r = h_{33}^r = \dots = h_{nn}^r, \forall r \in \overline{n + 1, m}$.

Proof. Consider $x \in M$, $\{e_1, e_2, \dots, e_n\}$ an orthonormal frame in $T_x M$, $\{e_{n+1}, e_{n+2}, \dots, e_m\}$ an orthonormal frame in $T_x^\perp M$ and $a \in (-1, 1)$.

From Gauss' equation it follows

$$\begin{aligned} \tau - ak(e_1 \wedge e_2) &= \frac{(n^2 - n - 2a)c}{2} \\ &+ \sum_{r=n+1}^m \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) - a \sum_{r=n+1}^m (h_{11}^r h_{22}^r - (h_{12}^r)^2). \end{aligned}$$

Using the fact that $a \in (-1, 1)$, we obtain

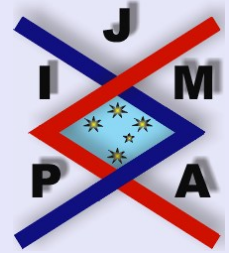
$$(3.1) \quad \tau - ak(e_1 \wedge e_2) \leq \frac{(n^2 - n - 2a)c}{2} + \sum_{r=n+1}^m \sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r - a \sum_{r=n+1}^m h_{11}^r h_{22}^r.$$

For $r \in \overline{n + 1, m}$, let us consider the quadratic form

$$\begin{aligned} f_r &: R^n \rightarrow R, \\ f_r(h_{11}^r, h_{22}^r, \dots, h_{nn}^r) &= \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r) - ah_{11}^r h_{22}^r \end{aligned}$$

and the constrained extremum problem

$$\begin{aligned} &\max f_r, \\ &\text{subject to } P : h_{11}^r + h_{22}^r + \dots + h_{nn}^r = k^r, \end{aligned}$$



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where k^r is a real constant.

The first three partial derivatives of the function f_r are

$$(3.2) \quad \frac{\partial f_r}{\partial h_{11}^r} = \sum_{2 \leq j \leq n} h_{jj}^r - ah_{22}^r,$$

$$(3.3) \quad \frac{\partial f_r}{\partial h_{22}^r} = \sum_{j \in \overline{1, n} \setminus \{2\}} h_{jj}^r - ah_{11}^r,$$

$$(3.4) \quad \frac{\partial f_r}{\partial h_{33}^r} = \sum_{j \in \overline{1, n} \setminus \{3\}} h_{jj}^r.$$

As for a solution $(h_{11}^r, h_{22}^r, \dots, h_{nn}^r)$ of the problem in question, the vector $(\text{grad})(f_1)$ being normal at P , from (3.2) and (3.3) we obtain

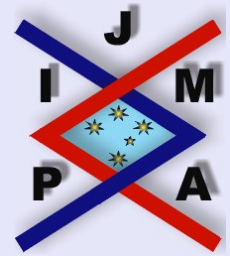
$$\sum_{j=1}^n h_{jj}^r - h_{11}^r - ah_{22}^r = \sum_{j=1}^n h_{jj}^r - h_{22}^r - ah_{11}^r,$$

therefore

$$(3.5) \quad h_{11}^r = h_{22}^r = b^r.$$

From (3.2) and (3.4), it follows

$$\sum_{j=1}^n h_{jj}^r - h_{11}^r - ah_{22}^r = \sum_{j=1}^n h_{jj}^r - h_{33}^r.$$



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By using (3.5) we obtain $h_{33}^r = b^r(a + 1)$. Similarly one gets

$$(3.6) \quad h_{jj}^r = b^r(a + 1), \quad \forall j \in \overline{3, n}.$$

As $h_{11}^r + h_{22}^r + \dots + h_{nn}^r = k^r$, from (3.5) and (3.6) we obtain

$$(3.7) \quad b^r = \frac{k^r}{n(a + 1) - 2a}.$$

We fix an arbitrary point $p \in P$.

The 2-form $\alpha : T_p P \times T_p P \rightarrow R$ has the expression

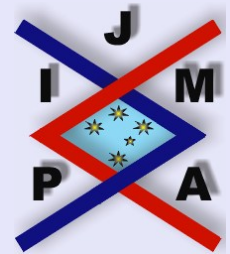
$$\alpha(X, Y) = \text{Hess } f_r(X, Y) + \langle h'(X, Y), (\text{grad } f_r)(p) \rangle,$$

where h' is the second fundamental form of P in R^n and $\langle \cdot, \cdot \rangle$ is the standard inner-product on R^n .

In the standard frame of R^n , the Hessian of f_r has the matrix

$$\text{Hess } f_r = \begin{pmatrix} 0 & 1 - a & 1 & \cdots & 1 \\ 1 - a & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

As P is totally geodesic in R^n , considering a vector X tangent to P at the



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arbitrary point p , that is, verifying the relation $\sum_{i=1}^n X^i = 0$, we have

$$\begin{aligned}
 \alpha(X, X) &= 2 \sum_{1 \leq i < j \leq n} X^i X^j - 2aX^1 X^2 \\
 &= \left(\sum_{i=1}^n X^i \right)^2 - \sum_{i=1}^n (X^i)^2 - 2aX^1 X^2 \\
 &= - \sum_{i=1}^n (X^i)^2 - a(X^1 + X^2)^2 + a(X^1)^2 + a(X^2)^2 \\
 &= - \sum_{i=3}^n (X^i)^2 - a(X^1 + X^2)^2 - (1-a)(X^1)^2 - (1-a)(X^2)^2 \\
 &\leq 0.
 \end{aligned}$$

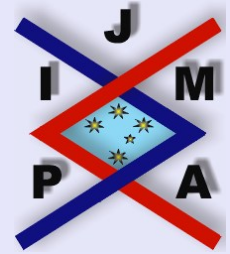
So $\text{Hess } f|_M$ is everywhere negative semidefinite, therefore the point $(h_{11}^r, h_{22}^r, \dots, h_{nn}^r)$, which satisfies (3.5), (3.6), (3.7) is a global maximum point.

From (3.5) and (3.6), it follows

$$\begin{aligned}
 (3.8) \quad f_r &\leq (b^r)^2 + 2b^r(n-2)b^r(a+1) + C_{n-2}^2(b^r)^2(a+1)^2 - a(b^r)^2 \\
 &= \frac{(b^r)^2}{2} [n^2(a+1)^2 - n(a+1)(5a+1) + 6a^2 + 2a] \\
 &= \frac{(b^r)^2}{2} [n(a+1) - 3a - 1][n(a+1) - 2a].
 \end{aligned}$$

By using (3.7) and (3.8), we obtain

$$(3.9) \quad f_r \leq \frac{(k^r)^2}{2[n(a+1) - 2a]} [n(a+1) - 3a - 1]$$



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$$= \frac{n^2(H^r)^2}{2} \cdot \frac{n(a+1) - 3a - 1}{n(a+1) - 2a}.$$

The relations (3.1) and (3.9) imply

$$(3.10) \quad \tau - ak(e_1 \wedge e_2) \leq \frac{(n^2 - n - 2a)c}{2} + \frac{n(a+1) - 3a - 1}{n(a+1) - 2a} \cdot \frac{n^2 \|H\|^2}{2}.$$

In (3.10) we have equality if and only if the same thing occurs in the inequality (3.1) and, in addition, (3.5) and (3.6) occur. Therefore

$$(3.11) \quad h_{ij}^r = 0, \quad \forall r \in \overline{n+1, m}, \forall i, j \in \overline{1, n}, \quad \text{with } i \neq j$$

and

$$(3.12) \quad (a+1)h_{11}^r = (a+1)h_{22}^r = h_{33}^r = \dots = h_{nn}^r, \forall r \in \overline{n+1, m}.$$

The relations (3.10), (3.11) and (3.12) imply the conclusion of the theorem. \square

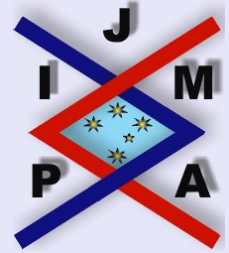
Remark 2.

i) Making a to converge at 1 in the previous inequality, we obtain **Chen's Inequality**. The conditions for which we have equality are obtained in [1] and [7].

ii) For $a = 0$ we obtain the well-known inequality

$$\tau \leq \frac{n(n-1)}{2} (\|H\|^2 + c).$$

The equality is attained at the point $x \in M$ if and only if x is a totally umbilical point.



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iii) Making a converge at -1 in the previous inequality, we obtain

$$\delta_M^{-1} \leq \frac{(n^2 - n + 2)c}{2} + \frac{n^2 \|H\|^2}{2}.$$

The equality is attained at the point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_x M$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ in which the Weingarten operators take the following form

$$A_r = \begin{pmatrix} h_{11}^r & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

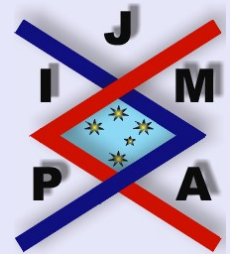
with $h_{11}^r = h_{22}^r, \forall r \in \overline{n+1, m}$.

Corollary 3.2. *If the Riemannian manifold (M, g) , of dimension $n \geq 3$, admits a minimal isometric immersion into a real space form $\widetilde{M}(c)$, then*

$$\tau - \frac{(n-2)(n+1)c}{2} \leq k \leq -\tau + \frac{(n^2 - n + 2)c}{2}.$$

Corollary 3.3. *If the Riemannian manifold (M, g) , of dimension $n \geq 3$, admits a minimal isometric immersion into a Euclidean space, then*

$$\tau \leq k \leq -\tau.$$



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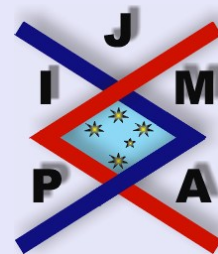
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