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## ON MINKOWSKI AND HARDY INTEGRAL INEQUALITIES

LAZHAR BOUGOFFA

Faculty of Computer Science and Information  
Al-Imam Muhammad Ibn Saud Islamic University  
P.O. Box 84880, Riyadh 11681

*EMail:* [lbougoffa@ccis.imamu.edu.sa](mailto:lbougoffa@ccis.imamu.edu.sa)



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## Abstract

The reverse Minkowski's integral inequality:

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq c \left(\int_a^b (f(x) + g(x))^p dx\right)^{\frac{1}{p}}, \quad p > 1,$$

where  $c$  is a positive constant, and the following Hardy's inequality:

$$\int_0^\infty \left(\frac{F_1(x)F_2(x)\cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \leq \left(\frac{p}{ip-i}\right)^p \int_0^\infty (f_1(x) + f_2(x) + \cdots + f_i(x))^p dx, \quad p > 1,$$

where

$$F_k(x) = \int_a^x f_k(t)dt, \quad \text{where } k = 1, \dots, i$$

are proved.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Minkowski's inequality, Hardy's inequality.

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# 1. The Reverse Minkowski Integral Inequality

In [1, 3, 4], the well-known Minkowski integral inequality is given as follows:

**Theorem 1.1.** Let  $p \geq 1$ ,  $0 < \int_a^b f^p(x)dx < \infty$  and  $0 < \int_a^b g^p(x)dx < \infty$ . Then

$$(1.1) \quad \left( \int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}} \leq \left( \int_a^b f^p(x)dx \right)^{\frac{1}{p}} + \left( \int_a^b g^p(x)dx \right)^{\frac{1}{p}}.$$

In this section we establish the following reverse Minkowski integral inequality

**Theorem 1.2.** Let  $f$  and  $g$  be positive functions satisfying

$$(1.2) \quad 0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a, b].$$

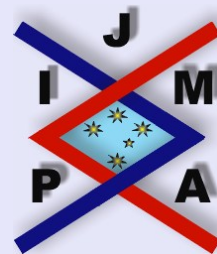
Then

$$(1.3) \quad \left( \int_a^b f^p(x)dx \right)^{\frac{1}{p}} + \left( \int_a^b g^p(x)dx \right)^{\frac{1}{p}} \leq c \left( \int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}},$$

where  $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ .

*Proof.* Since  $\frac{f(x)}{g(x)} \leq M$ ,  $f \leq M(f + g) - Mf$ . Therefore

$$(1.4) \quad (M + 1)^p f^p \leq M^p (f + g)^p$$



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and so,

$$(1.5) \quad \left( \int_a^b f^p(x) dx \right)^{\frac{1}{p}} \leq \frac{M}{M+1} \left( \int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}}$$

On the other hand, since  $mg \leq f$ . Hence

$$(1.6) \quad g \leq \frac{1}{m} (f(x) + g(x)) - \frac{1}{m} g(x).$$

Therefore,

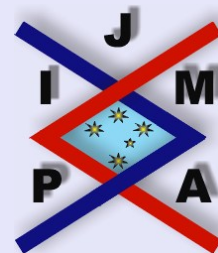
$$(1.7) \quad \left( \frac{1}{m} + 1 \right)^p g^p(x) \leq \left( \frac{1}{m} \right)^p (f(x) + g(x))^p,$$

and so,

$$(1.8) \quad \left( \int_a^b g^p(x) dx \right)^{\frac{1}{p}} \leq \frac{1}{m+1} \left( \int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}}.$$

Now add the inequalities (1.5) and (1.8) to get the desired inequality (1.1).

Thus, (1.1) is proved.  $\square$



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## 2. Hardy Integral Inequality Involving Many Functions

Hardy's inequality [2, 5] reads:

**Theorem 2.1.** Let  $f$  be a nonnegative integrable function. Define  $F(x) = \int_a^x f(t)dt$ . Then

$$(2.1) \quad \int_0^\infty \left( \frac{F(x)}{x} \right)^p dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty (f(x))^p dx, \quad p > 1.$$

Our purpose in this section is to prove the Hardy inequality for several functions.

**Theorem 2.2.** Let  $f_1, f_2, \dots, f_i$  be nonnegative integrable functions. Define  $F_k(x) = \int_a^x f_k(t)dt$ , where  $k = 1, \dots, i$ . Then

$$(2.2) \quad \int_0^\infty \left( \frac{F_1(x)F_2(x) \cdots F_i(x)}{x^i} \right)^{\frac{p}{i}} dx \leq \left( \frac{p}{ip-i} \right)^p \int_0^\infty (f_1(x) + f_2(x) + \cdots + f_i(x))^p dx.$$

*Proof.* By using Jensen's inequality [6, 7]

$$(2.3) \quad (F_1(x)F_2(x) \cdots F_i(x))^{\frac{1}{i}} \leq \frac{\sum_{k=1}^i F_k(x)}{i},$$



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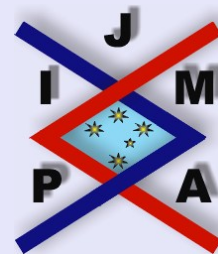
and so,

$$(2.4) \quad (F_1(x)F_2(x) \cdots F_i(x))^{\frac{p}{i}} \leq \frac{\left(\sum_{k=1}^i F_k(x)\right)^p}{i^p}.$$

Divide both sides of (2.4) by  $x^p$  and integrate resulting the inequality to get

$$(2.5) \quad \int_0^\infty \left(\frac{F_1(x)F_2(x) \cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \leq \frac{1}{i^p} \int_0^\infty \left(\frac{F_1(x) + F_2(x) + \cdots + F_i(x)}{x}\right)^p dx.$$

Applying inequality (2.1) to the right hand side of (2.5) we get (2.2). □



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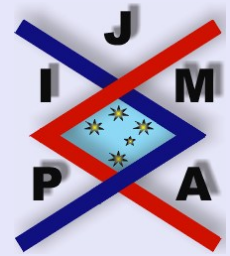
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## References

- [1] M. ABRAMOWITZ AND I.A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 11, 1972.
- [2] T.A.A. BROADBENT, A proof of Hardy's convergence theorem, *J. London Math. Soc.*, **3** (1928), 232–243.
- [3] I.S. GRADSHTEYN AND I.M. RYZHIK, *Tables of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic Press, pp. 1092 and 1099, 2000.
- [4] G.H. HARDY, J.E. LITTLEWOOD, AND G. PÓLYA, “Minkowski's Inequality” and “Minkowski's Inequality for Integrals”, §2.11, 5.7, and 6.13 in *Inequalities*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 30–32, 123, and 146–150, 1988.
- [5] G.H. HARDY, Note on a theorem of Hilbert, *Math. Z.*, **6** (1920), 314–317.
- [6] S.G. KRANTZ, Jensen's Inequality, §9.1.3 in *Handbook of Complex Variables*, Boston, MA: Birkhäuser, p. 118, 1999.
- [7] J.L.W.V. JENSEN, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.*, **30** (1906), 175–193.



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