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ON NEW INEQUALITIES OF HADAMARD-TYPE FOR LIPSCHITZIAN MAPPINGS AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we study some new inequalities of Hadamard's Type for Lipschitzian mappings. some applications are also included.

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1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}$ $(a < b)$ be a continuous function.
If f is convex on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities in (1.1) are known as the Hermite-Hadamard inequality [1].

For some recent results which generalize, improve, and extend this classic inequality, see references of [2] – [7]. In order to refine inequalities of (1.1), the author of this paper in [2] defined the following some notations, symbols and mappings. we list these notations and symbols by

$Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $\mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$, $T_n = t_1 + t_2 + \cdots + t_n$; $\mathbf{0} = (0, 0, \dots, 0)$, $\mathbf{1} = (1, 1, \dots, 1)$, $\frac{\mathbf{1}}{\mathbf{n}} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ and $(1_i, 0) = (0, \dots, 0, 1, 0, \dots, 0)$ (1 is ith component, $i = 1, 2, \dots, n$) are special points in $\mathbb{R}^n$; $G = [0, \frac{1}{n}] \times [0, \frac{1}{n}] \times \cdots \times [0, \frac{1}{n}]$, $I = [0,1]\times[0,1]\times\cdots\times[0,1]$, $V = [a,b]\times[a,b]\times\cdots\times[a,b]$, $D = [a,x_1]\times[x_1,x_2]\times\cdots\times[x_{n-1},b]$ ($x_i = a + \frac{(b-a)i}{n}, i = 0, 1, \dots, n; x_0 = a, x_n = b$), $H = \{\mathbf{t} \in I | T_n \le 1\}$ and $L = \{\mathbf{t} \in I | T_n = 1\}$ are subsets in $\mathbb{R}^n$.

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We list these mappings by

$$R_n : I \mapsto \mathbb{R}, \quad R_n(\mathbf{t}) \triangleq \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^n \left(t_i y_i + (1-t_i)\frac{x_{i-1}+x_i}{2}\right)\right) dY,$$

$$S_n : H \mapsto \mathbb{R}, \quad S_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n}\int_V f\left(\sum_{i=1}^n t_i y_i + (1-T_n)\frac{a+b}{2}\right) dY$$

and

$$P_n : L \mapsto \mathbb{R}, \quad P_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n}\int_V f\left(\sum_{i=1}^n t_i y_i\right) dY.$$

We write P_{n+1} in the following equivalent form

$$P_{n+1} : H \mapsto \mathbb{R}, \quad P_{n+1}(\mathbf{t}) \triangleq \frac{1}{(b-a)^{n+1}}\int_V \left[\int_a^b f\left(\sum_{i=1}^n t_i y_i + (1-T_n)x\right) dx\right] dY.$$

Let $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$. For all $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \dots, t_n^{(j)}\right) \in A (j = 1, 2)$ with $t_i^{(1)} \leq t_i^{(2)}$ $(i = 1, 2, \dots, n)$, if $g(\mathbf{t}^{(1)}) \leq g(\mathbf{t}^{(2)})$, then we call g increasing on A.

For these mappings and if f is convex on $[a, b]$, L.-C. Wang in [2] gave the following properties and inequalities:

P_n is convex on L; R_n and S_n are convex, increasing on I and G, respectively;

$$f\left(\frac{a+b}{2}\right) = R_n(\mathbf{0}) \leq R_n(\mathbf{t}) \leq R_n(\mathbf{1}) = \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^n y_i\right) dY \leq \frac{1}{b-a}\int_a^b f(x)dx \tag{1.2}$$

for any $\mathbf{t} \in I$,

$$f\left(\frac{a+b}{2}\right) = S_n(\mathbf{0}) \leq S_n(\mathbf{t}) \leq S_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) = \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right) dY \tag{1.3}$$

for all $\mathbf{t} \in G$,

$$S_n(\mathbf{t}) \leq P_{n+1}(\mathbf{t}) \tag{1.4}$$

for all $\mathbf{t} \in H$, and

$$S_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) = P_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \leq P_n(\mathbf{t}) \leq P_n(1_i, 0) = \frac{1}{b-a}\int_a^b f(x)dx \tag{1.5}$$

for all $\mathbf{t} \in L$.

(1.2) – (1.5) are refinements of (1.1).

Recently, Dragomir *et al.* [3], Yang and Tseng [5], Matic and Pečarić [6] and L.-C. Wang [7] proved some results for Lipschitzian mappings related to (1.1). In this paper, we will prove some new inequalities for Lipschitzian mappings related to the mappings R_n (or (1.2)), S_n (or (1.3)) and P_n (or (1.5) and (1.4)). Finally, some applications are given.

2. MAIN RESULTS

A function $f : [a, b] \to \mathbb{R}$ is called an $M-$Lipschitzian mapping, if for every two elements $x, y \in [a, b]$ and $M > 0$ we have

$$|f(x) - f(y)| \leq M|x - y|.$$

For the mapping $R_n(\mathbf{t})$, we have the following theorem:

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an $M-$Lipschitzian mapping, then we have*

$$\left|R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)})\right| \leq \frac{M}{4n^2}(b-a)\sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.1}$$

for any $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \ldots, t_n^{(j)}\right) \in I (j = 1, 2)$,

$$\left|f\left(\frac{a+b}{2}\right) - R_n(\mathbf{t})\right| \leq \frac{M}{4n^2}(b-a)T_n \tag{2.2}$$

and

$$\left|R_n(\mathbf{t}) - \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY\right| \leq \frac{M}{4n^2}(b-a)\left(n - T_n\right) \tag{2.3}$$

for all $t \in I$, and

$$\left|\left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M(n^2-1)}{3n^2}(b-a). \tag{2.4}$$

Proof. (1) For $x_i = \frac{(b-a)i}{n}$ ($i = 0, 1, \ldots, n$; $x_0 = a, x_n = b$), from integral properties, we have

$$\begin{aligned}
&\left|R_n\left(\mathbf{t}^{(2)}\right) - R_n\left(\mathbf{t}^{(1)}\right)\right| \\
&\leq \left(\frac{n}{b-a}\right)^n \int_D \left| f\left(\frac{1}{n}\sum_{i=1}^{n}\left(t_i^{(2)} y_i + (1 - t_i^{(2)})\frac{x_{i-1}+x_i}{2}\right)\right)\right. \\
&\qquad\qquad \left. - f\left(\frac{1}{n}\sum_{i=1}^{n}\left(t_i^{(1)} y_i + (1 - t_i^{(1)})\frac{x_{i-1}+x_i}{2}\right)\right)\right| dY \\
&\leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n}\int_D \left|\sum_{i=1}^{n}\left(t_i^{(2)} - t_i^{(1)}\right)\left(y_i - \frac{x_{i-1}+x_i}{2}\right)\right| dY \\
&\leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n}\sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \int_D \left|y_i - \frac{x_{i-1}+x_i}{2}\right| dY \\
&= \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n}\sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \left(\frac{b-a}{n}\right)^{n-1} \int_{x_{i-1}}^{x_i} \left|y_i - \frac{x_{i-1}+x_i}{2}\right| dy_i
\end{aligned}$$

$$\begin{aligned}
&= \frac{M}{b-a}\sum_{i=1}^{n}\left|t_i^{(2)}-t_i^{(1)}\right|\left[\int_{x_{i-1}}^{\frac{x_{i-1}+x_i}{2}}\left(\frac{x_{i-1}+x_i}{2}-y_i\right)dy_i\right.\\
&\qquad\left.+\int_{\frac{x_{i-1}+x_i}{2}}^{x_i}\left(y_i-\frac{x_{i-1}+x_i}{2}\right)dy_i\right]\\
&= \frac{M}{4n^2}(b-a)\sum_{i=1}^{n}\left|t_i^{(2)}-t_i^{(1)}\right|.
\end{aligned}$$

This completes the proof of (2.1).

(2) The inequalities (2.2) and (2.3) follow from (2.1) by choosing $\mathbf{t}^{(1)} = \mathbf{0}$, $\mathbf{t}^{(2)} = \mathbf{t}$ and $\mathbf{t}^{(1)} = \mathbf{1}$, $\mathbf{t}^{(2)} = \mathbf{t}$, respectively. This completes the proof of (2.2) and (2.3).

(3) From integral properties, we have

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(y_i)dy_i = \left(\frac{n}{b-a}\right)^{n-1}\sum_{i=1}^{n}\int_D f(y_i)dY. \tag{2.5}$$

Using (2.5) and integral properties, we obtain

$$\begin{aligned}
&\left|\left(\frac{n}{b-a}\right)^n\int_D f\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)dY-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le \left(\frac{n}{b-a}\right)^n\int_D\left|\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{1}{n}\sum_{j=1}^{n}y_j\right)-\frac{1}{n}\sum_{i=1}^{n}f(y_i)\right|dY\\
&\le \left(\frac{n}{b-a}\right)^n\cdot\frac{M}{n}\int_D\sum_{i=1}^{n}\left|\frac{1}{n}\sum_{j=1}^{n}y_j-y_i\right|dY\\
&\le \left(\frac{n}{b-a}\right)^n\cdot\frac{M}{n^2}\sum_{i=1}^{n}\int_D\sum_{j=1}^{n}|y_j-y_i|\,dY\\
&= \left(\frac{n}{b-a}\right)^n\cdot\frac{M}{n^2}\sum_{i=1}^{n}\int_D\left[\sum_{j=1}^{i-1}(y_i-y_j)+\sum_{j=i+1}^{n}(y_j-y_i)\right]dY\\
&= \frac{n}{b-a}\cdot\frac{M}{n^2}\sum_{i=1}^{n}\left[\sum_{j=1}^{i-1}\left(\int_{x_{i-1}}^{x_i}y_idy_i-\int_{x_{j-1}}^{x_j}y_jdy_j\right)\right.\\
&\qquad\left.+\sum_{j=i+1}^{n}\left(\int_{x_{j-1}}^{x_j}y_jdy_j-\int_{x_{i-1}}^{x_i}y_idy_i\right)\right]\\
&= \frac{n}{b-a}\cdot\frac{M}{n^2}\sum_{i=1}^{n}\left[\left(\frac{b-a}{n}\right)^2\left(\sum_{j=1}^{i-1}(i-j)+\sum_{j=i+1}^{n}(j-i)\right)\right]\\
&= \frac{M(n^2-1)}{3n^2}(b-a).
\end{aligned}$$

This completes the proof of (2.4).

This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let f be convex on $[a,b]$, with $f'_+(a)$ and $f'_-(b)$ existing. Then we obtain*

$$\begin{aligned} 0 &\le R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)}) \\ &\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)\sum_{i=1}^{n}\left(t_i^{(2)} - t_i^{(1)}\right) \end{aligned} \tag{2.6}$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in I (j = 1, 2)$ with $t_i^{(2)} \ge t_i^{(1)} (i = 1, 2, \dots, n)$,

$$0 \le R_n(\mathbf{t}) - f\left(\frac{a+b}{2}\right) \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)T_n \tag{2.7}$$

and

$$\begin{aligned} 0 &\le \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - R_n(\mathbf{t}) \\ &\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)(n-T_n) \end{aligned} \tag{2.8}$$

for all $\mathbf{t} \in I$, and

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)dx - \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY \\ &\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}(n^2-1)}{3n^2}(b-a). \end{aligned} \tag{2.9}$$

Proof. For any $x, y \in [a,b]$, from properties of convex functions, we have the following $\max\{|f'_+(a)|,|f'_-(b)|\}-$Lipschitzian inequality (see [8]):

$$|f(x) - f(y)| \le \max\{|f'_+(a)|,|f'_-(b)|\}|x-y|. \tag{2.10}$$

Since R_n is increasing on I, using (1.2), (2.10) and Theorem 2.1, we obtain (2.6)-(2.9).

This completes the proof of Corollary (2.2). □

For the mapping $S_n(\mathbf{t})$, we have the following theorem:

Theorem 2.3. *Let f be defined as in Theorem 2.1, then we obtain*

$$\left|S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)})\right| \le \frac{M}{4}(b-a)\sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.11}$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in H (j = 1, 2)$,

$$\left|f\left(\frac{a+b}{2}\right) - S_n(\mathbf{t})\right| \le \frac{M}{4}(b-a)T_n \tag{2.12}$$

and

$$\left|S_n(\mathbf{t}) - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY\right| \le \frac{M}{4n}(b-a)\sum_{i=1}^{n}|nt_i - 1| \tag{2.13}$$

for all $\mathbf{t} \in H$, and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY\right| \le \frac{M}{4}(b-a). \tag{2.14}$$

Proof. (1) From integral properties, we obtain

$$\begin{aligned}
&\left|S_n\left(\mathbf{t}^{(2)}\right)-S_n\left(\mathbf{t}^{(1)}\right)\right| \\
&\le \frac{1}{(b-a)^n}\int_V \left| f\left(\sum_{i=1}^n t_i^{(2)}y_i+\left(1-\sum_{i=1}^n t_i^{(2)}\right)\frac{a+b}{2}\right)\right. \\
&\qquad\qquad \left. -f\left(\sum_{i=1}^n t_i^{(1)}y_i+\left(1-\sum_{i=1}^n t_i^{(1)}\right)\frac{a+b}{2}\right)\right| dY \\
&\le \frac{M}{(b-a)^n}\int_V \left|\sum_{i=1}^n \left(t_i^{(2)}-t_i^{(1)}\right)\left(y_i-\frac{a+b}{2}\right)\right| dY \\
&\le \frac{M}{(b-a)^n}\sum_{i=1}^n \left|t_i^{(2)}-t_i^{(1)}\right|\int_V \left|y_i-\frac{a+b}{2}\right| dY \\
&= \frac{M}{b-a}\sum_{i=1}^n \left|t_i^{(2)}-t_i^{(1)}\right|\int_a^b \left|y_i-\frac{a+b}{2}\right| dy_i \\
&= \frac{M}{b-a}\sum_{i=1}^n \left|t_i^{(2)}-t_i^{(1)}\right|\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-y_i\right)dy_i+\int_{\frac{a+b}{2}}^b\left(y_i-\frac{a+b}{2}\right)dy_i\right] \\
&= \frac{M}{4}(b-a)\sum_{i=1}^n \left|t_i^{(2)}-t_i^{(1)}\right|.
\end{aligned}$$

This completes the proof of (2.11).

(2) The inequalities (2.12) and (2.13) follow from (2.11) by choosing $\mathbf{t}^{(1)}=\mathbf{0}$, $\mathbf{t}^{(2)}=\mathbf{t}$ and $\mathbf{t}^{(1)}=\frac{\mathbf{1}}{\mathbf{n}}$, $\mathbf{t}^{(2)}=\mathbf{t}$, respectively. The inequalities (2.14) follow from (2.12) by choosing $\mathbf{t}^{(1)}=\frac{\mathbf{1}}{\mathbf{n}}$. This completes the proof of (2.12)-(2.14).

This completes the proof of Theorem 2.3. □

Corollary 2.4. *Let f be defined as in Corollary 2.2, then we have*

$$0\le S_n(\mathbf{t}^{(2)})-S_n(\mathbf{t}^{(1)})\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4}(b-a)\sum_{i=1}^n\left(t_i^{(2)}-t_i^{(1)}\right) \tag{2.15}$$

for any $\mathbf{t}^{(j)}=(t_1^{(j)},\dots,t_n^{(j)})\in G(j=1,2)$ *with* $t_i^{(2)}\ge t_i^{(1)}$ $(i=1,2,\dots,n)$*,*

$$0\le S_n(\mathbf{t})-f\left(\frac{a+b}{2}\right)\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4}(b-a)T_n \tag{2.16}$$

and

$$\begin{aligned}
0&\le \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right)dY-S_n(\mathbf{t}) \\
&\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4}(b-a)(1-T_n)
\end{aligned} \tag{2.17}$$

for all $\mathbf{t}\in G$*, and*

$$\begin{aligned}
0&\le \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right)dY-f\left(\frac{a+b}{2}\right) \\
&\le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4}(b-a).
\end{aligned} \tag{2.18}$$

Proof. Since S_n is increasing on G, using (1.3), (2.10) and Theorem 2.3, we obtain (2.15) – (2.18).

This completes the proof of Corollary (2.4). $\square$

For the mapping $P_n(\mathbf{t})$, we have the following theorem:

Theorem 2.5. *Let f be defined as in Theorem 2.1. For $n \geq 2$, then we obtain*

$$|p_n(\mathbf{t}^{(2)}) - p_n(\mathbf{t}^{(1)})| \leq \frac{M}{3}(b-a)\sum_{i=1}^{n-1}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.19}$$

for any $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \dots, t_n^{(j)}\right) \in L (j = 1, 2)$,

$$\left|\frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - p_n(\mathbf{t})\right| \leq \frac{M}{3n}(b-a)\sum_{i=1}^{n-1}|nt_i - 1| \tag{2.20}$$

and

$$\left|p_n(\mathbf{t}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M}{3}(b-a)\sum_{i=1}^{n-1} t_i \tag{2.21}$$

for all $\mathbf{t} \in L$, *and*

$$\left|\frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M(n-1)}{3n}(b-a). \tag{2.22}$$

For $n \geq 1$ and all $\mathbf{t} \in H$, then we have

$$|S_n(\mathbf{t}) - P_{n+1}(\mathbf{t})| \leq \frac{M}{4}(b-a)(1 - T_n). \tag{2.23}$$

Proof. (1) Since $n \geq 2$ and $T_n = t_1 + \cdots + t_{n-1} + t_n = 1$, we can write $P_n(\mathbf{t})$ in the following equivalent form

$$P_n(\mathbf{t}) = \frac{1}{(b-a)^n}\int_V f\left(\sum_{i=1}^{n-1} t_i y_i + \left(1 - \sum_{i=1}^{n-1} t_i\right) y_n\right) dY. \tag{2.24}$$

Using (2.24) and integral properties, we obtain

$$\begin{aligned}
&\left|P_n\left(\mathbf{t}^{(2)}\right) - P_n\left(\mathbf{t}^{(1)}\right)\right| \\
&\leq \frac{1}{(b-a)^n}\int_V \left| f\left(\sum_{i=1}^{n-1} t_i^{(2)} y_i + \left(1 - \sum_{i=1}^{n-1} t_i^{(2)}\right) y_n\right)\right. \\
&\qquad\qquad \left. - f\left(\sum_{i=1}^{n-1} t_i^{(1)} y_i + \left(1 - \sum_{i=1}^{n-1} t_i^{(1)}\right) y_n\right)\right| dY \\
&\leq \frac{M}{(b-a)^n}\int_V \left|\sum_{i=1}^{n-1}\left(t_i^{(2)} - t_i^{(1)}\right)(y_i - y_n)\right| dY
\end{aligned}$$

$$\le \frac{M}{(b-a)^n}\sum_{i=1}^{n-1}\left|t_i^{(2)}-t_i^{(1)}\right|(b-a)^{n-2}\int_a^b\int_a^b |y_i-y_n|\,dy_i dy_n$$
$$=\frac{M}{(b-a)^2}\sum_{i=1}^{n-1}\left|t_i^{(2)}-t_i^{(1)}\right|\int_a^b\left[\int_a^x (x-y)dy+\int_x^b (y-x)dy\right]dx$$
$$=\frac{M}{3}(b-a)\sum_{i=1}^{n-1}\left|t_i^{(2)}-t_i^{(1)}\right|.$$

This completes the proof of (2.19).

(2) The inequalities (2.20) and (2.21) follow from (2.19) by choosing $\mathbf{t}^{(1)}=\frac{\mathbf{1}}{\mathbf{n}}$, $\mathbf{t}^{(2)}=\mathbf{t}$ and $\mathbf{t}^{(1)}=(1_n,0)=(0,\dots,0,1)$, $\mathbf{t}^{(2)}=\mathbf{t}$, respectively. The inequalities (2.22) follow from (2.21) by choosing $\mathbf{t}=\frac{\mathbf{1}}{\mathbf{n}}$. This completes the proof of (2.20) – (2.22).

(3) Using integral properties, we write $S_n(\mathbf{t})$ in the following equivalent form

$$S_n(\mathbf{t})=\frac{1}{(b-a)^{n+1}}\int_V\left[\int_a^b f\left(\sum_{i=1}^n t_i y_i+(1-T_n)\frac{a+b}{2}\right)dx\right]dY. \tag{2.25}$$

Using (2.25) and integral properties, we obtain

$$|S_n(\mathbf{t})-P_{n+1}(\mathbf{t})|$$
$$\le\frac{1}{(b-a)^{n+1}}\int_V\left[\int_a^b\left|f\left(\sum_{i=1}^n t_iy_i+(1-T_n)\frac{a+b}{2}\right)-f\left(\sum_{i=1}^n t_iy_i+(1-T_n)x\right)\right|dx\right]dY$$
$$\le\frac{M}{(b-a)^{n+1}}(1-T_n)\int_V\left[\int_a^b\left|\frac{a+b}{2}-x\right|dx\right]dY$$
$$=\frac{M}{b-a}(1-T_n)\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)dx+\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)dx\right]$$
$$=\frac{M}{4}(b-a)(1-T_n).$$

This completes the proof of (2.23).

This completes the proof of Theorem 2.5. □

Corollary 2.6. *Let f be defined as in Corollary 2.2. For $n\ge 2$, then we have*

$$\left|P_n(\mathbf{t}^{(2)})-P_n(\mathbf{t}^{(1)})\right|\le\frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{3}(b-a)\sum_{i=1}^{n-1}\left|t_i^{(2)}-t_i^{(1)}\right| \tag{2.26}$$

for any $\mathbf{t}^{(j)}=(t_1^{(j)},\dots,t_n^{(j)})\in L(j=1,2)$ *,*

$$0\le P_n(\mathbf{t})-\frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right)dY \tag{2.27}$$
$$\le\frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{3n}(b-a)\sum_{i=1}^{n-1}|nt_i-1|$$

and

$$0\le\frac{1}{b-a}\int_a^b f(x)dx-P_n(\mathbf{t})\le\frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{3}(b-a)\sum_{i=1}^{n-1}t_i \tag{2.28}$$

for all $\mathbf{t} \in L$, *and*

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right)dY \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}(n-1)}{3n}(b-a). \tag{2.29}$$

For $n \ge 1$ *and all* $\mathbf{t} \in H$, *we have*

$$0 \le P_{n+1}(\mathbf{t}) - S_n(\mathbf{t}) \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4}(b-a)(1-T_n). \tag{2.30}$$

Proof. Using (1.5), (1.4), (2.10) and Theorem 2.5, we obtain (2.26) – (2.30).

This completes the proof of Corollary (2.6). □

Remark 2.7. The condition in Corollary 2.2 (or Corollary 2.4 or 2.6) is better than the condition in Corollary 2.2 (or Corollary 4.2 or Theorem 3.3) of [3]. This is due to the fact that f is a differentiable convex function on $[a,b]$ with $M = \sup_{t\in[a,b]} |f'(t)| < \infty$.

Remark 2.8. When $n = 1$, (2.1) and (2.11), (2.2) and (2.12), (2.3) and (2.13) and (2.23) reduce to (3.4), (3.2), (3.1) and (4.3) of [3], respectively. When $n = 2$, (2.19), (2.20), and (2.21) reduce to (4.6), (4.1) and (4.2) of [3], respectively.

3. APPLICATIONS

In this section, we agree that when $t_i = 0$,

$$\frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] = \frac{\ln b - \ln a}{n^2} \quad \text{and} \quad \frac{b^{t_i} - a^{t_i}}{t_i} = \ln b - \ln a.$$

For $b > a > 0$, $1 \ge t_i^{(2)} \ge t_i^{(1)} \ge 0$ and $1 \ge t_i \ge 0$ $(i = 1, 2, \dots, n)$, we have

$$0 \le \prod_{i=1}^n \frac{1}{t_i^{(2)}}\left[\left(\frac{b}{a}\right)^{\frac{t_i^{(2)}}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i^{(2)}}{2n^2}}\right] - \prod_{i=1}^n \frac{1}{t_i^{(1)}}\left[\left(\frac{b}{a}\right)^{\frac{t_i^{(1)}}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i^{(1)}}{2n^2}}\right] \le \frac{1}{4}\left(\frac{\ln b - \ln a}{n^2}\right)^{n+1}\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^n\left(t_i^{(2)} - t_i^{(1)}\right), \tag{3.1}$$

$$0 \le \left(\frac{n^2}{\ln b - \ln a}\right)^n \prod_{i=1}^n \frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] - 1 \le \frac{\ln b - \ln a}{4n^2}\left(\frac{b}{a}\right)^{\frac{1}{2}} T_n \tag{3.2}$$

and

$$0 \le \prod_{i=1}^n \left[\left(\frac{b}{a}\right)^{\frac{1}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{1}{2n^2}}\right] - \prod_{i=1}^n \frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] \le \frac{1}{4}\left(\frac{\ln b - \ln a}{n^2}\right)^{n+1}\left(\frac{b}{a}\right)^{\frac{1}{2}}(n - T_n). \tag{3.3}$$

For $b > a > 0$, $\frac{1}{n} \geq t_i^{(2)} \geq t_i^{(1)} \geq 0$ and $\frac{1}{n} \geq t_i \geq 0$ $(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned} 0 &\leq (ab)^{\frac{1-\sum_{i=1}^n t_i^{(2)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left(b^{t_i^{(2)}} - a^{t_i^{(2)}}\right) - (ab)^{\frac{1-\sum_{i=1}^n t_i^{(1)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left(b^{t_i^{(1)}} - a^{t_i^{(1)}}\right) \\ &\leq \frac{b(\ln b - \ln a)^{n+1}}{4} \sum_{i=1}^n \left(t_i^{(2)} - t_i^{(1)}\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 &\leq \frac{1}{(\ln b - \ln a)^n} (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i}\right) - (ab)^{\frac{1}{2}} \\ &\leq \frac{b(\ln b - \ln a)}{4} T_n \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} 0 &\leq \left(nb^{\frac{1}{n}} - na^{\frac{1}{n}}\right)^n - (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i}\right) \\ &\leq \frac{b(\ln b - \ln a)^{n+1}}{4} (1 - T_n). \end{aligned} \tag{3.6}$$

For $b > a > 0$, $1 \geq t_i^{(j)} \geq 0$ $(i = 1, 2, \ldots, n; n \geq 2)$ and $t_1^{(j)} + t_2^{(j)} + \cdots + t_n^{(j)} = 1$ $(j = 1, 2)$, we have

$$\left| \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left(b^{t_i^{(2)}} - a^{t_i^{(2)}}\right) - \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left(b^{t_i^{(1)}} - a^{t_i^{(1)}}\right) \right| \leq \frac{b(\ln b - \ln a)^{n+1}}{3} \sum_{i=1}^{n-1} \left| t_i^{(2)} - t_i^{(1)} \right|. \tag{3.7}$$

For $b > a > 0$, $1 \geq t_i \geq 0$ $(i = 1, 2, \ldots, n;\ n \geq 2)$ and $T_n = 1$, we have

$$0 \leq \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i}\right) - \left(nb^{\frac{1}{n}} - na^{\frac{1}{n}}\right)^n \leq \frac{b(\ln b - \ln a)^{n+1}}{3n} \sum_{i=1}^{n-1} |nt_i - 1| \tag{3.8}$$

and

$$0 \leq \frac{b-a}{\ln b - \ln a} - \frac{1}{(\ln b - \ln a)^n} \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i}\right) \leq \frac{b(\ln b - \ln a)}{3} \sum_{i=1}^{n-1} t_i. \tag{3.9}$$

For $b > a > 0$, we have

$$\begin{aligned} 0 &\leq \frac{b-a}{\ln b - \ln a} - \left(\frac{n^2}{\ln b - \ln a}\right)^n (ab)^{\frac{1}{2}} \prod_{i=1}^n \left[\left(\frac{b}{a}\right)^{\frac{1}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{1}{2n^2}}\right] \\ &\leq \frac{b(n^2-1)}{3n^2} (\ln b - \ln a), \end{aligned} \tag{3.10}$$

$$0 \leq \left(\frac{nb^{\frac{1}{n}} - na^{\frac{1}{n}}}{\ln b - \ln a}\right)^n - (ab)^{\frac{1}{2}} \leq \frac{b(\ln b - \ln a)}{4} \tag{3.11}$$

and

$$0 \leq \frac{b-a}{\ln b - \ln a} - \left(\frac{n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right)}{\ln b - \ln a}\right)^n \leq \frac{b(n-1)}{3n} (\ln b - \ln a). \tag{3.12}$$

Indeed, (3.1) – (3.12) follow from (2.6) – (2.8), (2.15) – (2.17), (2.26) – (2.28), (2.9), (2.18) and (2.29) applied to the convex function $f : [\ln a, \ln b] \mapsto [a, b]$, $f(x) = e^x$, with some simple manipulations, respectively.

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