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# On a Family of Generalized Pascal Triangles Defined by Exponential Riordan Arrays

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**Abstract**

We introduce a family of number triangles defined by exponential Riordan arrays, which generalize Pascal's triangle. We characterize the row sums and central coefficients of these triangles, and define and study a set of generalized Catalan numbers. We establish links to the Hermite, Laguerre and Bessel polynomials, as well as links to the Narayana and Lah numbers.

## 1 Introduction

In [1], we studied a family of generalized Pascal triangles whose elements were defined by Riordan arrays, in the sense of [10, 13]. In this note, we use so-called "exponential Riordan arrays" to define another family of generalized Pascal triangles. These number triangles are easy to describe, and important number sequences derived from them are linked to both the Hermite and Laguerre polynomials, as well as being related to the Narayana and Lah numbers.

We begin by looking at Pascal's triangle, the binomial transform, exponential Riordan arrays, the Narayana numbers, and briefly summarize those features of the Hermite and Laguerre polynomials that we will require. We then introduce the family of generalized Pascal triangles based on exponential Riordan arrays, and look at a simple case in depth. We finish by enunciating a set of general results concerning row sums, central coefficients and generalized Catalan numbers for these triangles.

## 2 Preliminaries

Pascal's triangle, with general term $C(n,k) = \binom{n}{k}$, $n, k \geq 0$, has fascinated mathematicians by its wealth of properties since its discovery [6]. Viewed as an infinite lower-triangular matrix, it is invertible, with an inverse whose general term is given by $(-1)^{n-k}\binom{n}{k}$. Invertibility follows from the fact that $\binom{n}{n} = 1$. It is *centrally symmetric*, since by definition, $\binom{n}{k} = \binom{n}{n-k}$. All the terms of this matrix are integers.

By a generalized Pascal triangle we shall understand a lower-triangular infinite integer matrix $T = T(n,k)$ with $T(n,0) = T(n,n) = 1$ and $T(n,k) = T(n,n-k)$. We index all matrices in this paper beginning at the $(0,0)$-th element.

We shall encounter transformations that operate on integer sequences during the course of this note. An example of such a transformation that is widely used in the study of integer sequences is the so-called Binomial transform [17], which associates to the sequence with general term $a_n$ the sequence with general term $b_n$ where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{1}$$

If we consider the sequence with general term $a_n$ to be the vector $\mathbf{a} = (a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose $(n,k)$-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{2}$$

We note that $\mathbf{B}$ corresponds to Pascal's triangle. Its row sums are $2^n$, while its diagonal sums are the Fibonacci numbers $F(n+1)$. If $\mathbf{B}^m$ denotes the $m-$th power of $\mathbf{B}$, then the $n-$th term of $\mathbf{B}^m\mathbf{a}$ where $\mathbf{a} = \{a_n\}$ is given by $\sum_{k=0}^{n} m^{n-k}\binom{n}{k} a_k$.

If $\mathcal{A}(x)$ is the ordinary generating function of the sequence $a_n$, then the ordinary generating function of the transformed sequence $b_n$ is $\frac{1}{1-x}\mathcal{A}(\frac{x}{1-x})$. Similarly, if $\mathcal{G}(x)$ is the exponential generating function (e.g.f.) of the sequence $a_n$, then the exponential generating function of the binomial transform of $a_n$ is $\exp(x)\mathcal{G}(x)$.

The binomial transform is an element of the exponential Riordan group, which can be defined as follows.

The *exponential Riordan group* [5], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \ldots$ and $f(x) = f_1x + f_2x^2 + \ldots$ where $f_1 \neq 0$. The associated matrix is the matrix whose $k$-th column has exponential generating function $g(x)f(x)^k/k!$ (the first column being indexed by 0). The matrix corresponding to the pair $f, g$ is denoted by $(g, f)$ or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of $(g, f)$ is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of $f$.

If $\mathbf{M}$ is the matrix $(g, f)$, and $\mathbf{a} = \{a_n\}$ is an integer sequence with exponential generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has exponential generating function $g(x)\mathcal{A}(f(x))$.

We note at this juncture that the exponential Riordan group, as well as the group of 'standard' Riordan arrays [10] can be cast in the more general context of matrices of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ as found in [7, 8, 9]. Specifically, a matrix $C = (c_{nk})_{n,k=0,1,2,\ldots}$ is of type $R^q(\alpha_n, \beta_k; \phi, f, \psi)$ if its general term is defined by the formula

$$c_{nk} = \frac{\beta_k}{\alpha_n}\mathbf{res}_x(\phi(x)f^k(x)\psi^n(x)x^{-n+qk-1})$$

where $\mathbf{res}_x A(x) = a_{-1}$ for a given formal power series $A(x) = \sum_j a_jx^j$ is the formal residue of the series.

For the exponential Riordan arrays in this note, we have $\alpha_n = \frac{1}{n!}$, $\beta_k = \frac{1}{k!}$, and $q = 1$.

**Example 1.** The Binomial matrix $\mathbf{B}$ is the element $(e^x, x)$ of the exponential Riordan group. More generally, $\mathbf{B}^m$ is the element $(e^{mx}, x)$ of the Riordan group. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(e^{-mx}, x)$.

**Example 2.** The exponential generating function of the row sums of the matrix $(g, f)$ is obtained by applying $(g, f)$ to $e^x$, the e.g.f. of the sequence $1, 1, 1, \ldots$. Hence the row sums of $(g, f)$ have e.g.f. $g(x)e^{f(x)}$.

We shall frequently refer to sequences by their sequence number in the On-Line Encyclopedia of Integer Sequences [11, 12]. For instance, Pascal's triangle is A007318 while the Catalan numbers [18] $C(n) = \binom{2n}{n}/(n+1)$ are A000108.

**Example 3.** An example of a well-known centrally symmetric invertible triangle is the Narayana triangle $\tilde{\mathbf{N}}$, [14, 15], defined by

$$\tilde{N}(n, k) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}$$

for $n, k \geq 0$. Other expressions for $\tilde{N}(n, k)$ are given by

$$\tilde{N}(n, k) = \binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} = \binom{n+1}{k+1}\binom{n}{k} - \binom{n+1}{k}\binom{n}{k+1}.$$

This triangle begins

$$\tilde{\mathbf{N}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ 1 & 15 & 50 & 50 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that in the literature, it is often the triangle $\tilde{N}(n-1,k-1) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ that is referred to as the Narayana triangle. Alternatively, the triangle $\tilde{N}(n-1,k) = \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$ is referred to as the Narayana triangle. We shall denote this latter triangle by $N(n,k)$. We then have

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 20 & 10 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with row sums equal to the Catalan numbers $C(n)$.

Note that for $n,k \geq 1$, $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. We have, for instance,

$$\tilde{N}(n-1,k-1) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \binom{n}{k}^2 - \binom{n-1}{k}\binom{n+1}{k} = \binom{n}{k}\binom{n-1}{k-1} - \binom{n}{k-1}\binom{n-1}{k}.$$

The last expression represents a $2 \times 2$ determinant of adjacent elements in Pascal's triangle. The Narayana triangle is A001263.

The Hermite polynomials $H_n(x)$ [19] are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

They obey $H_n(-x) = (-1)^n H_n(x)$ and can be defined by the recurrence

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{3}$$

They have a generating function given by

$$e^{2tx-x^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} x^n.$$

A property that is related to the binomial transform is the following:

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)(2z)^{n-k} = H_n(x+z).$$

From this, we can deduce the following proposition.

**Proposition 4.** *For fixed $x$ and $y \neq 0$, the binomial transform of the sequence $n \to H_n(x)y^n$ is the sequence $n \to y^n H_n(x + \frac{1}{2y})$.*

*Proof.* Let $z = \frac{1}{2y}$. Then $2z = \frac{1}{y}$ and hence

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)(y)^{k-n} = H_n(x + \frac{1}{2y}).$$

That is,

$$\sum_{k=0}^{n} \binom{n}{k} H_k(x)y^k = y^n H_n(x + \frac{1}{2y})$$

as required. □

The Laguerre polynomials $L_n(x)$ [20] are defined by

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} x^n e^{-x}.$$

They have generating function

$$\frac{\exp(-\frac{tx}{1-x})}{1-x} = \sum_{n=0}^{\infty} \frac{L_n(t)}{n!} x^n.$$

They are governed by the following recurrence relationship:

$$(n+1)L_{n+1}(t) = (2n+1-t)L_n(t) - nL_{n-1}(t) \tag{4}$$

# 3 Introducing the family of centrally symmetric invertible triangles

We recall that the Binomial matrix $\mathbf{B}$, or Pascal's triangle, is the element $(e^x, x)$ of the Riordan group. For a given integer $r$, we shall denote by $\mathbf{B}_r$ the element $(e^x, x(1+rx))$ of the Riordan group. We note that $\mathbf{B} = \mathbf{B}_0$. We can characterize the general element of $\mathbf{B}_r$ as follows.

**Proposition 5.** *The general term $B_r(n, k)$ of the matrix $\mathbf{B}_r$ is given by*

$$B_r(n, k) = \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!}.$$

*Proof.* We have

$$\begin{aligned}
B_r(n,k) &= \frac{n!}{k!}[x^n](e^x(x(1+rx)^k) \\
&= \frac{n!}{k!}[x^n]\sum_{i=0}^{\infty}\frac{x^i}{i!}x^k\sum_{j=0}^{k}\binom{k}{j}r^jx^j \\
&= \frac{n!}{k!}[x^{n-k}]\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{i!}x^{i+j} \\
&= \frac{n!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{r^j}{(n-k-j)!}.
\end{aligned}$$

□

From the above expression we can easily establish that $B_r(n,k) = B_r(n,n-k)$ and $B_r(n,0) = B_r(n,n) = 1$.

An alternative derivation of these results can be obtained be observing that the matrix $\mathbf{B}_r$ may be defined as the array $R^1(\frac{1}{n!},\frac{1}{k!};e^x,(1+rx),1)$. Then we have

$$\begin{aligned}
B_r(n,k) &= \frac{1/k!}{1/n!}\mathbf{res}_x(e^x(1+rx)^kx^{-n+k-1}) \\
&= \frac{n!}{k!}\mathbf{res}_x(\sum_{i}^{\infty}\frac{x^i}{i!}\sum_{j}^{k}\binom{k}{j}r^jx^jx^{-n+k-1}) \\
&= \frac{n!}{k!}\mathbf{res}_x(\sum_{i}^{\infty}\sum_{j}^{k}\binom{k}{j}\frac{r^j}{i!}x^{i+j-n+k-1}) \\
&= \frac{n!}{k!}\sum_{j}^{k}\binom{k}{j}\frac{r^j}{(n-k-j)!}.
\end{aligned}$$

Thus $\mathbf{B_r}$ is a centrally symmetric lower-triangular matrix with $B_r(n,0) = B_r(n,n) = 1$. In this sense $\mathbf{B}_r$ can be regarded as a generalized Pascal matrix. Note that by the last property, this matrix is invertible.

**Proposition 6.** *The inverse of* $\mathbf{B_r}$ *is the element* $(e^{-u},u)$ *of the Riordan group, where*

$$u = \frac{\sqrt{1+4rx}-1}{2r}.$$

*Proof.* Let $(g^*,\bar{f})$ be the inverse of $(e^x, x(1+rx))$. Then

$$(g^*,\bar{f})(e^x,x(1+rx)) = (1,x) \Rightarrow \bar{f}(1+r\bar{f}) = x.$$

Solving for $\bar{f}$ we get

$$\bar{f} = \frac{\sqrt{1+4rx}-1}{2r}.$$

But $g^* = \frac{1}{g\circ\bar{f}} = e^{-\bar{f}}$. □

This result allows us to easily characterize the row sums of the inverse $\mathbf{B}_r^{-1}$.

**Corollary 7.** *The row sums of the inverse triangle $\mathbf{B}_r^{-1}$ are given by $0^n = 1, 0, 0, 0, \ldots$.*

*Proof.* We have $\mathbf{B}_r^{-1} = (e^{-u}, u)$ as above. Hence the e.g.f. of the row sums of $\mathbf{B}_r^{-1}$ is $e^{-u}e^{u} = 1$. The result follows from this. $\square$

**Example 8.** $\mathbf{B}_1 = (e^x, x(1+x))$ is given by

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 9 & 9 & 1 & 0 & 0 & \ldots \\ 1 & 16 & 42 & 16 & 1 & 0 & \ldots \\ 1 & 25 & 130 & 130 & 25 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The row sums of $\mathbf{B}_1$ are

$$1, 2, 6, 20, 76, 312, 1384, 6512, 32400, \ldots$$

or A000898.

From the above, the terms of this sequence are given by

$$s_1(n) = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!}.$$

with e.g.f. $g(x)e^{f(x)} = e^x e^{x(1+x)} = e^{2x+x^2}$. What is less evident is that

$$s_1(n) = H_n(-i)i^n$$

where $i = \sqrt{-1}$. This follows since

$$\begin{aligned} e^{2x+x^2} &= e^{2(-i)(ix)-(ix)^2} \\ &= \sum_{n=0} \frac{H_n(-i)}{n!}(ix)^n \\ &= \sum_{n=0} \frac{H_n(-i)i^n}{n!}x^n \end{aligned}$$

and hence $e^{2x+x^2}$ is the e.g.f. of $H_n(-i)i^n$. We therefore obtain the identity

$$H_n(-i)i^n = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!}.$$

We can characterize the row sums of $\mathbf{B}_1$ in terms of the diagonal sums of another related special matrix. For this, we recall [16] that

$$\text{Bessel}(n, k) = \frac{(n+k)!}{2^k(n-k)!k!} = \binom{n+k}{2}(2k-1)!!$$

defines the triangle A001498 of coefficients of Bessel polynomials that begins

$$\mathbf{Bessel} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 3 & 0 & 0 & 0 & \dots \\ 1 & 6 & 15 & 15 & 0 & 0 & \dots \\ 1 & 10 & 45 & 105 & 105 & 0 & \dots \\ 1 & 15 & 105 & 420 & 945 & 945 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We then have

**Proposition 9.** *The row sums of the matrix* $\mathbf{B}_1$ *are equal to the diagonal sums of the matrix with general term* $\mathrm{Bessel}(n,k)2^n$*. That is*

$$H_n(-i)i^n = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(n-k-j)!} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{Bessel}(n-k,k)2^{n-k}.$$

*Proof.* We shall prove this in two steps. First, we shall show that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{Bessel}(n-k,k)2^{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{k!} \binom{n}{2k} 2^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2k-1)!! \binom{n}{2k} 2^{n-k}.$$

We shall then show that this is equal to $H_n(-i)i^n$. Now

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{Bessel}(n-k,k)2^{n-k} &= \sum_{k=0}^{n} \mathrm{Bessel}(n-\frac{k}{2},\frac{k}{2})2^{n-\frac{k}{2}}(1+(-1)^k)/2 \\ &= \sum_{k=0}^{n} \frac{(n-\frac{k}{2}+\frac{k}{2})!2^{n-\frac{k}{2}}}{2^{\frac{k}{2}}(n-\frac{k}{2}-\frac{k}{2})!(\frac{k}{2})!}(1+(-1)^k)/2 \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{2^{n-k}}{(\frac{k}{2})!}(1+(-1)^k)/2 \\ &= \sum_{k=0}^{n} \frac{k!}{(\frac{k}{2})!} \binom{n}{k} 2^{n-k}(1+(-1)^k)/2 \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{k!} \binom{n}{2k} 2^{n-2k} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2k)!}{2^k k!} \binom{n}{2k} 2^{n-k} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2k-1)!! \binom{n}{2k} 2^{n-k}. \end{aligned}$$

establishes the first part of the proof. The second part of the proof is a consequence of the following more general result, when we set $a=2$ and $b=1$. □

**Proposition 10.** *The sequence with e.g.f. $e^{ax+bx^2}$ has general term $u_n$ given by*

$$u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k)(k+1)! a^{n-2k} b^k.$$

*Proof.* We have

$$\begin{aligned}
n![x^n]e^{ax+bx^2} &= n![x^n]e^{ax}e^{bx^2} \\
&= n![x^n] \sum_{i=0}^{\infty} \frac{a^i x^i}{i!} \sum_{k=0}^{\infty} \frac{b^k x^{2k}}{k!} \\
&= n![x^n] \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{a^i b^k}{i! k!} x^{i+2k} \\
&= n! \sum_{k=0}^{\infty} \frac{a^{n-2k} b^k}{(n-2k)! k!} \\
&= \sum_{k=0}^{\infty} \frac{n!}{(n-2k)!(2k)!} \frac{(2k)!}{k!} a^{n-2k} b^k \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k.
\end{aligned}$$

∎

**Corollary 11.**

$$H_n(-\frac{a}{2\sqrt{b}} i)(\sqrt{b} i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k.$$

**Corollary 12.** *Let $u_n$ be the sequence with e.g.f. $e^{ax+bx^2}$. Then $u_n$ satisfies the recurrence*

$$u_n = a u_{n-1} + 2(n-1) b u_{n-2}$$

*with $u_0 = 1$, $u_1 = a$.*

*Proof.* Equation 3 implies that

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x).$$

Thus

$$H_n(-\frac{a}{2\sqrt{b}} i) = -2 \frac{a}{2\sqrt{b}} i H_{n-1}(-\frac{a}{2\sqrt{b}} i) - 2(n-1) H_{n-2}(-\frac{a}{2\sqrt{b}} i).$$

Now multiply both sides by $(\sqrt{b} i)^n$ to obtain

$$u_n = a u_{n-1} + 2(n-1) b u_{n-2}.$$

Since

$$u_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k$$

we obtain the initial values $u_0 = 1$, $u_1 = a$. ∎

**Corollary 13.** *The binomial transform of* $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} a^{n-2k} b^k$ *is given by*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} (a+1)^{n-2k} b^k.$$

*Proof.* The e.g.f. of the binomial transform of the sequence with e.g.f. $e^{ax+cx^2}$ is $e^x e^{ax+bx^2} = e^{(a+1)x+bx^2}$. ∎

Equivalently, the binomial transform of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k)(k+1)! a^{n-2k} b^k$ is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k)(k+1)!(a+1)^{n-2k} b^k.$$

We note that in [1], it was shown that the binomial transform of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k) a^{n-2k} b^k$ is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C(k)(a+1)^{n-2k} b^k.$$

**Corollary 14.** *The row sums of* $\mathbf{B}_1$ *satisfy the recurrence equation*

$$u_n = 2u_{n-1} + 2(n-1)u_{n-2}$$

*with* $u_0 = 1$, $u_1 = 2$.

We can use **Proposition 4** to study the inverse binomial transform of $s_1(n)$. By that proposition, the inverse binomial transform of $H_n(-i)i^n$ is given by $i^n H_n(-i + \frac{1}{2i}) = H_n(-\frac{i}{2})i^n$. This is the sequence

$$1, 1, 3, 7, 25, 81, 331, 1303, 5937, \ldots$$

with e.g.f. $e^{x+x^2}$. This is A047974 which satisfies the recurrence $a_n = a_{n-1} + 2(n-1)a_{n-2}$. It is in fact equal to $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k, k)2^k$. The second inverse binomial transform of $s_1(n)$ is the sequence

$$1, 0, 2, 0, 12, 0, 120, 0, 1680, 0, 30240, \ldots$$

with e.g.f. $e^{x^2}$. This is an "aerated" version of the quadruple factorial numbers $C(n)(n+1)! = \frac{(2n)!}{n!}$, or A001813.

We now look at the central coefficients $B_1(2n, n)$ of $\mathbf{B}_1$. We have

$$\begin{aligned}
B_1(2n, n) &= \frac{(2n)!}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{(n-j)!} \\
&= C(n)(n+1)! \sum_{j=0}^{n} \binom{n}{j} \frac{1}{(n-j)!} \\
&= C(n)(n+1) \sum_{j=0}^{n} \binom{n}{j}^2 j! \\
&= C(n)(n+1)! L_n(-1).
\end{aligned}$$

Hence

$$\frac{B_1(2n,n)}{C(n)(n+1)!} = L_n(-1).$$

We note that this is the rational sequence $1, 2, \frac{7}{2}, \frac{17}{3}, \frac{209}{24}, \ldots$. Two other ratios are of interest.

1. $\frac{B_1(2n,n)}{C(2n,n)} = n!L_n(-1)$ is A002720. It has e.g.f. $\frac{1}{1-x}\exp(\frac{x}{1-x})$. It is equal to the number of partial permutations of an $n$-set, as well as the number of matchings in the bipartite graph $K(n,n)$. Using Equation (4) we can show that these numbers obey the following recurrence:

   $$u_n = 2nu_{n-1} - (n-1)^2 u_{n-2}$$

   with $u_0 = 1$, $u_1 = 2$.

2. $\frac{B_1(2n,n)}{C(n)} = (n+1)!L_n(-1)$ is A052852$(n+1)$. It has e.g.f. given by

   $$\frac{d}{dx}\frac{x}{1-x}\exp(\frac{x}{1-x}) = \frac{1}{(1-x)^3}\exp(\frac{x}{1-x}).$$

   Again using Equation (4) we can show that these numbers obey the following recurrence:

   $$v_n = 2(n+1)v_{n-1} - (n^2-1)v_{n-2}$$

   with $v_0 = 1$, $v_1 = 4$.

   This sequence counts the number of $(121, 212)$-avoiding $n$-ary words of length $n$. Specifically,

   $$\frac{B_1(2n,n)}{C(n)} = f_{121,212}(n+1, n+1)$$

   where

   $$f_{121,212}(n,k) = \sum_{j=0}^{k} \binom{k}{j}\binom{n-1}{j-1} j!$$

   is defined in [4].

From this last point, we find the following expression

$$B_1(2n,n) = C(n)\sum_{j=0}^{n+1} \binom{n+1}{j}\binom{n}{j-1} j! \tag{5}$$

Based on the fact that $C(n) = \binom{2n}{n} - \binom{2n}{n-1}$ we define

$$C_1(n) = B_1(2n,n) - B_1(2n,n-1) = B_1(2n,n) - B_1(2n,n+1)$$

to be *the generalized Catalan numbers associated with the triangle* $\mathbf{B}_1$. We calculate $B_1(2n, n-1)$ as follows:

$$\begin{aligned} B_1(2n, n-1) &= \frac{(2n)!}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{1}{(n-j+1)!} \\ &= \frac{(2n)!}{(n-1)!}\sum_{j=0}^{n-1}\binom{n}{j}\frac{n-j}{n}\frac{1}{(n-j+1)!} \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}\frac{n-j}{n-j+1}. \end{aligned}$$

Hence

$$\begin{aligned} B_1(2n, n) - B_1(2n, n-1) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}(1-\frac{n-j}{n-j+1}) \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j)!}\frac{1}{n-j+1} \\ &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j+1)!}. \end{aligned}$$

Starting from the above, we can find many expressions for $C_1(n)$. For example,

$$\begin{aligned} C_1(n) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{(n-j+1)!} \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(n+1-j)!} \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j}j! \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}^2\frac{n+1}{n-j+1}j! \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}\frac{(j+1)!}{n-j+1}. \end{aligned}$$

where we have used the fact that $\frac{(2n)!}{n!} = C(n)(n+1)!$. This is the sequence A001813 of quadruple factorial numbers with e.g.f. $\frac{1}{\sqrt{1-4x}}$.

Recognizing that the terms after $C(n)$ represent convolutions, we can also write

$$\begin{aligned}
C_1(n) &= C(n)\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(j+1)!}\\
&= C(n)\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}(n-j)!\\
&= C(n)\sum_{j=0}^{n}\binom{n}{j}^2\frac{n+1}{j+1}(n-j)!
\end{aligned}$$

We note that the first expression immediately above links $C_1(n)$ to the Lah numbers A008297.

The ratio $\frac{C_1(n)}{C(n)}$, or $\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(k+1)!}$, is the sequence

$$1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, \ldots$$

or A000262$(n+1)$. This is related to the number of partitions of $[n] = \{1, 2, 3, \ldots, n\}$ into any number of lists, where a list means an ordered subset. It also has applications in quantum physics [2]. The sequence has e.g.f.

$$\frac{d}{dx}e^{\frac{x}{1-x}} = \frac{e^{\frac{x}{1-x}}}{(1-x)^2}.$$

We can in fact describe this ratio in terms of the Narayana numbers $\tilde{N}(n,k)$ as follows:

$$\begin{aligned}
\frac{C_1(n)}{C(n)} &= \sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}(n-j)!\\
&= \sum_{j=0}^{n}\frac{n-j+1}{n+1}\binom{n+1}{j}\binom{n+1}{j+1}(n-j)!\\
&= \sum_{j=0}^{n}\frac{1}{n+1}\binom{n+1}{j}\binom{n+1}{j+1}(n-j+1)!\\
&= \sum_{j=0}^{n}\tilde{N}(n,j)(n-j+1)!\\
&= \sum_{j=0}^{n}\tilde{N}(n,n-j)(j+1)!\\
&= \sum_{j=0}^{n}\tilde{N}(n,j)(j+1)!
\end{aligned}$$

Hence we have

$$\frac{C_1(n)}{C(n)} = \sum_{j=0}^{n}\tilde{N}(n,j)(n-j+1)! = \sum_{j=0}^{n}\tilde{N}(n,j)(j+1)! = \sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(j+1)!}$$

# 4 The General Case

We shall now look at the row sums, central coefficients and generalized Catalan numbers associated with the general matrix $\mathbf{B}_r$. In what follows, proofs follow the methods developed in the last section.

**Proposition 15.** *The row sums $s_r(n)$ of $\mathbf{B}_r$ are given by $H_n(-\frac{i}{\sqrt{r}})(\sqrt{r}i)^n$.*

*Proof.* The row sums of $\mathbf{B}_r$ are given by the sequence

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!}$$

with e.g.f. $g(x)e^{f(x)} = e^x e^{x(1+rx)} = e^{2x+rx^2}$. Now

$$\begin{aligned} e^{2x+rx^2} &= e^{2(\frac{-i}{\sqrt{r}})(i\sqrt{r}x)-(i\sqrt{r}x)^2} \\ &= \sum_{n=0}^{\infty} \frac{H_n(-\frac{i}{\sqrt{r}})}{n!}(i\sqrt{r}x)^n \\ &= \sum_{n=0}^{\infty} \frac{H_n(-\frac{i}{\sqrt{r}})(i\sqrt{r})^n}{n!}x^n \end{aligned}$$

□

**Corollary 16.** *We have the identity*

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!} = H_n(-\frac{i}{\sqrt{r}})(\sqrt{r}i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} 2^{n-2k} r^k.$$

As before, we can rewrite this using the fact that $\frac{(2k)!}{k!} = C(k)(k+1)! = 2^k(2k-1)!!$.

We note that the second inverse binomial transform of $s_r(n)$ has e.g.f. $e^{rx^2}$.

**Proposition 17.** *The row sums of $\mathbf{B}_r$ are equal to the diagonal sums of the matrix with general term* $\text{Bessel}(n,k)2^n r^k$. *That is,*

$$\sum_{k=0}^{n} \frac{n!}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{r^j}{(n-k-j)!} = H_n(-\frac{i}{\sqrt{r}})(\sqrt{r}i)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k,k)2^{n-k}r^k.$$

**Proposition 18.** *The row sums of $\mathbf{B_r}$ obey the recurrence*

$$u_n = 2u_{n-1} + 2r(n-1)u_{n-2}$$

*with $u_0 = 1$, $u_1 = 2$.*

We now turn our attention to the central coefficients of $\mathbf{B}_r$.

**Proposition 19.** $B_r(2n,n) = C(n)(n+1)!\sum_{j=0}^{n}\binom{k}{j}^2 j! r^j$

*Proof.* The proof is the same as the calculation for $B_1(2n,n)$ in Example 8, with the extra factor of $r^j$ to be taken into account. □

**Corollary 20.**

$$\frac{B_r(2n,n)}{C(n)(n+1)!} = r^n L_n(-\frac{1}{r}) = \sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j)!}$$

*for $r \neq 0$.*

We note that the above expressions are not integers in general.

For instance, $\frac{B_2(2n,n)}{C(2n,n)} = n!2^n L_n(-\frac{1}{2})$ is A025167, and $\frac{B_3(2n,n)}{C(2n,n)} = n!3^n L_n(-\frac{1}{3})$ is A102757. In general, we have

**Proposition 21.** $\frac{B_r(2n,n)}{C(2n,n)} = n!r^n L_n(-1/r)$ *has e.g.f.* $\frac{1}{1-rx}\exp(\frac{x}{1-rx})$*, and satisfies the recurrence relation*

$$u_n = ((2n-1)r+1)u_{n-1} - r^2(n-1)^2 u_{n-2}$$

*with $u_0 = 1$, $u_1 = r+1$.*

*Proof.* We have

$$\begin{aligned}
n![x^n]\frac{e^{\frac{x}{1-rx}}}{(1-rx)} &= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\frac{x^i}{(1-rx)^i}(1-rx)^{-1} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-i}(1-rx)^{-1} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-(i+1)} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i\sum_{j=0}\binom{-(i+1)}{j}(-1)^j r^j x^j \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}\binom{i+j}{j}r^j x^{i+j} \\
&= n!\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j)!}.
\end{aligned}$$

To prove the second assertion, we use Equation (4) with $t = -\frac{1}{r}$. Multiplying by $n!r^{n+1}$, we obtain

$$(n+1)!r^{n+1}L_{n+1}(-\frac{1}{r}) = (2n+1+\frac{1}{r})r^{n+1}n!L_n(-\frac{1}{r}) - r^{n+1}n^2(n-1)!L_{n-1}(-\frac{1}{r}).$$

Simplifying, and letting $n \to n-1$, gives the result. □

**Corollary 22.** $\frac{B_r(2n,n)}{C(n)} = (n+1)!r^n L_n(-1/r)$ *has e.g.f.* $\frac{d}{dx}\frac{x}{1-rx}\exp(\frac{x}{1-rx})$*, and satisfies the recurrence*

$$w_n = ((2n-1)+r)\frac{n+1}{n}w_{n-1} - r^2(n^2-1)w_{n-2}$$

*for* $n > 1$*, with* $w_0 = 1$ *and* $w_1 = 2r+2$*.*

We can generalize Equation (5) to get

$$B_r(2n,n) = C(n)\sum_{j=0}^{n+1}\binom{n+1}{j}\binom{n}{j-1}j!r^{j-1}.$$

We define *the generalized Catalan numbers associated with the triangles* $\mathbf{B}_r$ to be the numbers

$$C_r(n) = B_r(2n,n) - B_r(2n,n-1).$$

Using the methods of **Example** 8, we have

**Proposition 23.** *We have the following equivalent expressions for* $C_r(n)$*:*

$$\begin{aligned} C_r(n) &= \frac{(2n)!}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j+1)!} \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}\frac{(n+1)!}{(j+1)!}r^{n-j} \\ &= C(n)\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j+1}(n-j)!r^{n-j} \\ &= C(n)\sum_{j=0}^{n}\frac{n+1}{j+1}\binom{n}{j}^2(n-j)!r^{n-j} \\ &= C(n)\sum_{j=0}^{n}\tilde{N}(n,j)(j+1)!r^j. \end{aligned}$$

For instance, $C_2(n)/C(n)$ is A025168.

**Proposition 24.** $\frac{C_r(n)}{C(n)}$ *has e.g.f.*

$$\frac{d}{dx}e^{\frac{x}{1-rx}} = \frac{e^{\frac{x}{1-rx}}}{(1-rx)^2}.$$

*Proof.* We have

$$\begin{aligned}
n![x^n]\frac{e^{\frac{x}{1-rx}}}{(1-rx)^2} &= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\frac{x^i}{(1-rx)^i}(1-rx)^{-2} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-i}(1-rx)^{-2} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i(1-rx)^{-(i+2)} \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}x^i\sum_{j=0}^{\infty}\binom{-(i+2)}{j}(-1)^j r^j x^j \\
&= n![x^n]\sum_{i=0}^{\infty}\frac{1}{i!}\sum_{j=0}^{\infty}\binom{i+j+1}{j}r^j x^{i+j} \\
&= n!\sum_{j=0}^{\infty}\binom{n+1}{j}\frac{r^j}{(n-j)!} \\
&= (n+1)!\sum_{j=0}^{n}\binom{n}{j}\frac{r^j}{(n-j+1)!}.
\end{aligned}$$

□

# 5 The case $r=\frac{1}{2}$

The assumption so far has been that $r$ is an integer. In this section, we indicate that $r=\frac{1}{2}$ also produces a generalized Pascal triangle. We have $\mathbf{B}_{\frac{1}{2}}=(e^x, x(1+x/2))$. This begins

$$\mathbf{B}_{\frac{1}{2}}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 6 & 6 & 1 & 0 & 0 & \dots \\ 1 & 10 & 21 & 10 & 1 & 0 & \dots \\ 1 & 15 & 55 & 55 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is triangle A100862. Quoting from A100862, $B_{\frac{1}{2}}(n,k)$ "is the number of $k$-matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$; in other words, $K'(n)$ is the graph constructed from $K(n)$ by adding for each vertex $v$ a new vertex $v'$ and the edge $vv'$". The row sums of this triangle, A005425, are given by

$$1, 2, 5, 14, 43, 142, 499, 1850, 7193, \dots$$

These have e.g.f. $e^{2x+x^2/2}$ and general term

$$H_n(-\sqrt{2}i)(i/\sqrt{2})^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} 2^{n-3k}.$$

They obey the recurrence

$$u_n = 2u_{n-1} + (n-1)u_{n-2}$$

with $u_0 = 1$, $u_1 = 2$.

[3] provides an example of their use in quantum physics. Using Proposition 4 or otherwise, we see that the inverse binomial transform of this sequence, with e.g.f. $e^{x+x^2/2}$, is given by

$$H_n(-\sqrt{2}i + \frac{i}{\sqrt{2}})(i/\sqrt{2})^n = H_n(-\frac{i}{\sqrt{2}})(i/\sqrt{2})^n.$$

This is the sequence

$$1, 1, 2, 4, 10, 26, 76, 232, 765, \ldots$$

or A000085. It has many combinatorial interpretations, including for instance the number of matchings in the complete graph $K(n)$. These numbers are the diagonal sums of the Bessel triangle **Bessel**:

$$H_n(-\frac{i}{\sqrt{2}})(i/\sqrt{2})^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \text{Bessel}(n-k, k).$$

The row sums of $\mathbf{B}_{\frac{1}{2}}$ are the second binomial transform of the sequence

$$1, 0, 1, 0, 3, 0, 15, 0, 105, 0, \ldots$$

with e.g.f. $e^{x^2/2}$. This is an "aerated" version of the double factorial numbers $(2n-1)!!$, or A001147. These count the number of perfect matchings in the complete graph $K(2n)$. The row sums count the number of $12-3$ and $214-3$-avoiding permutations, as well as the number of matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$.

This example prompts us to define a new family $\tilde{\mathbf{B}}_r$ where $\tilde{\mathbf{B}}_r$ is the element $(e^x, x(1+\frac{rx}{2}))$ of the exponential Riordan group. Then we have $\tilde{\mathbf{B}}_0 = \mathbf{B}$, $\tilde{\mathbf{B}}_1 = \mathbf{B}_{\frac{1}{2}}$, $\tilde{\mathbf{B}}_2 = \mathbf{B}_1$ etc. We can then show that $\tilde{\mathbf{B}}_r$ is the product of the binomial matrix $\mathbf{B}$ and the matrix with general term $\text{Bessel}(k, n-k)r^{n-k}$. We have

$$\tilde{\mathbf{B}}_r(n,k) = \frac{n!}{k!} \sum_{j=0}^{k} \frac{1}{2^j} \binom{n}{k} \frac{r^j}{(n-k-j)!} = \sum_{j=0}^{n} \binom{n}{j} \frac{j! r^{j-k}}{(2k-j)! 2^{j-k} (j-k)!}.$$

Thus

$$\tilde{\mathbf{B}}_r(n,k) = \binom{n}{k} \sum_{j=0}^{n} \binom{n-k}{n-j} \frac{k!}{(2k-j)!} \frac{r^{j-k}}{2^{j-k}}$$

and in particular

$$\tilde{\mathbf{B}}_r(2n,n) = \binom{2n}{n} \sum_{j=0}^{2n} \binom{n}{j-n} \frac{n!}{(2n-j)!} \frac{r^{j-n}}{2^{j-n}}.$$

Finally,

$$\tilde{\mathbf{B}}_r(n,k) = \tilde{\mathbf{B}}_r(n-1,k-1) + \tilde{\mathbf{B}}_r(n-1,k) + r(n-1)\tilde{\mathbf{B}}_r(n-2,k-1).$$

# 6 Conclusion

The foregoing has shown that the triangles $\mathbf{B}_r$, and more generally $\tilde{\mathbf{B}}_r$, defined in terms of exponential Riordan arrays, are worthy of further study. Many of the sequences linked to them have significant combinatorial interpretations. $\mathbf{B}_{\frac{1}{2}}$ as documented in A100862 by Deutsch has a clear combinatorial meaning. This leaves us with the challenge of finding combinatorial interpretations for the general arrays $\tilde{\mathbf{B}}_r$, $r \in \mathbf{Z}$.

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