Journal of Integer Sequences, Vol. 10 (2007),
Article 07.1.1

Generalized Catalan Numbers and Generalized Hankel Transformations

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Abstract

Cvetković, Rajković and Ivković proved that the Hankel transformation of the sequence of sums of adjacent Catalan numbers is a sequence of every other Fibonacci number. In this paper, an elementary proof is given and a generalization to sequences of generalized Catalan numbers.

1 Introduction

Given a sequence of numbers $S = \{s_0, s_1, s_2, \ldots\}$, the Hankel matrix of order n generated by the sequence S is the $n \times n$-matrix whose (i, j)-entry is given by s_{i+j} for $0 \leq i, j \leq n-1$. The Hankel transform of the sequence S is the sequence of determinants of the Hankel matrices generated by S.

Suppose that

$$a_n = \frac{1}{n+1}\binom{2n}{n} + \frac{1}{n+2}\binom{2n+2}{n+1},$$

so that a_n is the sum of the n^{th} and $n+1^{\text{st}}$ Catalan numbers. Then the Hankel transform of $\{a_0, a_1, a_2, \ldots\}$ begins $2, 5, 13, 34, \ldots$. Layman first conjectured in the On-Line Encyclopedia of Integer Sequences ([4], see sequence A001906) that this sequence consists of every other Fibonacci number, and subsequently Cvetković, Rajković and Ivković [2] proved this conjecture. The current paper arose out of an attempt to understand and generalize this result.

The Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$ uniquely satisfy the nonlinear recurrence relation

$$c_{n+1} = \sum_{r=0}^{n} c_{n-r}c_r, \quad c_0 = 1.$$

This sequence has been generalized to the recurrence relation

$$c_{n+1,k} = \sum_{r=0}^{\lfloor \frac{n}{k-1} \rfloor} (c_{n-r(k-1),k}) * (c_{r(k-1),k}), \quad c_{0,k} = 1.$$

When $k = 2$, $c_{n,k}$ is simply the n^{th} Catalan number. It can be shown ([3]) that

$$c_{(k-1)n+l-1,k} = \frac{l}{kn+l}\binom{kn+l}{n}, \quad 1 \leq l \leq k-1.$$

Some notation is necessary to state our more general result. Let $a'_{n,k} = c_{n,k} + c_{n+1,k}$ and $a''_{n,k} = c_{n,k} + c_{n+k-1,k}$. Note that $a'_{n,2}$ and $a''_{n,2}$ both coincide with the sequence a_n described above. Let $A'_{n,k}$ and $A''_{n,k}$ be the $n \times n$-matrices whose (i,j)-entries are given respectively by $a'_{(k-1)i+j,k}$ and $a''_{(k-1)i+j,k}$ for $0 \leq i,j \leq n-1$. Let $F'_{n,k}$ be the sequence determined by the recurrence relation

$$F'_{n+1,k} = F'_{n-(k-2),k} + F'_{n-(k-1),k}$$

with initial conditions

$$F'_{1,k} = F'_{2,k} = \cdots = F'_{k,k} = 1,$$

and let $F''_{n,k}$ be the sequence determined by

$$F''_{n+1,k} = F''_{n,k} + F''_{n-(k-1),k}$$

with the same initial conditions. Note that $F'_{n,2} = F''_{n,2} = F_n$ (the n^{th} Fibonacci number).

We can now state our main theorem:

Theorem 1.1.

$$\det(A'_{n,k}) = F'_{kn+1,k}, \ \textit{and} \ \det(A''_{n,k}) = F''_{kn+1,k}.$$

Note that when $k = 2$, our theorem reduces to the result of Cvetković, Rajković and Ivković [2].

In Section 2, we find LU decompositions of the inverses of a sequence of matrices $C_{n,k}$ obtained from the generalized Catalan numbers. It turns out that these take surprisingly simple forms, and can be used to prove our main result, as seen in Section 3.

2 Generalized Catalan numbers

Definition 2.1. *Let $C_{n,k}$ be the $n \times n$ matrix whose (i,j) entry is given by $c_{(k-1)i+j,k}$ for $0 \leq i,j \leq n-1$. Let $L_{n,k}$ be the $n \times n$ matrix whose (i,j) entry is given by $(-1)^{i-j}\binom{i+(k-1)j}{i-j}$ for $0 \leq i,j \leq n-1$. Let $U_{n,k}$ be the $n \times n$ matrix whose (i,j) entry is given by $(-1)^{j-i}\binom{j+\lfloor \frac{i}{k-1} \rfloor}{j-i}$ for $0 \leq i,j \leq n-1$.*

It is easy to see that $L_{n,k}$ is lower triangular with 1's on the diagonal and $U_{n,k}$ is upper triangular with 1's on the diagonal. Our goal in this section is to prove that the product $L_{n,k}C_{n,k}U_{n,k}$ is equal to the identity matrix.

Our first step is to show that the product $L_{n,k}C_{n,k}$ is upper triangular with 1's on the diagonal. We will then show that $C_{n,k}U_{n,k}$ is lower triangular with 1's on the diagonal. From these two facts, the result will follow formally.

The proof makes use of certain generating functions.

Definition 2.2. *For $1 \le l \le k-1$, let $g_l(z) = \sum_{n=0}^{\infty} c_{(k-1)n+l-1} z^n$, and let $g(z) = g_1(z)$.*

It follows from the recurrence relation defining $c_{n,k}$ that $g_l(z)g(z) = g_{l+1}(z)$ for $1 \le l \le k-2$, and also that $g_{k-1}(z)g(z) = \frac{g(z)-1}{z}$. Thus,

$$g(z)^l = g_l(z), \ \ 1 \le l \le k-1 \tag{1}$$

and

$$g(z)^k = \frac{g(z)-1}{z}. \tag{2}$$

Bajunaid, Cohen, Colonna, and Signman [1] prove that the function

$$f(z) := (1-z)g(z(1-z)^{k-1}) \tag{3}$$

converges to the constant function at 1 for z close to 0. This is then used to show that

$$\sum_{n=\lceil m/k \rceil}^{m} (-1)^n c_{(k-1)n,k} \binom{kn-n}{kn-m} = (-1)^m.$$

(Note that the cited reference refers to $c_{(k-1)n,k}$ as $a_{n,k}$.) We use exactly the same technique to prove the following slightly stronger result.

Lemma 2.1. *Suppose i and j are nonnegative integers. Then*

$$\sum_{m=0}^{i} (-1)^{i-m} \binom{i+(k-1)m}{i-m} c_{(k-1)m+j,k} = \begin{cases} 0, & j < i; \\ 1, & j = i. \end{cases}$$

Proof. It follows from Equation 1 that,

$$f_l(z) := (1-z)^l g_l(z(1-z)^{k-1}) \tag{4}$$

is equal to $f(z)^l$, and therefore (by 3) converges to 1 for z close to 0. From this, we find that for any $s \ge 0$, the series

$$\sum_{m=0}^{\infty} c_{(k-1)m+(l-1),k} [z(1-z)^{k-1}]^m (1-z)^s = (1-z)^{s-l} f_l(z)$$

converges to $(1-z)^{s-l}$ for z close to 0. Expanding $(1-z)^{m(k-1)+s}$, we can rewrite this sum as

$$\sum_{m=0}^{\infty} c_{(k-1)m+(l-1),k} \left(\sum_{t=0}^{mk-m+s} (-1)^t \binom{(k-1)m+s}{t} z^{t+m} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=\lceil \frac{n-s}{k} \rceil}^{n} (-1)^{n-m} \binom{(k-1)m+s}{n-m} c_{(k-1)m+(l-1),k} \right) z^n.$$

Therefore, if $n > s - l$

$$\sum_{m=\lceil \frac{n-s}{k} \rceil}^{n} (-1)^{n-m} \binom{(k-1)m+s}{n-m} c_{(k-1)m+(l-1),k} = \begin{cases} 0, & s > l-1; \\ 1, & s = l-1. \end{cases}$$

Now, by the division algorithm, we may write j as $(k-1)r+(l-1)$, for some nonnegative integer r and some l with $1 \le l \le k-1$. Letting $s = i-(k-1)r$ and $n = i+r$, gives

$$\sum_{m=r}^{i+r} (-1)^{i-(m-r)} \binom{(k-1)(m-r)+i}{n-m} c_{(k-1)(m-r)+j,k} = \begin{cases} 0, & j < i; \\ 1, & j = i. \end{cases}$$

The result now follows by an index shift. □

Corollary 2.1. *The product $L_{n,k}C_{n,k}$ is upper triangular with 1's on the diagonal.*

Proof. The (i, j) entry of $L_{n,k}C_{n,k}$ is

$$\sum_{m=0}^{n-1} (-1)^{i-m} \binom{i+(k-1)m}{i-m} c_{(k-1)m+j,k}.$$

□

We now turn to consider the product $C_{n,k}U_{n,k}$.

Lemma 2.2.

$$\sum_{m=0}^{j} c_{(k-1)i+m,k} (-1)^{j-m} \binom{j+\lfloor \frac{m}{k-1} \rfloor}{j-m} = \begin{cases} 0, & i < j; \\ 1, & i = j. \end{cases}$$

Proof. By Equation 2

$$\sum_{l=0}^{k-1} \left(zg(z^{k-1}(1-z)) \right)^l = \frac{(zg(z^{k-1}(1-z)))^k - 1}{zg(z^{k-1}(1-z)) - 1} = \frac{z^k \frac{g(z^{k-1}(1-z))-1}{z^{k-1}(1-z)} - 1}{zg(z^{k-1}(1-z)) - 1} = \frac{1}{1-z}$$

for z close to 0. It follows by Equation 1 that

$$\sum_{l=0}^{k-1} z^l g_l(z^{k-1}(1-z)) = \frac{1}{1-z},$$

so subtracting 1, dividing by z, and multiplying by $(1-z)^s$ gives

$$\sum_{l=1}^{k-1} (1-z)^s z^{l-1} g_l(z^{k-1}(1-z)) = (1-z)^{s-1}.$$

Now,

$$(1-z)^s z^{l-1} g_l(z^{k-1}(1-z)) = \sum_{p=0}^{\infty} c_{(k-1)p+l-1,k}(1-z)^{p+s} z^{p(k-1)+l-1}$$

$$= \sum_{p=0}^{\infty} c_{(k-1)p+l-1,k} \sum_{t=0}^{p+s} (-1)^t \binom{p+s}{t} z^{p(k-1)+l-1+t}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{p} (-1)^{n-p(k-1)-(l-1)} \binom{p+s}{n-p(k-1)-(l-1)} c_{(k-1)p+l-1,k} \right) z^n.$$

Therefore,

$$\sum_{l=1}^{k-1} (1-z)^s z^{l-1} g_l(z^{k-1}(1-z)) = \sum_{n=0}^{\infty} \left(\sum_{m} (-1)^{n-m} \binom{s+\lfloor \frac{m}{k-1} \rfloor}{n-m} c_{m,k} \right) z^n.$$

Here, we have used the division algorithm to substitute $m = p(k-1)+l-1$, where $1 \leq l \leq k-1$. So,

$$\sum_{m} (-1)^{n-m} \binom{s+\lfloor \frac{m}{k-1} \rfloor}{n-m} c_{m,k} = \begin{cases} 0, & n \geq s > 0; \\ 1, & s = 0. \end{cases}$$

Letting $s = j - i$, $n = j + (k-1)i$ and shifting index yields the result. $\square$

Corollary 2.2. *The product $C_{n,k}U_{n,k}$ is lower triangular with 1's on the diagonal.*

Theorem 2.1. *The product $L_{n,k}C_{n,k}U_{n,k}$ is equal to the identity matrix.*

Proof. By Corollaries 2.1 and 2.2, the products $L_{n,k}^{-1}(L_{n,k}C_{n,k})$ and $(C_{n,k}U_{n,k})U_{n,k}^{-1}$ are both LU decompositions of $C_{n,k}$. By uniqueness of LU decompositions, $L_{n,k}^{-1} = C_{n,k}U_{n,k}$. $\square$

3 Proof of the Main Theorem

In this section, we prove our main result, which will follow from two additional lemmas.

Lemma 3.1. *The determinant of the $(n-1) \times (n-1)$ minor of $C_{n,k}$ obtained by removing the final column and the jth row is $\binom{n-1+(k-1)j}{n-1-j}$. The determinant of the $(n-1) \times (n-1)$ minor of $C_{n,k}$ obtained by removing the final row and the ith column is $\binom{n-1+\lfloor \frac{i}{k-1} \rfloor}{n-1-i}$.*

Proof. Since the determinant of $L_{n,k}$ and $U_{n,k}$ are both 1, the determinant of $C_{n,k}$ is 1, so $C_{n,k}^{-1}$ is equal to the adjoint of $C_{n,k}$. Since $C_{n,k}^{-1} = U_{n,k}L_{n,k}$, the final row of the adjoing of $C_{n,k}$ is equal to the final row of $L_{n,k}$ and the final column of the adjoing of $C_{n,k}$ is equal to the final column of $U_{n,k}$. Now the (i,j) entry in the adjoint of $C_{n,k}$ is the product of $(-1)^{i+j}$ and the determinant of the $(n-1) \times (n-1)$ minor of $C_{n,k}$ obtained by removing the ith column and the jth row. The claim follows. $\square$

Lemma 3.2. *The determinants of $A'_{n,k}$ and $A''_{n,k}$ are respectively given by*

$$\sum_{i=0}^{n} \binom{n + \lfloor \frac{i}{k-1} \rfloor}{n - i}$$

and

$$\sum_{j=0}^{n} \binom{n + (k-1)j}{n - j}.$$

Proof. We consider only the determinant of $A'_{n,k}$, the other argument being similar. For each j between 1 and $n+1$, let $\mathbf{c}_j$ be the column vector consisting of the first n terms in the jth row of $C_{n+1,k}$. Then the jth column vector of $A'_{n,k}$ is $\mathbf{c}_j + \mathbf{c}_{j+1}$. Therefore, the determinant of $A'_{n,k}$ could be written as the sum of the determinants of 2^n matrices, where the jth column vector of each matrix is either $\mathbf{c}_j$ or $\mathbf{c}_{j+1}$. Most of these determinants are zero, since the determinant of any matrix with two identical column vectors is zero. The nonzero determinants belong to those matrices whose column vectors are n distinct vectors from the set $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n+1}\}$, in order. But these are just the determinants of the minors of $C_{n+1,k}$ obtained by removing the final row and one of the columns. The result follows by Lemma 3.1.

□

We now prove Theorem 1.1.

Proof. For $n \geq 0$ and $1 \leq l \leq k$, let $G'_{kn+l,k} = \sum_{i=0}^{n} \binom{n+\lfloor \frac{i+l-1}{k-1} \rfloor}{n-i}$. We now show that $G'_{kn+l,k} = F'_{kn+l}$ for all $n \geq 0$, $1 \leq l \leq k$. This follows from the following three observations.

1. For $1 \leq l \leq k$, $G'_{l,k} = 1$.

2. For $1 \leq l \leq k-1$,

$$G'_{kn+l,k} + G'_{kn+l+1,k} = \sum_{i=0}^{n} \binom{n + \lfloor \frac{i+l-1}{k-1} \rfloor}{n - i} + \sum_{i=0}^{n} \binom{n + \lfloor \frac{i+l}{k-1} \rfloor}{n - i}$$

$$= \sum_{i=0}^{n} \binom{n + \lfloor \frac{i+l-1}{k-1} \rfloor}{n - i} + \sum_{i=1}^{n+1} \binom{n + \lfloor \frac{i-1+l}{k-1} \rfloor}{n - (i-1)} = \sum_{i=0}^{n+1} \binom{n + 1 + \lfloor \frac{i+l-1}{k-1} \rfloor}{n - i} = G'_{k(n+1)+l,k}.$$

3.

$$G'_{kn+k,k} + G'_{k(n+1)+1,k} = \sum_{i=0}^{n} \binom{n + \lfloor \frac{i+k-1}{k-1} \rfloor}{n - i} + \sum_{i=0}^{n+1} \binom{n + 1 + \lfloor \frac{i}{k-1} \rfloor}{n + 1 - i}$$

$$= \sum_{i=0}^{n+1} \left(\binom{n + 1 + \lfloor \frac{i}{k-1} \rfloor}{n - i} + \binom{n + 1 + \lfloor \frac{i}{k-1} \rfloor}{n + 1 - i} \right)$$

$$= \sum_{i=0}^{n+1} \binom{n + 2 + \lfloor \frac{i}{k-1} \rfloor}{n + 1 - i} = \sum_{i=0}^{n+1} \binom{n + 1 + \lfloor \frac{i+k-1}{k-1} \rfloor}{n + 1 - i} = G'_{k(n+1)+k,k}.$$

Now for $n \geq 0$ and $1 \leq l \leq k$, let $G''_{kn+l,k} = \sum_{j=0}^{n} \binom{n+(k-1)j+l-1}{n-j}$. We now show that $G''_{kn+l,k} = F''_{kn+l}$ for all $n \geq 0$, $1 \leq l \leq k$. This follows from the following three observations.

1. For $1 \leq l \leq k$, $G''_{l,k} = 1$.

2. For $1 \leq l \leq k-1$,

$$G''_{kn+l,k} + G''_{k(n-1)+(l+1),k} = \sum_{j=0}^{n} \left(\binom{n+(k-1)j+l-1}{n-j} + \binom{n-1+(k-1)j+l}{n-1-j} \right)$$
$$= \sum_{j=0}^{n} \binom{n+(k-1)j+l}{n-j} = G''_{kn+l+1,k}.$$

3.

$$G''_{kn+k,k} + G''_{kn+1,k} = \sum_{j=0}^{n} \left(\binom{n+(k-1)j+k-1}{n-j} + \binom{n+(k-1)j}{n-j} \right)$$
$$= \sum_{j=1}^{n+1} (\binom{n+(k-1)j}{n-(j-1)}) + \sum_{j=0}^{n} \binom{n+(k-1)j}{n-j} = \sum_{j=0}^{n+1} (\binom{n+1+(k-1)j}{n+1-j}) = G''_{k(n+1)+1,k}.$$

The statement of the theorem now follows from Lemma 3.2. □

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2000 *Mathematics Subject Classification*: Primary 11B37; Secondary 11B39, 11B75.

Keywords: Generalized Catalan numbers, Hankel transform.

(Concerned with sequence A001906.)

Received August 28 2006; revised version received December 6 2006. Published in *Journal of Integer Sequences*, December 8 2006.

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