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On a Class of Polynomials with Integer Coefficients

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Abstract

We define a certain class of polynomials denoted by $P_{n,m,p}(x)$, and give the combinatorial meaning of the coefficients. Chebyshev polynomials are special cases of $P_{n,m,p}(x)$. We first show that $P_{n,m,p}(x)$ can be expressed in terms of $P_{n,0,p}(x)$. From this we derive that $P_{n,2,2}(x)$ can be obtained in terms of trigonometric functions, from which we obtain some of its important properties. Some questions about orthogonality are also addressed. Furthermore, it is shown that $P_{n,2,2}(x)$ fulfills the same three-term recurrence as the Chebyshev polynomials. We also obtain some other recurrences for $P_{n,m,p}(x)$ and its coefficients. Finally, we derive a formula for the coefficients of Chebyshev polynomials of the second kind.

1 Introduction

In the paper [1] I proved the following result:

Theorem 1. *If a finite set X consists of n blocks of size p and an additional block of size m then, for $n \geq 0$, $k \geq 0$, the number $f(n,k,m,p)$ of $(n+k)$-subsets of X intersecting each block of size p is*

$$f(n,k,m,p) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{np+m-ip}{n+k}.$$

I also proved the following relations for the function f [1]:

$$f(n,k,m,p) = \sum_{i=0}^{m} \binom{m}{i} f(n,k-i,0,p), \tag{1}$$

$$f(n,k,m,p)=\sum_{i=0}^{t}(-1)^i\binom{t}{i}f(n,k+t,m+t-i,p), \tag{2}$$

$$f(n,k,m,p)=\sum_{i=1}^{p}\binom{p}{i}f(n-1,k-i+1,m,p), \tag{3}$$

$$f(n,k,m,p)=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}f(n-j,k-i+j,m,p-1). \tag{4}$$

Furthermore, I showed that $(-1)^k f(n,k,0,2)$ is the coefficient of the Chebyshev polynomial $U_{n+k}(x)$ by x^{n-k}, and that $(-1)^k f(n,k,1,2)$ is the coefficient of the Chebyshev polynomial $T_{n+k-1}(x)$ by x^{n-k+1}.

Definition 2. We define the set of coefficients

$$\{c(n,k,m,p):\ n=m,m+1,\ldots;k=0,1,\ldots,n\}$$

such that

$$c(n,k,m,p)=(-1)^{\frac{n-k}{2}}f\left(\frac{n+k-2m}{2},\frac{n-k}{2},m,p\right),$$

if n and k are of the same parity, and $c(n,k,m,p)=0$ otherwise. The polynomials $P_{n,m,p}(x)$ are defined to be

$$P_{n,m,p}(x)=\sum_{k=0}^{n}c(n,k,m,p)x^k.$$

Example 3. The Chebyshev polynomials are particular cases of $P_{n,m,p}(x)$, obtained for $m=1,\ p=2$ and $m=0,\ p=2$, that is,

$$U_n(x)=P_{n,0,2}(x),\ T_n(x)=P_{n,1,2}(x).$$

The polynomial $P_{n,2,2}(x)$ is the closest to the Chebyshev polynomials, and will be denoted simply by $P_n(x)$.

In the next table we state the first few values of $P_n(x)$.

$$\begin{gathered}x^2\\ 2x^3-2x\\ 4x^4-5x^2+1\\ 8x^5-12x^3+4x\\ 16x^6-28x^4+13x-1\\ 32x^7-64x^5+38x^3-6x.\end{gathered}$$

Among coefficients of the above polynomials the following sequences from [2] appear: A024623, A049611, A055585, A001844, A035597.

Triangles of coefficients for $P_{n,m,2}(x),(\ m=2,3,4,5,6)$ are given in A136388, A136389, A136390, A136397, and A136398, respectively.

2 Reduction to the case $m = 0$.

We shall first prove an analog of the formula (1) for polynomials.

Theorem 4. *The following equation is fulfilled:*

$$P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} P_{n-m-i,0,p}(x).$$

Proof. The following equation holds:

$$P_{n,m,p}(x) = \sum_{k=0}^{n} (-1)^{\frac{n-k}{2}} f\left(\frac{n+k-2m}{2}, \frac{n-k}{2}, m, p\right) x^k.$$

Using (1) one obtains

$$P_{n,m,p}(x) = \sum_{k=0}^{n} \sum_{i=0}^{m} (-1)^{\frac{n-k}{2}} \binom{m}{i} f\left(r, s, 0, p\right) x^k.$$

where

$$r = \frac{n+k-2m}{2}, \ \ s = \frac{n-k}{2} - i.$$

Changing the order of summation yields

$$P_{n,m,p}(x) = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} \sum_{k=0}^{n} (-1)^{\frac{n-k}{2}} f\left(r, s, 0, p\right) x^{k-m+i}.$$

Terms in the sum on the right side of the preceding equation produce nonzero coefficients only in the case $0 \le s \le r$, that is,

$$m - i \le k \le n - 2i.$$

It follows that

$$P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} \sum_{k=m-i}^{n-2i} (-1)^{\frac{n-k}{2}-i} f\left(r, s, 0, p\right) x^{k-m+i}.$$

Denoting $k - m + i = j$ we obtain

$$P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} \sum_{j=0}^{n-m-i} c(n-m-i, j, 0, p) x^j,$$

which means that

$$P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} P_{n-m-i,0,p}(x).$$

$\square$

According the the preceding theorem we can express $P_n(x)$ in terms of Chebyshev polynomials of the second kind. Namely, for $m = 2,\ n \geq 4$ we have

$$P_n(x) = x^2 U_{n-2}(x) - 2xU_{n-3}(x) + U_{n-4}(x). \tag{5}$$

This allow us to express $P_n(x)$ in terms of trigonometric functions.

Theorem 5. *For each $n \geq 3$ we have*

$$P_n(\cos\theta) = -\sin\theta \sin(n-1)\theta. \tag{6}$$

Proof. According to (5) and well-known property of Chebyshev polynomials we obtain

$$\sin\theta P_n(\cos\theta) = \cos^2\theta \sin(n-1)\theta - 2\cos\theta \sin(n-2)\theta + \sin(n-3)\theta.$$

From the identity

$$2\cos\theta \sin(n-2)\theta = \sin(n-1)\theta + \sin(n-3)\theta$$

follows

$$\sin\theta P_n(\cos\theta) = \cos^2\theta \sin(n-1)\theta - \sin(n-1)\theta = -\sin^2\theta \sin(n-1)\theta.$$

Dividing by $\sin\theta \neq 0$ we prove the theorem. $\square$

Note that this proof is valid for $n \geq 4$. The case $n = 3$ can be checked directly.

In the following theorem we prove that $P_n(x)$ have the same important property concerning zeroes as Chebyshev polynomials do.

Theorem 6. *For $n \geq 3$, the polynomial $P_n(x)$ has all simple zeroes lying in the segment $[-1, 1]$.*

Proof. Since

$$U_n(1) = n + 1,\ U_n(-1) = (-1)^n(n+1)$$

the equation (5) implies

$$P_n(1) = U_{n-2}(1) - 2U_{n-3}(1) + U_{n-4}(1) = n - 1 - 2(n-2) + n - 3 = 0,$$

and

$$\begin{aligned} P_n(-1) = U_{n-2}(-1) + 2U_{n-3}(-1) + U_{n-4}(-1) = \\ = (-1)^{n-2}(n-1) + 2(-1)^{n-3}(n-2) + (-1)^{n-4}(n-3) = 0. \end{aligned}$$

Thus, $x = -1$ and $x = 1$ are zeroes of $P_n(x)$. The remaining $n - 2$ zeroes are obtained from the equation

$$\sin(n-1)\theta = 0,$$

and they are

$$x_k = \cos\frac{k\pi}{n-1},\ (k = 1, 2, \ldots, n-2).$$

$\square$

We shall now state an immediate consequence of (6) which shows that values of $P_n(x),\ (x \in [-1,1])$ lie inside the unit circle.

Corollary 7. *For $n \geq 3$ and $x \in [-1,1]$ we have*

$$P_n(x)^2 + x^2 \leq 1.$$

Example 8. Dividing $P_n(x)$ by 2^{n-2} we obtain a polynomial with the leading coefficient 1. Thus, its sup norm on $[-1,1]$ is $\leq \frac{1}{2^{n-2}}$, which means that $\frac{1}{2^{n-2}}P_n(x)$ has at most 2 times greater sup norm, comparing with the sup norm of $T_n(x)$, which is minimal.

Taking the derivative in the equation (6) we obtain the following equation for extreme points of $P_n(x)$:

$$(n-1)\tan\theta + \tan(n-1)\theta = 0.$$

The values $\theta = 0$, and $\theta = \pi$ obviously satisfied this equation, which implies that end-points $x = -1$ and $x = 1$ are extreme points. The remaining extreme points of $P_3(x)$ are $x = \arctan\sqrt{2}$ and $x = -\arctan\sqrt{2}$.

3 Orthogonality

In this section we investigate the set $\{P_n(x) : n = 2,3,4,\ldots\}$ concerning to the problem of orthogonality, with respect to some standard Jacobi's weights.

The first result is for the weight $\frac{1}{\sqrt{1-x^2}}$ of Chebyshev polynomials of the first kind.

So, when $n = m = 2$, $\int_{-1}^{1} \frac{[P_n(x)]^2}{\sqrt{1-x^2}}dx = \frac{3}{8}\pi$.

Theorem 9. *The following equation holds:*

$$\int_{-1}^{1} \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} \frac{\pi}{4}, & \text{if } m = n > 2; \\ -\frac{\pi}{8}, & \text{if } |n-m| = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Puting $x = \cos\theta$ implies

$$I = \int_{-1}^{1} \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}}dx = \int_{0}^{\pi} P_n(\cos\theta)P_m(\cos\theta)d\theta.$$

Using (6) we obtain

$$I = \int_{0}^{\pi} \sin^2\theta \sin(n-1)\theta \sin(m-1)\theta d\theta.$$

Transforming the integrating function we obtain

$$\sin^2\theta \sin(n-1)\theta \sin(m-1)\theta = \frac{1}{4}\cos(n-m)\theta - \frac{1}{4}\cos(n+m-2)\theta -$$

$$-\frac{1}{8}\cos(n-m-2)\theta - \frac{1}{8}\cos(n-m+2)\theta + \frac{1}{8}\cos(n+m-4)\theta + \frac{1}{8}\cos(n+m)\theta.$$

Taking into account that $m, n \geq 3$ we conclude that integrals of the terms on the right side of the preceding equation are zero if $n \neq m$ and $|n-m| \neq 2$. If $n = m$ we obtain $I = \frac{\pi}{4}$, and $I = -\frac{\pi}{8}$, if $|n-m| = 2$. □

Corollary 10. *Each subset of the set $\{P_n(x) : n \geq 3\}$, not containing polynomials $P_k(x)$ and $P_m(x)$ such that $|k-m| = 2$ and $k \neq m$ is orthogonal.*

The next result concerns the weight $\sqrt{1-x^2}$ of Chebyshev polynomials of the second kind. The result is similar to the result of the preceding theorem.

First of all we have

$$\int_{-1}^{1} \sqrt{1-x^2} P_n(x) P_m(x) dx = \begin{cases} \frac{5\pi}{32}, & \text{if } (n,m) = (3,3); \\ -\frac{\pi}{32}, & \text{if } (n,m) = (2,4) \text{ or } (4,2). \end{cases}$$

Theorem 11. *For m, n such that $(m,n) \neq (2,4)$ and $(4,2)$ we have*

$$\int_{-1}^{1} \sqrt{1-x^2} P_n(x) P_m(x) dx = \begin{cases} \frac{3\pi}{16}, & \text{if } m = n > 3; \\ -\frac{\pi}{8}, & \text{if } |n-m| = 2; \\ \frac{\pi}{32}, & \text{if } |n-m| = 4; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In this case we have

$$\int_{-1}^{1} \sqrt{1-x^2} P_n(x) P_m(x) dx = \int_{0}^{\pi} \sin^2\theta P_n(\cos\theta) P_m(\cos\theta) d\theta.$$

We therefore need to calculate the integral

$$\int_{0}^{\pi} \sin^4\theta \sin(n-1)\theta \sin(m-1)\theta d\theta.$$

We have

$$\sin^4\theta \sin(n-1)\theta \sin(m-1)\theta = \frac{3}{16}\cos(n-m)\theta - \frac{3}{16}\cos(n+m-2)\theta +$$

$$+\frac{1}{32}\cos(n-m-4)\theta + \frac{1}{32}\cos(n-m+4)\theta - \frac{1}{32}\cos(n+m-6)\theta - \frac{1}{32}\cos(n+m+2)\theta -$$

$$-\frac{1}{8}\cos(n-m-2)\theta - \frac{1}{8}\cos(n-m+2)\theta + \frac{1}{8}\cos(n+m-4)\theta + \frac{1}{8}\cos(n+m)\theta.$$

The integral of each term on the right side with $m \neq n,\ |m-n| \neq 2,\ |n-m| \neq 4$ is zero. For these particular values we easily obtain the desired result. □

Taking, for instance, the weight $(1-x^2)^{\frac{3}{2}}$ in a similar way one obtains

$$\int_{-1}^{1} (1-x^2)^{\frac{3}{2}} P_n(x) P_m(x) dx = \begin{cases} \frac{7\pi}{64}, & \text{if } (m,n) = (3,3); \\ \frac{21\pi}{128}, & \text{if } (m,n) = (4,4); \\ -\frac{\pi}{128}, & \text{if } (m,n) = (4,2) \text{ or } (2,4); \\ -\frac{7\pi}{64}, & \text{if } (m,n) = (3,5) \text{ or } (5,3). \end{cases}$$

For m, n such that $(m,n) \neq (2,4),\ (4,2),\ (3,5)$, and $(5,3)$ we have

$$\int_{-1}^{1} (1-x^2)^{\frac{3}{2}} P_n(x) P_m(x) dx = \begin{cases} \frac{5\pi}{32}, & \text{if } m = n > 4; \\ -\frac{15\pi}{128}, & \text{if } |n-m| = 2; \\ \frac{3\pi}{64}, & \text{if } |n-m| = 4; \\ -\frac{\pi}{128}, & \text{if } |n-m| = 6; \\ 0, & \text{otherwise.} \end{cases}$$

Considering the weight 1 leads to the following result:

Theorem 12. *If m and n are of different parity then*

$$\int_{-1}^{1} P_n(x) P_m(x) dx = 0.$$

Proof. In this case we need to calculate the integral

$$\int_{0}^{\pi} \sin^3\theta \sin(n-1)\theta \sin(m-1)\theta d\theta.$$

We have

$$\sin^3\theta \sin(n-1)\theta \sin(m-1)\theta = -\frac{1}{16}\sin(n-m+3)\theta + \frac{1}{16}\sin(n-m-3)\theta +$$
$$+\frac{1}{16}\sin(n+m+1)\theta - \frac{1}{16}\sin(n+m-5)\theta + \frac{3}{16}\sin(n-m+1)\theta -$$
$$-\frac{3}{16}\sin(n-m-1)\theta - \frac{3}{16}\sin(n+m-1)\theta + \frac{1}{16}\sin(n+m-3)\theta.$$

Since m and n are of different parity each function on the right is of the form $\sin 2k\theta$, which implies that its integral equals zero. □

4 Some recurrence relations

In this section we prove some recurrence relation for $P_{n,m,p}(x)$ as well as some recurrence relations for their coefficients.

Theorem 13. *For each integer $t \geq 0$ we have*

$$P_{n,m,p}(x) = \sum_{i=0}^{t} (-1)^{t-i} \binom{t}{i} x^i P_{n+2t-i,m+t-i,p}(x).$$

Proof. Translating (2) into the equation for coefficients we obtain

$$c(n,k,m,p) = \sum_{i=0}^{t} (-1)^{i+t} \binom{t}{i} c(n+2t-i, k-i, m+t-i, p).$$

Multiplying by x^k yields

$$c(n,k,m,p)x^k = \sum_{i=0}^{t} (-1)^{i+t} \binom{t}{i} x^i c(n+2t-i, k-i, m+t-i, p) x^{k-i},$$

which easily implies the claim of the theorem. □

In the case $t = 1,\ m = 1,\ p = 2$ we obtain the following formula, expressing $P_n(x)$ in terms of Chebyshev polynomials of the first kind:

$$P_n(x) = xT_{n-1}(x) - T_{n-2}(x).$$

From this we easily conclude that $P_n(x)$ satisfies the same three-term recurrence as Chebyshev polynomials.

Corollary 14. *The polynomials $P_{n,m,2}(x)$ satisfy the following equation:*

$$P_{n,m,2}(x) = 2xP_{n-1,m,2}(x) - P_{n-2,m,2}(x),$$

with initial conditions

$$P_{0,m,2}(x) = x^m,\ P_{1,m,2}(x) = 2x^{m+1} - mx^{m-1}.$$

Combining equations (1) and (4) we obtain

$$f(n,k,m,p) = \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{t=0}^{m} \binom{n}{i} \binom{m}{t} \binom{i}{j} f(n-j, k-i+j-t, 0, p-1).$$

Translating this equation into an equation for the coefficients, we obtain

$$c(n,k,m,p) = \sum_{i=0}^{\tilde{n}} \sum_{j=0}^{i} \sum_{t=0}^{m} \binom{\tilde{n}}{i} \binom{m}{t} \binom{i}{j} (-1)^{i-j+t} c(n-m-i-t, k-m+i-2j+t, 0, p-1),$$

where $\tilde{n} = \frac{n+k-2m}{2}$.

Applying the preceding equation several times we obtain the following:

Corollary 15. *The coefficients of $P_{n,m,p}(x)$ can be expressed as functions of coefficients of Chebyshev polynomials of the second kind.*

Converting (3) into the equation for coefficients we obtain

$$c(n,k,m,p) = \sum_{i=1}^{p} (-1)^{i-1} \binom{p}{i} c(n-i, k+i-2, m, p).$$

This equation implies the following:

Corollary 16. *The coefficients of $P_{n,m,p}(x)$ can be expressed in terms of coefficients of the polynomials $P_{n',m,p}(x)$, where $n' < n$.*

We shall finish the paper with a formula for coefficients of Chebyshev polynomials of the second kind. Taking $p = 2$ in (4) we obtain

$$f(n,k,m,2) = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}f(n-j,k-i+j,m,1).$$

Since $f(r,s,m,1) = \binom{m}{s}$ we have

$$f(n,k,m,2) = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\binom{m}{k-i+j}.$$

For $m = 0$, in the sum on the right side of this equation only terms with $k = i - j$ remains. We thus obtain

$$f(n,k,0,2) = \sum_{s=0}^{n-k}\binom{n}{s}\binom{n-s}{k}.$$

Accordingly, the following formula follows.

Corollary 17. *For coefficients $c(n,k)$ of Chebyshev polynomial $U_n(x)$ hold*

$$c(n,k) = (-1)^{\frac{n-k}{2}}\sum_{i=0}^{k}\binom{\frac{n+k}{2}}{i}\binom{\frac{n+k}{2}-i}{\frac{n-k}{2}},$$

if n i k are of the same parity and $c(n,k) = 0$ otherwise.

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