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Some Remarks on a Paper of L. Toth

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Abstract

Consider the functions $P(n) := \sum_{k=1}^{n} \gcd(k,n)$ (studied by Pillai in 1933) and $\widetilde{P}(n) := n\prod_{p|n}(2-1/p)$ (studied by Toth in 2009). From their results, one can obtain asymptotic expansions for $\sum_{n\le x} P(n)/n$ and $\sum_{n\le x} \widetilde{P}(n)/n$. We consider two wide classes of functions $\mathcal{R}$ and $\mathcal{U}$ of arithmetical functions which include $P(n)/n$ and $\widetilde{P}(n)/n$ respectively. For any given $R\in\mathcal{R}$ and $U\in\mathcal{U}$, we obtain asymptotic expansions for $\sum_{n\le x} R(n)$, $\sum_{n\le x} U(n)$, $\sum_{p\le x} R(p-1)$ and $\sum_{p\le x} U(p-1)$.

1 Introduction

In 1933, Pillai [6] introduced the function

$$P(n) := \sum_{k=1}^{n} \gcd(k,n)$$

and showed that

$$P(n)=\sum_{d|n} d\varphi(n/d) \quad \text{ and } \sum_{d|n} P(d)=n\tau(n)=\sum_{d|n}\sigma(d)\varphi(n/d),$$

where φ stands for the Euler function and where $\tau(n)$ and $\sigma(n)$ stand for the number of divisors of n and the sum of the divisors of n respectively.

It is easily shown that

$$P(n)=n\tau(n)\prod_{p^a\|n}\left(1-\frac{a/(a+1)}{p}\right).$$

In 1985, Chidambaraswamy and Sitaramachandrarao [2] showed that, given an arbitrary $\varepsilon>0$,

$$\sum_{n\le x} P(n)=e_1x^2\log x+e_2x^2+O\left(x^{1+\theta+\varepsilon}\right), \tag{1}$$

where $e_1=\dfrac{1}{2\zeta(2)}$ and $e_2=\dfrac{1}{2\zeta(2)}\left(2\gamma-\dfrac{1}{2}-\dfrac{\zeta'(2)}{\zeta(2)}\right)$, and where θ is the constant appearing below in Lemma 5, ζ stands for the Riemann Zeta Function and γ stands for Euler's constant.

Using partial summation, one easily deduces from (1) that

$$\sum_{n\le x}\frac{P(n)}{n}=e_1x\log x+(2e_2-e_1)x+O\left(x^{\theta+\varepsilon}\right). \tag{2}$$

In [10], Toth introduced the function

$$\widetilde{P}(n)=n\prod_{p|n}\left(2-\frac{1}{p}\right)=n\cdot 2^{\omega(n)}\prod_{p|n}\left(1-\frac{1}{2p}\right),$$

where $\omega(n)$ stands for the number of distinct prime factors of n. Toth obtained an estimate for $\sum_{n\le x}\widetilde{P}(n)$, analogous to (1) and from which one can easily derive an asymptotic expansion for $\sum_{n\le x}\widetilde{P}(n)/n$.

In this paper, we consider two wide classes of arithmetical functions $\mathcal{R}$ and $\mathcal{U}$, the first of which includes the function $P(n)/n$, and the second of which includes $\widetilde{P}(n)/n$. Given $R\in\mathcal{R}$, we obtain an asymptotic expansion for $\sum_{n\le x}R(n)$; similarly for $U\in\mathcal{U}$. We then examine the behavior of $\sum_{p\le x}R(p-1)$ and $\sum_{p\le x}U(p-1)$.

More precisely, the class $\mathcal{R}$ is made of the following functions R. First, let $\gamma(n)$ stand for the kernel of $n\ge 2$, that is $\gamma(n)=\prod_{p|n}p$ (with $\gamma(1)=1$). Then, given an arbitrary positive constant c, an arbitrary real number $\alpha>0$ and a multiplicative function $\kappa(n)$ satisfying $|\kappa(n)|\le\dfrac{c}{\gamma(n)^\alpha}$ for all $n\ge 2$, let $R\in\mathcal{R}$ be defined by

$$R(n)=R_{\kappa,c,\alpha}(n):=\tau(n)\sum_{d\|n}\kappa(d)=\tau(n)\prod_{p^a\|n}(1+\kappa(p^a)). \tag{3}$$

Here $d\|n$ means that the sum runs over the unitary divisors of n, that is over all divisors d of n for which $(d, n/d) = 1$.

It is easily seen that if we let $\kappa(p^a) = -\dfrac{a/(a+1)}{p}$, then the corresponding function $R(n)$ is precisely $P(n)/n$.

As for the class of functions $\mathcal{U}$, it is made of the functions

$$U(n) = U_{h,c,\alpha}(n) := 2^{\omega(n)} \sum_{d|n} h(d), \tag{4}$$

where h is a multiplicative function satisfying $|h(d)| \leq \dfrac{c}{\gamma(d)^\alpha}$ for each integer $d \geq 2$, where $\alpha > 0$ is a given number. It is easily seen that by taking $h(p) = -\frac{1}{2p}$ and $h(p^a) = 0$ for $a \geq 2$, we obtain the particular case $U(n) = \widetilde{P}(n)/n$.

Throughout this paper, $c_1, c_2, \ldots$ denote absolute positive constants.

2 Main results

Theorem 1. *Let R be as in (3). For any arbitrary $\varepsilon > 0$, as $x \to \infty$,*

$$T(x) := \sum_{n \leq x} R(n) = A_0 x \log x + B_0 x + O\left(x^\beta\right),$$

with

$$\beta = \begin{cases} \theta + \varepsilon, & \text{if } \alpha \geq 1 - \theta; \\ 1 - \alpha + \varepsilon, & \text{if } \alpha < 1 - \theta; \end{cases}$$

where θ is the number mentioned in Lemma 5 below and where

$$A_0 = \sum_{d \geq 1} \frac{\lambda(d)}{d} \qquad \text{and} \qquad B_0 = \sum_{d \geq 1} \frac{\lambda(d)}{d}(2\gamma - 1 - \log d), \tag{5}$$

the function λ being defined below in (20) and (21).

Theorem 2. *Let U be as in (4). As $x \to \infty$,*

$$S(x) := \sum_{n \leq x} U(n) = t_1 x \log x + t_2 x + O\left(\frac{x}{\log x}\right),$$

where

$$\begin{aligned} t_1 &= \sum_{\delta=1}^{\infty} \frac{h(\delta) 2^{\omega(\delta)} B_1(\delta) \eta_\delta}{\delta} \sum_{\substack{n=1 \\ \gamma(n)|\delta}}^{\infty} \frac{1}{n}, \\ t_2 &= \sum_{\substack{\delta=1 \\ b \in G_\delta}}^{\infty} \frac{h(\delta) 2^{\omega(\delta)}}{\delta b} \big(B_2(\delta)\eta_\delta - 2B_1(\delta)\mu_\delta - B_1(\delta)\eta_\delta \log(\delta b)\big), \end{aligned}$$

where $B_1(\delta)$ and $B_2(\delta)$ are defined below in Lemma 7, while η_δ and μ_δ are defined respectively in (35) and (36).

Theorem 3. *Let R be as in (3). As $x \to \infty$,*

$$M(x) := \sum_{p \le x} R(p-1) = K_1 x + O\left(\frac{x}{\log\log x}\right), \tag{6}$$

where $K_1 = \frac{1}{2} \sum_{d=1}^{\infty} \frac{\kappa(d)\tau(d)c_d}{d}$, where c_d is itself defined in Lemma 15.

Theorem 4. *Let U be as in (4). As $x \to \infty$,*

$$N(x) := \sum_{p \le x} U(p-1) = K_2 x + O\left(\frac{x}{\log\log x}\right), \tag{7}$$

where K_2 is a positive constant which may depend on the function κ.

3 Preliminary results

Lemma 5. *As $x \to \infty$,*

$$D(x) := \sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^\theta) \qquad (x \to \infty), \tag{8}$$

for some positive constant $\theta < 1/3$.

Proof. A proof can be found in the book of Ivić [4], where one can also find a history of the improvements concerning the size of θ. $\square$

Lemma 6. *Given $1 \le \ell < k$ with $\gcd(\ell, k) = 1$,*

$$\sum_{\substack{n \le x \\ n \equiv \ell \pmod k}} \tau(n) = A_1(k) x \log x + A_2(k) x + O\left(k^{7/3} x^{1/3} \log x\right),$$

where

$$A_1(k) = \frac{\varphi(k)}{k^2}, \qquad A_2(k) = (2\gamma - 1)\frac{\varphi(k)}{k^2} - \frac{2}{k} \sum_{d \mid k} \frac{\mu(d) \log d}{d}.$$

Proof. This result is due to Tolev [9]. $\square$

Lemma 7. *Given a positive integer k,*

$$\sum_{\substack{n \le x \\ (n,k)=1}} \tau(n) = B_1(k) x \log x + B_2(k) x + O\left(k^{10/3} x^{1/3} \log x\right),$$

where $B_1(k) = \left(\frac{\varphi(k)}{k}\right)^2$ and $B_2(k) = \varphi(k) A_2(k)$.

Proof. Observing that

$$\sum_{\substack{n\le x\\ \gcd(n,k)=1}} \tau(n) = \sum_{\substack{\ell=1\\ \gcd(\ell,k)=1}}^{k} \sum_{\substack{n\le x\\ n\equiv \ell \pmod k}} \tau(n),$$

and using the trivial fact that $\varphi(k)\le k$, the result follows immediately from Lemma 6. □

Lemma 8. *Given an arbitrary positive real number $\alpha<1$,*

$$\sum_{n>x}\frac{1}{n\gamma(n)^\alpha} \ll \frac{1}{x^\alpha}.$$

Proof. Writing each number n as $n=rm$, where r and m are the square-full and square-free parts of n respectively with $(r,m)=1$, we may write

$$\begin{aligned}
\sum_{n>x}\frac{1}{n\gamma(n)^\alpha} &= \sum_{\substack{rm>x\\ \gcd(r,m)=1\\ r\ \text{square-full}}} \frac{\mu^2(m)}{r\gamma(r)^\alpha m^{1+\alpha}} = \sum_{\substack{r\ge 1\\ r\ \text{square-full}}} \frac{1}{r\gamma(r)^\alpha} \sum_{\substack{m>x/r\\ \gcd(m,r)=1}} \frac{\mu^2(m)}{m^{1+\alpha}}\\
&\ll \sum_{\substack{r\ge 1\\ r\ \text{square-full}}} \frac{1}{r\gamma(r)^\alpha}\frac{\varphi(r)}{r}\int_{x/r}^{\infty}\frac{dt}{t^{1+\alpha}} = \frac{1}{x^\alpha}\sum_{\substack{r\ge 1\\ r\ \text{square-full}}} \frac{\varphi(r)}{r^2}\left(\frac{r}{\gamma(r)}\right)^\alpha\\
&= \frac{1}{x^\alpha}\prod_p\left(1+\frac{p-1}{p^2(p^{1-\alpha}-1)}\right) \ll \frac{1}{x^\alpha}.
\end{aligned}$$

□

Lemma 9. *Given any fixed number $z>0$,*

$$\sum_{n\le x} z^{\omega(n)} \ll x\ \log^{z-1} x. \tag{9}$$

Remark 10. This result is a weak form of the well known Selberg-Sathe formula

$$\sum_{n\le x} z^{\omega(n)} = C(t)x\log^{t-1}x + O(x\log^{t-2}x),$$

an estimate which holds uniformly for $x\ge 2$ and $|t|\le 1$, where $C(t)$ is a constant depending only on t (see Selberg [7]). Here we give a simple direct proof of (9).

Proof. For each positive integer k, let

$$\pi_k(x) := \#\{n\le x : \omega(n)=k\}.$$

From a classical result of Hardy and Ramanujan [3], we know that

$$\pi_k(x) \leq \frac{c_1 x}{(k-1)!\log x}(\log\log x + c_2)^{k-1} \qquad (x \geq 3),$$

for all $k \geq 1$, where c_1 and c_2 are some absolute constants. Using this estimate, it follows that

$$\begin{aligned} \sum_{n\leq x} z^{\omega(n)} &= \sum_{k\geq 1} \pi_k(x) z^k < c_1 z \frac{x}{\log x} \sum_{k\geq 1} \frac{(z(\log\log x + c_2))^{k-1}}{(k-1)!} \\ &= c_1 z \frac{x}{\log x} e^{z\log\log x + zc_2} \ll \frac{x}{\log x} e^{z\log\log x} = x \log^{z-1} x, \end{aligned}$$

as required. □

Lemma 11. *(*Brun-Titchmarsh Theorem*) Let $\pi(y;k,\ell) := \#\{p \leq y : p \equiv \ell \pmod{k}\}$. Given a fixed positive number $\beta < 1$, then, uniformly for $k \in [1, x^\beta]$, there exists a positive constant ξ_1 such that*

$$\pi(x;k,\ell) < \xi_1 \frac{x}{\varphi(k)\log(x/k)}.$$

Proof. For a proof, see Titchmarsh [8]. □

Lemma 12. *(*Bombieri-Vinogradov Theorem*). Given an arbitrary positive constant A, and let $B = \frac{3}{2}A + 17$. Then*

$$\sum_{k \leq \frac{\sqrt{x}}{\log^B x}} \max_{\gcd(k,\ell)=1} \max_{y\leq x} \left| \pi(y;k,\ell) - \frac{li(x)}{\varphi(k)} \right| \ll \frac{x}{\log^A x}.$$

Proof. For a proof, see Cheng-Dong [1]. □

Lemma 13. *Given a fixed positive number $\beta < 1$, then, uniformly for $d \in [1, x^{1-\beta}]$, there exists a positive constant ξ_3 such that*

$$\sum_{\substack{p\leq x \\ p\equiv 1 \mod d}} \tau(p-1) \leq \xi_3 x \frac{\tau(d)}{\varphi(d)}.$$

Proof. Since $\tau(n) \le 2 \sum_{\substack{d|n \\ d\le\sqrt{n}}} 1$ and $\tau(mn) \le \tau(m)\tau(n)$ for all positive integers m, n, it follows that

$$\begin{aligned} E : &= \sum_{\substack{p\le x \\ p\equiv 1 \pmod d}} \tau(p-1) \\ &\le \tau(d) \sum_{\substack{p\le x \\ p\equiv 1 \pmod d}} \tau(\frac{p-1}{d}) \\ &\le 2\tau(d) \sum_{u\le\sqrt{x/d}} \pi(x; du, 1). \end{aligned} \tag{10}$$

On the other hand, since $du = \sqrt{d}\cdot u\sqrt{d} \le \sqrt{d}\cdot\sqrt{x} \le x^{\frac{1}{2}-\frac{\beta}{2}}\cdot x^{\frac{1}{2}} = x^{1-\frac{\beta}{2}}$, we have, in light of Lemma 11, that

$$\pi(x; du, -1) \le \xi_1 \frac{x}{\varphi(du)\log(x/du)} \le \xi_2 \frac{x}{\varphi(du)\log x}, \tag{11}$$

with $\xi_2 = 2\xi_1/\beta$. Using (11) in (10) and keeping in mind that $\varphi(d)\varphi(u) \le \varphi(du)$ for positive integers d, u, we obtain that

$$\begin{aligned} E &\le 2\xi_2 \frac{x}{\log x}\tau(d) \sum_{u\le\sqrt{x/d}} \frac{1}{\varphi(du)} \\ &\le 2\xi_2 \frac{x}{\log x}\frac{\tau(d)}{\varphi(d)} \sum_{u\le\sqrt{x/d}} \frac{1}{\varphi(u)} \\ &\le \xi_3 x \frac{\tau(d)}{\varphi(d)}, \end{aligned}$$

for some positive constant ξ_3, where we used the fact that $\sum_{n\le y} \frac{1}{\varphi(n)} \ll \log y$, thus completing the proof of the lemma. $\square$

Lemma 14. *Given fixed numbers $A > 0$ and $\kappa < \alpha$, then*

$$\sum_{d\ge\log^A x} \frac{\tau(d)}{d\gamma(d)^\alpha} \ll \frac{1}{\log^{A\kappa} x}.$$

Proof. Clearly we have

$$\begin{aligned}
\sum_{d\geq \log^A x} \frac{\tau(d)}{d\gamma(d)^\alpha} &\leq \sum_{d\geq \log^A x} \frac{\tau(d)}{d\gamma(d)^\alpha}\left(\frac{d}{\log^A x}\right)^\kappa \\
&= \frac{1}{\log^{A\kappa} x}\sum_{d\geq \log^A x} \frac{\tau(d)}{d^{1-\kappa}\gamma(d)^\alpha} \\
&< \frac{1}{\log^{A\kappa} x}\prod_p\left(1+\frac{2}{p^{1-\kappa}p^\alpha}+\frac{2}{p^{2(1-\kappa)}p^\alpha}+\dots\right) \\
&\ll \frac{1}{\log^{A\kappa} x},
\end{aligned}$$

since the above infinite product converges in light of the fact that $\kappa < \alpha$. □

Lemma 15. *Given a fixed positive integer D, then*

$$\sum_{\substack{n\leq x\\ \gcd(n,D)=1}} \frac{1}{\varphi(n)} = c_D \log x + O(1),$$

where

$$c_D = \prod_p\left(1+\frac{1}{p(p-1)}\right)\cdot\prod_{p|D}\left(1+\frac{p}{(p-1)^2}\right)^{-1}.$$

Remark 16. Note that the result of Lemma 15 is known in a more precise form (see the book of Montgomery and Vaughan [5, pp. 42–43]).

Proof. We first compute the generating series of $n/\varphi(n)$. We have

$$\begin{aligned}
\sum_{\substack{n=1\\ \gcd(n,D)=1}}^\infty \frac{n/\varphi(n)}{n^s} &= \prod_{p\nmid D}\left(1+\frac{(1-1/p)^{-1}}{p^s}+\frac{(1-1/p)^{-1}}{p^{2s}}+\dots\right) \\
&= \frac{\prod_p\left(1+\frac{(1-1/p)^{-1}}{p^s}+\frac{(1-1/p)^{-1}}{p^{2s}}+\dots\right)}{\prod_{p|D}\left(1+\frac{(1-1/p)^{-1}}{p^s}+\frac{(1-1/p)^{-1}}{p^{2s}}+\dots\right)} \\
&= \zeta(s)\prod_p\left(1-\frac{1}{p^s}\right)\frac{\prod_p\left(1+\frac{(1-1/p)^{-1}}{p^s}+\frac{(1-1/p)^{-1}}{p^{2s}}+\dots\right)}{\prod_{p|D}\left(1+\frac{(1-1/p)^{-1}}{p^s}+\frac{(1-1/p)^{-1}}{p^{2s}}+\dots\right)} \\
&= \zeta(s)\prod_p\left(1+\frac{1}{p^s(p-1)}\right)\prod_{p|D}\left(1+\frac{p}{(p-1)(p^s-1)}\right)^{-1},
\end{aligned}$$

which by Wintner's Theorem yields

$$\sum_{\substack{n\leq x\\ \gcd(n,D)=1}} \frac{n}{\varphi(n)} = c_D x + O(x^{1/2}\log x), \tag{12}$$

where

$$c_D = \prod_p \left(1 + \frac{1}{p(p-1)}\right) \cdot \prod_{p|D} \left(1 + \frac{p}{(p-1)^2}\right)^{-1}.$$

Then, using partial summation, we get that

$$\sum_{\substack{n\le x\\ \gcd(n,D)=1}} \frac{1}{\varphi(n)} = c_D + O\left(\frac{\log x}{x^{1/2}}\right) + \int_1^x c_D \frac{dt}{t} + O(1) = c_D \log x + O(1),$$

as required. □

Lemma 17. *For each integer $\delta \ge 2$, let G_δ be the semigroup generated by the prime factors of δ, i.e. for $\delta = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, let $G_\delta = \{q_1^{\beta_1} \dots q_r^{\beta_r} : \beta_i \ge 0\}$. Then*

$$\sum_{n \in G_\delta} \frac{1}{n} = \sum_{\substack{n=1\\ \gamma(n)|\delta}}^{\infty} \frac{1}{n} \ll \log\log \delta.$$

Proof. Using the well known result of Landau

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^\gamma,$$

we certainly have that there exists some constant $C > 0$ such that

$$\frac{n}{\varphi(n)} < C \log\log n \qquad (n \ge 3),$$

from which is follows easily that

$$\begin{aligned}\sum_{\substack{n=1\\ \gamma(n)|\delta}}^{\infty} \frac{1}{n} &= \prod_{p|\delta}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) = \prod_{p|\delta}\left(1 - \frac{1}{p}\right)^{-1}\\ &= \frac{\delta}{\varphi(\delta)} < C\log\log\delta,\end{aligned}$$

which proves our result. □

Lemma 18. *Let G_δ be as in Lemma 17. Then*

$$\sum_{\delta=1}^{\infty} \frac{|h(\delta)|2^{\omega(\delta)}}{\delta} \sum_{b\in G_\delta} \frac{1}{b} < +\infty.$$

Proof. In light of Lemma 17, the fact that $|h(\delta)| \le \dfrac{c}{\gamma(\delta)^\alpha}$ and since given any $\varepsilon > 0$, $\log \delta < \delta^\varepsilon$ provided $\delta \ge \delta_0(\varepsilon)$, we have

$$\begin{aligned}
\sum_{\delta=1}^{\infty} \frac{|h(\delta)|2^{\omega(\delta)}}{\delta} \sum_{b\in G_\delta} \frac{1}{b} &\ll \sum_{\delta=1}^{\infty} \frac{|h(\delta)|2^{\omega(\delta)} \log\log\delta}{\delta} \\
&\ll \sum_{\delta=1}^{\infty} \frac{2^{\omega(\delta)} \log\log\delta}{\delta\gamma(\delta)^\alpha} \\
&\ll \sum_{\delta=\delta_0}^{\infty} \frac{2^{\omega(\delta)}}{\delta^{1-\varepsilon}\gamma(\delta)^\alpha} \\
&< \prod_p \left(1 + \frac{2}{p^{1-\varepsilon}p^\alpha} + \frac{2}{p^{2-2\varepsilon}p^\alpha} + \dots\right) \\
&= \prod_p \left(1 + \frac{2}{p^\alpha(p^{1-\varepsilon}-1)}\right) < +\infty,
\end{aligned}$$

provided $\varepsilon < \alpha$. □

Lemma 19. *Let G_δ be as in Lemma 17 and let $\rho(x)$ be a real function which tends to $+\infty$ as $x \to \infty$. Given any fixed positive constant $\kappa < \alpha$, we have*

$$Z(x) := \sum_{\substack{\rho(x)<\delta b\le x \\ b\in G_\delta}} \frac{h(\delta)2^{\omega(\delta)}}{\delta b} \ll \frac{1}{\rho(x)^\kappa}.$$

Proof. We have

$$|Z(x)| \le \sum_{\substack{\delta b>\rho(x) \\ b\in G_\delta}} \frac{|h(\delta)|\cdot 2^{\omega(\delta)}}{\delta b} = \sum_{\substack{\delta b>\rho(x) \\ \delta<\rho(x)}} + \sum_{\substack{\delta b>\rho(x) \\ \delta\ge\rho(x)}} = Z_1 + Z_2, \tag{13}$$

say. We first estimate Z_2. Recalling that $|h(\delta)| \le c/\gamma(\delta)^\alpha$,

$$\begin{aligned}
Z_2 &\le \sum_{\delta\ge\rho(x)} \frac{|h(\delta)|\cdot 2^{\omega(\delta)}}{\delta} \prod_{p|\delta}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \\
&\le \sum_{\delta=1}^{\infty} \frac{|h(\delta)|\cdot 2^{\omega(\delta)}}{\delta} \prod_{p|\delta} \frac{1}{1-1/p} \cdot \left(\frac{\delta}{\rho(x)}\right)^\kappa \\
&\le \frac{c}{\rho(x)^\kappa} \sum_{\delta=1}^{\infty} \frac{2^{\omega(\delta)}}{\delta^{1-\kappa}\gamma(\delta)^\alpha} \prod_{p|\delta} \frac{1}{1-1/p}.
\end{aligned} \tag{14}$$

Define

$$f(\delta) := \frac{2^{\omega(\delta)}}{\delta^{1-\kappa}\gamma(\delta)^\alpha} \prod_{p|\delta} \frac{1}{1-1/p}.$$

Clearly f is a multiplicative function with the following values on prime powers:

$$f(q^a) = \frac{2}{q^\alpha \cdot q^{a(1-\kappa)}(1-1/q)}.$$

Moreover, f is such that $\sum_{\delta=1}^{\infty} f(\delta)$ is bounded provided $\kappa < \alpha$. Hence, taking this into account in (14), we obtain that

$$Z_2 \ll \frac{1}{\rho(x)^\kappa}. \tag{15}$$

We now move on to estimate Z_1. We have

$$\begin{aligned} Z_1 &\le \sum_{\delta<\rho(x)} \frac{|h(\delta)|2^{\omega(\delta)}}{\delta} \sum_{\substack{b>\rho(x)/\delta \\ b\in G_\delta}} \frac{1}{b} \le \sum_{\delta<\rho(x)} \frac{|h(\delta)|2^{\omega(\delta)}}{\delta} \sum_{b\in G_\delta} \frac{1}{b}\left(\frac{b\delta}{\rho(x)}\right)^\kappa \\ &= \frac{1}{\rho(x)^\kappa} \sum_{\delta<\rho(x)} \frac{|h(\delta)|2^{\omega(\delta)}}{\delta^{1-\kappa}} \sum_{b\in G_\delta} b^{\kappa-1}. \end{aligned} \tag{16}$$

Since

$$\sum_{b\in G_\delta} b^{\kappa-1} \le \prod_{p|\delta}\left(1+\frac{1}{p^{1-\kappa}}+\frac{1}{p^{2(1-\kappa)}}+\ldots\right) = \prod_{p|\delta}\left(1-\frac{1}{p^{1-\kappa}}\right)^{-1},$$

it follows from (16) and the fact that $|h(\delta)| \le c/\gamma(\delta)^\alpha$, that

$$Z_1 \le \frac{c}{\rho(x)^\kappa} \sum_{\delta<\rho(x)} \frac{2^{\omega(\delta)}}{\gamma(\delta)^\alpha \delta^{1-\kappa}} \prod_{p|\delta}\left(1-\frac{1}{p^{1-\kappa}}\right)^{-1} = \frac{c}{\rho(x)^\kappa} \sum_{\delta<\rho(x)} g(\delta), \tag{17}$$

say, where g is clearly a multiplicative function whose values at the prime powers are given by

$$g(q^a) = \frac{2}{q^\alpha \cdot q^{a(1-\kappa)}}\left(1+\frac{1}{q^{1-\kappa}}+\frac{1}{q^{2(1-\kappa)}}+\ldots\right).$$

Since we assumed that $\kappa < \alpha$and since $\displaystyle\sum_{a=1}^{\infty} g(q^a) < \frac{c_3}{q^{\alpha+1-\kappa}}$, with a suitable constant $c_3 > 0$, it follows that

$$\sum_{\delta=1}^{\infty} g(\delta) = \prod_p (1+g(p)+g(p^2)+\ldots) < +\infty.$$

Using this information, (17) yields

$$Z_1 \ll \frac{1}{\rho(x)^\kappa}. \tag{18}$$

Substituting (15) and (18) in (13), the lemma is proved.

□

4 Proof of Theorem 1

One can easily see that

$$F(s) := \sum_{n=1}^{\infty} \frac{R(n)}{n^s}$$

can be written as

$$F(s) = A(s)\zeta^2(s), \tag{19}$$

where

$$A(s) = \prod_p \left(1 + \frac{\lambda(p)}{p^s} + \frac{\lambda(p^2)}{p^{2s}} + \dots\right),$$

with

$$\lambda(p) = 2\kappa(p), \qquad \lambda(p^2) = 3\kappa(p^2) - 4\kappa(p), \tag{20}$$

and more generally, for each $\beta \geq 3$, by

$$\lambda(p^\beta) = (\beta+1)\kappa(p^\beta) - 2\beta\kappa(p^{\beta-1}) + (\beta-1)\kappa(p^{\beta-2}). \tag{21}$$

Hence,

$$|\lambda(p^\beta)| \leq \frac{4\beta c}{p^\alpha} \qquad (\beta \geq 1).$$

Consequently,

$$|\lambda(d)| \leq \frac{\tau(d)^{c_4}}{\gamma(d)^\alpha}, \tag{22}$$

where c_4 is a suitable constant ($c_4 = 4c$ will do).

Now observe that, in light of (19) and of Lemma 5,

$$\begin{aligned} T(x) &= \sum_{d\leq x} \lambda(d) D(x/d) \\ &= \sum_{d\leq x} \lambda(d) \left(\frac{x}{d}\log(\frac{x}{d}) + (2\gamma-1)\frac{x}{d}\right) + O\left(x^\theta \sum_{d\leq x} \frac{|\lambda(d)|}{d^\theta}\right). \end{aligned} \tag{23}$$

It follows from (22) and Lemma 8 that

$$\sum_{d>x} \frac{|\lambda(d)|}{d} \leq \sum_{d>x} \frac{c\tau(d)^{c_4}}{d\gamma(d)^\alpha} \ll x^{\varepsilon-\alpha}. \tag{24}$$

Indeed, we have

$$S_U := \sum_{d\in[U,2U]} \frac{\tau(d)^{c_4}}{d\gamma(d)} < U^\varepsilon \sum_{d\in[U,2U]} \frac{1}{d\gamma(d)} = U^\varepsilon K_U,$$

say, provided U is sufficiently large. Now, from Lemma 8, it follows that $K_U \leq \frac{c}{U^\alpha}$. Hence, if we set $U_j = 2^j x$ for $j = 0, 1, \dots$, then, if x is large enough,

$$\sum_{d>x} \frac{|\lambda(d)|}{d} \leq c \sum_{j=0}^{\infty} \frac{U_j^\varepsilon}{U_j^\alpha} = c\, x^{\varepsilon-\alpha} \sum_{j=0}^{\infty} \frac{1}{2^{j(\alpha-\varepsilon)}} \ll x^{\varepsilon-\alpha}.$$

Similarly, observing that $\tau(d)^{c_4}(c+\log d) \le d^{\varepsilon}$ and arguing in a similar way, one can easily proves that

$$\sum_{d>x} \frac{|\lambda(d)| \cdot |c+\log d|}{d} \ll x^{\varepsilon-\alpha}. \tag{25}$$

Substituting (24) and (25) in (23), one obtains

$$T(x) = A_0 x \log x + B_0 x + O\left(x^{1-\alpha+\varepsilon}\right) + O\left(x^{\theta} \sum_{d\le x} \frac{|\lambda(d)|}{d^{\theta}}\right), \tag{26}$$

where $A_0 = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d}$ and $B_0 = \sum_{d=1}^{\infty} \frac{\lambda(d)}{d}(2\gamma - 1 - \log d)$.

Since

$$\sum_{d\le x} \frac{|\lambda(d)|}{d^{\theta}} \le \sum_{d\le x} \frac{c\tau(d)^{c_4}}{d^{\theta}\gamma(d)^{\alpha}} \le \begin{cases} x^{\varepsilon}, & \text{if } \theta+\alpha \ge 1; \\ x^{1-(\theta+\alpha)}, & \text{if } \theta+\alpha < 1; \end{cases}$$

it follows that

$$x^{\theta} \sum_{d\le x} \frac{|\lambda(d)|}{d^{\theta}} \le \begin{cases} x^{\theta+\varepsilon}, & \text{if } \theta+\alpha \ge 1; \\ x^{1-\alpha}, & \text{if } \theta+\alpha < 1. \end{cases}$$

Using this last estimate in (26) completes the proof of Theorem 1.

5 Proof of Theorem 2

Since

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} \qquad (\Re(s) > 1),$$

we have that

$$2^{\omega(n)} = \sum_{d^2 e = n} \mu(d)\tau(e). \tag{27}$$

Let G_δ be as in the statement of Lemma 17. If $\delta | n$, then let m be the largest divisor of n/δ which is co-prime with δ, and let $b = b(\delta) \in G_\delta$ be defined implicitly by $n = \delta \cdot b \cdot m$. Note that with this setup, the numbers m and b are uniquely determined, and we therefore have $2^{\omega(n)} = 2^{\omega(\delta)} \cdot 2^{\omega(m)}$. Hence,

$$\begin{aligned} S(x) &= \sum_{\delta\le x} h(\delta) 2^{\omega(\delta)} \sum_{\substack{b\le x/\delta \\ b\in G_\delta}} \sum_{\substack{m\le \frac{x}{\delta b} \\ \gcd(m,\delta)=1}} 2^{\omega(m)} \\ &= \sum_{\delta\le x} h(\delta) 2^{\omega(\delta)} \sum_{\substack{b\le x/\delta \\ b\in G_\delta}} E_\delta\left(\frac{x}{\delta b}\right), \end{aligned} \tag{28}$$

say. We therefore need to estimate

$$E_k(X) := \sum_{\substack{n\le X \\ \gcd(n,k)=1}} 2^{\omega(n)}.$$

Observe that it follows from (27) that

$$E_k(X) = \sum_{\substack{d\le\sqrt{X}\\ \gcd(d,k)=1}} \mu(d) \sum_{\substack{d^2e\le X\\ \gcd(e,k)=1}} \tau(e) = \sum_{\substack{d\le\sqrt{X}\\ \gcd(d,k)=1}} \mu(d)V_k\left(\frac{X}{d^2}\right), \tag{29}$$

where

$$V_k(y) := \sum_{\substack{n\le y\\ \gcd(n,k)=1}} \tau(n),$$

while we trivially have

$$E_k(X) \ll X\log X. \tag{30}$$

We shall now make use of a function $\rho(x)$ satisfying

$$\exp\left\{\sqrt{\log x}\right\} \le \rho(x) \le \sqrt{x} \tag{31}$$

and which we will later determine more precisely.

We first write (29) as

$$\begin{aligned} E_k(X) &= \sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}} \mu(d)V_k\left(\frac{X}{d^2}\right) + \sum_{\substack{\rho(X)<d\le\sqrt{X}\\ \gcd(d,k)=1}} \mu(d)V_k\left(\frac{X}{d^2}\right)\\ &= W_1(X)+W_2(X), \end{aligned} \tag{32}$$

say.

It follows from Lemma 7 that

$$\begin{aligned} W_2(X) &\ll \sum_{\rho(X)<d\le\sqrt{X}} \frac{X}{d^2}\log(X/d^2) \le X\log X \sum_{d>\rho(X)} \frac{1}{d^2}\\ &\ll \frac{X}{\rho(X)}\log X \ll \frac{X}{\log X}. \end{aligned} \tag{33}$$

On the other hand, again using Lemma 7, we have

$$\begin{aligned} W_1(X) &= \sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}} \mu(d)\left\{B_1(k)\frac{X}{d^2}\log\left(\frac{X}{d^2}\right)\right.\\ &\qquad\left. +B_2(k)\frac{X}{d^2} + O\left(k^{10/3}\left(\frac{X}{d^2}\right)^{1/3}\log\left(\frac{X}{d^2}\right)\right)\right\}\\ &= X\log X B_1(k) \sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}} \frac{\mu(d)}{d^2} - 2XB_1(k)\sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}} \frac{\mu(d)\log d}{d^2}\\ &\quad + XB_2(k)\sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}} \frac{\mu(d)}{d^2} + O\left(k^{10/3}X^{1/3}\log X\right). \end{aligned} \tag{34}$$

Since

$$\sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}}\frac{\mu(d)}{d^2} = \sum_{\substack{d=1\\ \gcd(d,k)=1}}^{\infty}\frac{\mu(d)}{d^2}+O\left(\frac{1}{\rho(X)}\right)=\frac{6}{\pi^2}\prod_{p|k}\left(1-\frac{1}{p^2}\right)^{-1}+O\left(\frac{1}{\rho(X)}\right)$$
$$= \eta_k+O\left(\frac{1}{\rho(X)}\right) \tag{35}$$

say, and since

$$\sum_{\substack{d\le\rho(X)\\ \gcd(d,k)=1}}\frac{\mu(d)\log d}{d^2}=\sum_{\substack{d=1\\ \gcd(d,k)=1}}^{\infty}\frac{\mu(d)\log d}{d^2}+O\left(\frac{1}{\rho(X)}\right)=\mu_k+O\left(\frac{\log X}{\rho(X)}\right), \tag{36}$$

say, it follows that (34) can be written as

$$W_1(X)=X\log XB_1(k)\eta_k-2XB_1(k)\mu_k+XB_2(k)\eta_k+O\left(\frac{X}{\log X}\right)+O\left(k^{10/3}X^{1/3}\log X\right). \tag{37}$$

Let us then write (28) as

$$S(x) = \sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}}h(\delta)2^{\omega(\delta)}E_\delta\left(\frac{x}{\delta b}\right)+\sum_{\substack{\rho(x)<\delta b\le x\\ b\in G_\delta}}h(\delta)2^{\omega(\delta)}E_\delta\left(\frac{x}{\delta b}\right)$$
$$= Z_1(x)+Z_2(x), \tag{38}$$

say.

First of all, we have using (30) and Lemma 19,

$$Z_2(x)\ll\sum_{\substack{\rho(x)<\delta b\le x\\ b\in G_\delta}}\frac{h(\delta)2^{\omega(\delta)}}{\delta b}x\log(x/\delta b)\ll x\log x\frac{1}{\rho(x)^\kappa}\ll\frac{x}{\log x}. \tag{39}$$

On the other hand, substituting (33) and (37) in (32), we get that $Z_1(x)$ from (38) can be written as

$$
\begin{aligned}
Z_1(x) &= \sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}} h(\delta)2^{\omega(\delta)}\left\{\frac{x}{\delta b}\log\left(\frac{x}{\delta b}\right)B_1(\delta)\eta_\delta+(B_2(\delta)\eta_\delta-2B_1(\delta)\mu_\delta)\frac{x}{\delta b}\right.\\
&\qquad\left.+O\left(\frac{\frac{x}{\delta b}}{\log(\frac{x}{\delta b})}\right)+\left(\delta^{10/3}\left(\frac{x}{\delta b}\right)^{1/3}\log\left(\frac{x}{\delta b}\right)\right)\right\}\\
&= x\log x\sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}}\frac{h(\delta)2^{\omega(\delta)}}{\delta b}B_1(\delta)\eta_\delta\\
&\quad +x\sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}}\frac{h(\delta)2^{\omega(\delta)}}{\delta b}\big(B_2(\delta)\eta_\delta-2B_1(\delta)\mu_\delta-B_1(\delta)\eta_\delta\log(\delta b)\big)\\
&\qquad +O\left(x\sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}}\frac{h(\delta)2^{\omega(\delta)}}{\delta b\log(x/(\delta b))}\right)\\
&\qquad +O\left(\sum_{\substack{\delta b\le\rho(x)\\ b\in G_\delta}}h(\delta)2^{\omega(\delta)}\delta^{10/3}\left(\frac{x}{\delta b}\right)^{1/3}\log\left(\frac{x}{\delta b}\right)\right)\\
&= (x\log x)T_1(x)+x\,T_2(x)+O(T_3(x))+O(T_4(x)),
\end{aligned}\tag{40}
$$

say.

First, it follows from Lemma 18 that

$$
T_3(x)\ll\frac{x}{\log x}\tag{41}
$$

and that

$$
\begin{aligned}
T_4(x) &\ll x^{1/3}\log x\sum_{\delta b\le\rho(x)}\sum_{b\in G_\delta}\frac{h(\delta)2^{\omega(\delta)}}{\delta b}\delta^4 b^{2/3}\\
&\ll x^{1/3}\log x(\rho(x))^4\sum_{\delta=1}^{\infty}\sum_{b\in G_\delta}\frac{h(\delta)2^{\omega(\delta)}}{\delta b}\\
&\ll x^{1/3}\log x(\rho(x))^4\ll\frac{x}{\log x}.
\end{aligned}\tag{42}
$$

On the other hand, using Lemma 19, we have

$$
\begin{aligned}
T_1(x) &= \sum_{\substack{\delta=1\\ b\in G_\delta}}^{\infty}\frac{h(\delta)2^{\omega(\delta)}B_1(\delta)\eta_\delta}{\delta b}-\sum_{\substack{\rho(x)<\delta b\le x\\ b\in G_\delta}}\frac{h(\delta)2^{\omega(\delta)}B_1(\delta)\eta_\delta}{\delta b}\\
&= t_1+O\left(\frac{1}{\rho(x)^\kappa}\right),
\end{aligned}\tag{43}
$$

say. Similarly, one can prove that

$$T_2(x) = t_2 + O\left(\frac{1}{\rho(x)^\kappa}\right), \tag{44}$$

say.

Substituting (41), (42), (43) and (44) in (40), we get

$$Z_1(x) \ll \frac{x \log x}{\rho(x)^\kappa}. \tag{45}$$

Gathering (39) and (45) in (38), we obtain

$$Z(x) \ll \frac{x \log x}{\rho(x)^\kappa}.$$

Choosing $\rho(x)$, already introduced in (31), in such a way that

$$\frac{\log x}{\rho(x)^\kappa} \ll \frac{1}{\log x}$$

completes the proof of Theorem 2.

6 Proofs of Theorems 3 and 4

Defining

$$S_d(x) := \sum_{\substack{p \le x \\ p \equiv 1 \mod d \\ \gcd(\frac{p-1}{d}, d) = 1}} \tau(p-1),$$

it is clear that

$$M(x) = \sum_{d \le x-1} \kappa(d) S_d(x). \tag{46}$$

Moreover, let

$$E_d(x) := \sum_{\substack{p \le x \\ p \equiv 1 \pmod d}} 2^{\omega(p-1)} \qquad \text{and} \qquad H_{d,\delta}(x) := \sum_{\substack{p \le x \\ p \equiv 1 \pmod d \\ p \equiv 1 \pmod{\delta^2}}} \tau\left(\frac{p-1}{\delta^2}\right).$$

Since

$$U(n) = 2^{\omega(n)} \sum_{d \mid n} h(d),$$

it follows that

$$N(x) = \sum_{d \le x-1} h(d) E_d(x). \tag{47}$$

On the other hand, since

$$2^{\omega(n)} = \sum_{\delta^2 e = n} \mu(\delta) \tau(e),$$

we have

$$E_d(x) = \sum_{\delta \le \sqrt{x-1}} \mu(\delta) H_{d,\delta}(x).$$

It follows from Lemma 13 that

$$E_d(x), S_d(x) \le c_5 \frac{\tau(d)}{\varphi(d)} x \qquad \text{say, for } d \le \sqrt{x}, \tag{48}$$

while it is clear that

$$\sum_{\substack{p \le x \\ p \equiv 1 \pmod d}} \tau(p-1) \le \tau(d) \sum_{n \le x/d} \tau(n) \le \frac{c_6 x \log x \cdot \tau(d)}{d} \qquad \text{for } 1 \le d \le x,$$

and therefore,

$$E_d(x), S_d(x) \le c_7 \frac{\tau(d)}{d} x \log x. \tag{49}$$

Let A be a large constant. We shall now estimate

$$T_1 := \sum_{\log^A x \le d \le x-1} |\kappa(d)| S_d(x) \qquad \text{and} \qquad T_2 := \sum_{\log^A x \le d \le x-1} |h(d)| E_d(x).$$

From (48) and (49), we have

$$T_1 \ll x \log x \sum_{d \ge \sqrt{x}} \frac{\tau(d)}{d\gamma(d)^\alpha} + x \sum_{d \ge \log^A x} \frac{\tau(d)}{d\gamma(d)^\alpha} = x \log x \cdot U_1 + x \cdot U_2, \tag{50}$$

say. Given a positive constant $\kappa < \alpha$, we have that

$$\begin{aligned} U_1 &\le \sum_{d=1}^{\infty} \frac{\tau(d)}{d\gamma(d)^\alpha} \left(\frac{d}{\sqrt{x}} \right)^\kappa = \frac{1}{x^{\kappa/2}} \sum_{d=1}^{\infty} \frac{\tau(d)}{\gamma(d)^\alpha d^{1-\kappa}} \\ &\le \frac{1}{x^{\kappa/2}} \prod_p \left(1 + \sum_{a=1}^{\infty} \frac{\tau(p^a)}{p^\alpha \cdot p^{a(1-\kappa)}} \right), \end{aligned} \tag{51}$$

where this last product, which we denote by Q, is convergent provided $\kappa < \alpha$.

In a similar way, we obtain

$$U_2 \le \sum_{d=1}^{\infty} \frac{\tau(d)}{d\gamma(d)^\alpha} \left(\frac{d}{\log^A x} \right)^\kappa \le \frac{1}{\log^{A\kappa} x} Q. \tag{52}$$

Substituting (51) and (52) in (50), we get that

$$T_1 \ll \frac{x}{\log^{\kappa A - 1} x} \qquad (0 < \kappa < \alpha). \tag{53}$$

In a similar manner, we obtain

$$T_2 \ll \frac{x}{\log^{\kappa A - 1} x} \qquad (0 < \kappa < \alpha). \tag{54}$$

We further define

$$M_1(x) = \sum_{d\le \log^A x} \kappa(d) S_d(x), \tag{55}$$

$$N_1(x) = \sum_{d\le \log^A x} h(d) E_d(x). \tag{56}$$

Let us first fix $d \le \log^A x$ and move on to estimate $S_d(x)$ and $E_d(x)$, by using the Bombieri-Vinogradov Theorem.

It turns out that it is more convenient to estimate

$$S_d(x) - S_d(x/2) \qquad \text{and} \qquad E_d(x) - E_d(x/2)$$

and similarly for $x/2$, $x/2^2$, …in place of x. This is why, for any fixed integer $r \ge 1$, we write

$$M_1(x) = \sum_{d\le \log^A x} \kappa(d) \sum_{j=0}^{r} A_d(x/2^j) + O\left(\frac{x}{2^r}\right) + O\left(\frac{x}{\log^{A\kappa} x}\right), \tag{57}$$

where

$$A_d(x/2^j) = S_d(x/2^j) - S_d(x/2^{j+1}).$$

Similarly, for any fixed integer $r \ge 1$, we may write

$$N_1(x) = \sum_{d\le \log^A x} h(d) \sum_{j=0}^{r} B_d(x/2^j) + O\left(\frac{x}{2^r}\right) + O\left(\frac{x}{\log^{A\kappa} x}\right), \tag{58}$$

where

$$B_d(x/2^j) = E_d(x/2^j) - E_d(x/2^{j+1}).$$

For now, fix x and set

$$r := \left\lfloor \frac{A\kappa \log\log x}{\log 2} \right\rfloor \qquad \text{so that} \qquad 2^r \approx \log^{A\kappa} x. \tag{59}$$

We now proceed to estimate $A_d(x)$ and $B_d(x)$.

Clearly we have

$$A_d(x) = \sum_{\substack{\frac{x}{2} < p \le x \\ \gcd(\frac{p-1}{d}, d)=1 \\ p\equiv 1 \pmod d}} \tau(p-1) = \tau(d) \sum_{\substack{\frac{x}{2} < p \le x \\ \gcd(\frac{p-1}{d}, d)=1 \\ p\equiv 1 \pmod d}} \tau(\frac{p-1}{d}). \tag{60}$$

Observe that

$$\tau(n) = 2\#\{(u,v) : u < v \text{ and } uv = n\} + \theta_n,$$

where

$$\theta_n = \begin{cases} 1, & \text{if } n = \text{square;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in light of (60),

$$A_d(x) = \tau(d) \sum_{\substack{\frac{x}{2} < p \le x \\ p \equiv 1 \pmod d}} \#\{(u,v) : u < v,\ \gcd(uv,d)=1,\ uv = \frac{p-1}{d}\} + O\left(\sqrt{x}\right), \tag{61}$$

where the error term is there to account for those p for which $\frac{p-1}{d} = u^2$.

It is clear that in the sum appearing in (61), we have $u \le \sqrt{x}$. On the other hand, the contribution of those u's for which $u > \sqrt{x}/(\log^B x)$ can be bounded above, using Lemma 11, by

$$\begin{aligned} \tau(d) \sum_{\substack{\sqrt{x}/(\log^B x) < u \le \sqrt{x} \\ \gcd(u,d)=1}} \pi(x; du, 1) &\le c_8 \tau(d) \frac{\mathrm{li}(x)}{\varphi(d)} \sum_{\substack{\sqrt{x}/(\log^B x) < u \le \sqrt{x} \\ \gcd(u,d)=1}} \frac{1}{\varphi(u)} \\ &\le c_9 \tau(d) \frac{\mathrm{li}(x)}{\varphi(d)} B \log\log x, \end{aligned} \tag{62}$$

where we also used Lemma 15.

Concerning the equation $\frac{p-1}{d} = uv$, the condition $\gcd(uv,d)=1$ is satisfied if and only if $\gcd(u,d)=1$ and $v \equiv \ell \pmod d$ for some positive integer ℓ co-prime to d, meaning that $v = \ell + dt$ and $p - 1 = du(\ell + dt) = du\ell + d^2ut$ for some integer t.

In light of these observations and of (62), relation (61) can be replaced by

$$\begin{aligned} A_d(x) &= \tau(d) \sum_{\substack{u \le \sqrt{x}/(\log^B x) \\ \gcd(u,d)=1}} \sum_{\gcd(\ell,d)=1} \left(\pi(x; d^2u, 1 + du\ell) - \pi(\frac{x}{2}; d^2u, 1 + du\ell)\right) \\ &\quad + O\left(\frac{\tau(d)}{\varphi(d)} (\mathrm{li}(x)) \log\log x\right). \end{aligned} \tag{63}$$

We shall assume that B is sufficiently large to insure that $d^2u \le \sqrt{x}/(\log^\Delta x)$, for some large number Δ.

Using Lemma 12, (63) becomes

$$\begin{aligned} A_d(x) &= \tau(d)\left(\mathrm{li}(x) - \mathrm{li}(x/2)\right) \sum_{\substack{u \le \sqrt{x}/(\log^B x) \\ \gcd(u,d)=1}} \sum_{\gcd(\ell,d)=1} \frac{1}{\varphi(d^2u)} \\ &\quad + O\left(\frac{\tau(d)}{\varphi(d)} (\mathrm{li}(x)) \log\log x\right). \end{aligned}$$

But observe that in the above sum, we have

$$\sum_{\gcd(\ell,d)=1} \frac{1}{\varphi(d^2u)} = \frac{1}{\varphi(u)} \cdot \frac{\varphi(d)}{d\varphi(d)} = \frac{1}{\varphi(u)} \cdot \frac{1}{d},$$

so that

$$\begin{aligned} A_d(x) &= \frac{\tau(d)}{d} \left(\mathrm{li}(x) - \mathrm{li}(x/2)\right) \sum_{\substack{u \le \sqrt{x}/(\log^B x) \\ \gcd(u,d)=1}} \frac{1}{\varphi(u)} + O\left(\frac{\tau(d)}{\varphi(d)} (\mathrm{li}(x)) \log\log x\right) \\ &= \frac{1}{4} \frac{\tau(d) c_d}{d} x + O\left(\frac{x \log\log x}{\log x}\right), \end{aligned} \tag{64}$$

(c_d being the constant appearing in the statement of Lemma 15) where we used Lemma 15 and the fact that

$$(\mathrm{li}(x) - \mathrm{li}(x/2)) \cdot \left(\frac{1}{2} \log x - B \log\log x \right) = \frac{1}{4} x + O\left(\frac{x \log\log x}{\log x} \right)$$

and that

$$\frac{\tau(d)}{\varphi(d)} \mathrm{li}(x) \log\log x = O\left(\frac{x \log\log x}{\log x} \right).$$

Substituting (64) in (57) and taking into account the choice of r made in (59), we get

$$\begin{aligned} M_1(x) &= \sum_{d \le \log^A x} \kappa(d) \sum_{j=0}^{r} A_d(x/2^j) + O\left(\frac{x}{\log^{A\kappa} x} \right) \\ &= \frac{1}{4} \sum_{d \le \log^A x} \frac{\kappa(d)\tau(d)c_d}{d} \sum_{j=0}^{r} \frac{x}{2^j} + O\left(\frac{x(\log\log x)^2}{\log x} \right) \\ &= \frac{1}{2} x \sum_{d \le \log^A x} \frac{\kappa(d)\tau(d)c_d}{d} + O\left(\frac{x(\log\log x)^2}{\log x} \right) \end{aligned} \tag{65}$$

provided A is chosen so that $A\kappa > 1$. Finally, using Lemma 14, recalling the initial formulation of $M(x)$ given in (46) and using (65), we get

$$M(x) = x \frac{1}{2} \sum_{d=1}^{\infty} \frac{\kappa(d)\tau(d)c_d}{d} + O\left(\frac{x(\log\log x)^2}{\log x} \right),$$

thus completing the proof of Theorem 3.

We now move to complete the proof of Theorem 4. For this, we shall use essentially the same kind of technique to obtain an estimate for $E_d(x)$ for $d \le \log^A x$.

Since

$$Q_{d,\delta}(x) := \sum_{n \le x/\delta^2} \tau(n) \le \frac{c_{10} x \log x}{\delta^2},$$

it follows that

$$E_d(x) = \sum_{\delta \le \log^B x} \mu(\delta) Q_{d,\delta}(x) + O\left(\frac{x}{d} \frac{1}{\log^2 x} \right),$$

provided B is sufficiently large.

We shall now estimate $Q_{d,\delta}(x)$ assuming that $d \le \log^A x$ and $\delta \le \log^B x$.

We proceed to estimate

$$B(x) = B_{d,\delta}(x) := Q_{d,\delta}(x) - Q_{d,\delta}(x/2).$$

We have

$$B(x) = 2 \sum_{\substack{\frac{x}{2} < p \le x \\ p \equiv 1 \pmod{\delta^2}}} \#\{(u,v) : u < v,\ uv = \frac{p-1}{\delta^2},\ d | \delta^2 uv\} + O(\sqrt{x}).$$

As in the proof of Theorem 3, we can drop in the above solution count those pairs u, v for which $u > \sqrt{x}/(\log^{\Delta} x)$ for any fixed $\Delta > 0$, arbitrarily large.

Now let $K, L \in G_d$. Given u, v such that $uv = \frac{p-1}{\delta^2}$, write $u = K\widetilde{u}$ and $v = L\widetilde{v}$, where $\gcd(\widetilde{u}, d) = \gcd(\widetilde{v}, d) = 1$. We then have

$$\frac{p-1}{\delta^2} = KL\widetilde{u}\widetilde{v}. \tag{66}$$

And in order to guarantee that $\gcd(\widetilde{v}, d) = 1$, we seek $\widetilde{v}$ from the arithmetical progression $\widetilde{v} = \ell + td$ with $\gcd(\ell, d) = 1$. It follows from (66) that

$$p - 1 = \delta^2 KL\widetilde{u}(\ell + td) = \delta^2 KL\widetilde{u}\ell + \delta^2 KLd\widetilde{u}t.$$

With this set up, we have

$$\begin{aligned} B(x) \ = \ 2 \sum_{K,L \in G_d} \sum_{\gcd(\ell,d)=1} \sum_{\substack{\widetilde{u} \le (\sqrt{x}/K)/\log^{\Delta} x \\ \gcd(\widetilde{u},d)=1}} & \Big(\pi(x; \delta^2 KLd\widetilde{u}, \delta^2 KL\widetilde{u}\ell + 1) \\ & - \ \pi(x/2; \delta^2 KLd\widetilde{u}, \delta^2 KL\widetilde{u}\ell + 1)\Big) \ + \ O(\sqrt{x}). \end{aligned} \tag{67}$$

First we drop all pairs K, L for which $KL > \log^D x$, where D is a fixed large number. Indeed observe that

$$\begin{aligned} & \sum_{\substack{\log^D x < KL \le x^{1/10} \\ K,L \in G_d}} \sum_{\widetilde{u} \le \sqrt{x}} \sum_{\ell \pmod d} \pi(x; \delta^2 KLd\widetilde{u}, \delta^2 KL\widetilde{u}\ell + 1) \\ & \qquad + \sum_{\substack{x^{1/10} < KL \le x \\ K,L \in G_d}} \sum_{\widetilde{u} \le \sqrt{x}} \frac{x\varphi(d)}{\delta^2 KLd\widetilde{u}} \\ & \le \mathrm{li}(x) \sum_{\substack{KL > \log^D x \\ K,L \in G_d}} \sum_{\widetilde{u} \le \sqrt{x}} \frac{\varphi(d)}{\varphi(\delta^2 KLd\widetilde{u})} + \sum_{\substack{x^{1/10} < KL \le x \\ K < L \in G_d}} \frac{x}{KL} \frac{\log x}{\varphi(\delta^2)} \\ & \le \frac{\mathrm{li}(x)}{\varphi(\delta^2)} \cdot \log x \cdot \sum_{\substack{KL > \log^D x \\ K,L \in G_d}} \frac{1}{\varphi(KL)} + \frac{x \log x}{\varphi(\delta^2)} \sum_{\substack{KL > x^{1/10} \\ K,L \in G_d}} \frac{1}{KL} \\ & = \frac{\mathrm{li}(x)}{\varphi(\delta^2)} \cdot \log x \cdot \mathcal{H}_\delta(x) + \frac{x \log x}{\varphi(\delta^2)} \cdot \mathcal{J}_\delta(x), \end{aligned} \tag{68}$$

say.

First of all, using the fact that $\dfrac{1}{\varphi(n)} \ll \dfrac{\log\log n}{n}$, we have, for a fixed $\kappa > 0$,

$$\begin{aligned} \mathcal{H}_\delta(x) \ & \le \ c_{11} \log\log\log x \sum_{\substack{KL > \log^D x \\ K,L \in G_d}} \frac{1}{KL} \le c_{11} \log\log\log x \sum_{\substack{KL > \log^D x \\ K,L \in G_d}} \frac{1}{KL} \left(\frac{KL}{\log^D x}\right)^{\kappa} \\ & \le \ c_{11} \frac{\log\log\log x}{\log^{D\kappa} x} \prod_{p \mid d} \left(\frac{1}{1 - \frac{1}{p^{1-\kappa}}}\right)^2. \end{aligned} \tag{69}$$

Choosing $\kappa = 1/2$ and observing that for $d < \log^A x$, we have that $\omega(d) \ll \frac{\log d}{\log\log d}$. Therefore for such a number d,

$$\prod_{p|d}\left(\frac{1}{1-\frac{1}{\sqrt{p}}}\right)^2 < c_{12}4^{\omega(d)} < (\log x)^{\varepsilon},$$

for any $\varepsilon > 0$ arbitrarily small. Using this observation, we may conclude from (69) that

$$\mathcal{H}_\delta(x) \le \frac{c_{13}}{(\log x)^{D/3}}, \tag{70}$$

say. Proceeding in a somewhat similar manner, we get that

$$\begin{aligned}\mathcal{J}_\delta(x) &= \sum_{\substack{KL>x^{1/10}\\ K,L\in G_d}} \frac{1}{KL} \le \sum_{\substack{KL>x^{1/10}\\ K,L\in G_d}} \frac{1}{KL}\left(\frac{KL}{x^{1/10}}\right)^{1/2}\\ &\le \frac{1}{x^{1/20}}\prod_{p|d}\frac{1}{\left(1-\frac{1}{\sqrt{p}}\right)^2} < \frac{(\log x)^{\varepsilon}}{x^{1/20}}.\end{aligned} \tag{71}$$

Estimates (68), (70) and (71) therefore establish that we can drop from the sum in (67) those $K, L \in G_d$ for which $KL > \log^D x$ for a large D, the error thus created being no larger than $O\left(\frac{\mathrm{li}(x)}{(\log x)^{D/3}}\right)$. In light of these observations and using the fact that $\varphi(\delta^2 KLd\widetilde{u}) = KL\varphi(\delta^2 d\widetilde{u})$ and that

$$\sum_{\substack{K,L\in G_d\\ KL<(\log x)^D}} \frac{1}{KL} = \sum_{K,L\in G_d}\frac{1}{KL} - \sum_{\substack{K,L\in G_d\\ KL\ge(\log x)^D}}\frac{1}{KL} = \prod_{p|G_d}\left(1-\frac{1}{p}\right)^{-2} + O\left(\frac{\mathrm{li}(x)}{(\log x)^{D/3}}\right),$$

we may write (67) as

$$\begin{aligned}B(x) &= 2(\mathrm{li}(x)-\mathrm{li}(x/2))\sum_{\substack{KL<(\log x)^D\\ K,L\in G_d}}\ \sum_{\substack{\widetilde{u}<\sqrt{x}/(\log x)^{\Delta}\\ \gcd(\widetilde{u},d)=1}} \frac{\varphi(d)}{\varphi(\delta^2 KLd\widetilde{u})} + O\left(\frac{\mathrm{li}(x)}{(\log x)^{D/3}}\right)\\ &= 2(\mathrm{li}(x)-\mathrm{li}(x/2))\prod_{p\in G_d}\left(1-\frac{1}{p}\right)^{-2}\sum_{\substack{\widetilde{u}<\sqrt{x}/(\log x)^{\Delta}\\ \gcd(\widetilde{u},d)=1}}\frac{\varphi(d)}{\varphi(\delta^2 d\widetilde{u})} + O\left(\frac{\mathrm{li}(x)}{(\log x)^{D/3}}\right).\end{aligned} \tag{72}$$

Set

$$Y_d(x) := \sum_{\substack{\widetilde{u}<\sqrt{x}/(\log x)^{\Delta}\\ \gcd(\widetilde{u},d)=1}}\frac{1}{\varphi(\delta^2 d\widetilde{u})}.$$

In order to evaluate $Y_d(x)$, we let $\delta = \delta_1\delta_2$, where $\delta_1 \in G_d$, $\gcd(\delta_2, d) = 1$, and $\widetilde{u} = \widetilde{u_1}\cdot\widetilde{u_2}$, where $\widetilde{u_1} \in G_{\delta_2}$, $\gcd(\widetilde{u_2}, d\delta_2) = \gcd(\widetilde{u_2}, d\delta) = 1$. Then,

$$\varphi(\delta^2 d\widetilde{u}) = \varphi(\delta_1^2\delta_2^2\widetilde{u_1})\varphi(\widetilde{u_2}) = \varphi(\delta_1^2 d)\varphi(\delta_2^2\widetilde{u_1})\varphi(\widetilde{u_2}).$$

It follows from Lemma 15 that

$$\sum_{\substack{\gcd(\widetilde{u_2},d\delta)=1\\ \widetilde{u_2}\le y}} \frac{1}{\varphi(\widetilde{u_2})} = c_{d\delta}\log y + O(1). \tag{73}$$

Now, the contribution of those $\widetilde{u_1}$ in $Y_d(x)$ for which $\widetilde{u_1}\in G_{\delta_2}$ and $\widetilde{u_1} > (\log x)^{\Delta_0}$ is, using (73),

$$\begin{aligned}
&\ll \sum_{\substack{\widetilde{u_1}\in G_{\delta_2}\\ \widetilde{u_1}>(\log x)^{\Delta_0}}} \frac{1}{\varphi(\delta_1^2 d)}\frac{1}{\varphi(\delta_2^2)\widetilde{u_1}}\sum \frac{1}{\varphi(\widetilde{u_2})}\\
&\ll (\log x)\sum \frac{1}{\varphi(\delta_1^2 d)}\frac{1}{\varphi(\delta_2^2)} \sum_{\substack{\widetilde{u_1}\in G_{\delta_2}\\ \widetilde{u_1}>(\log x)^{\Delta_0}}} \frac{1}{\widetilde{u_1}}\\
&\ll \frac{\log x}{\varphi(\delta_1^2 d)\varphi(\delta_2^2)}\frac{1}{(\log x)^{\Delta_0/2}} \sum_{\widetilde{u_1}\in G_{\delta_2}} \frac{1}{\sqrt{\widetilde{u_1}}}\\
&\ll (\log x)^{1-\Delta_0/2}\frac{1}{\delta_1^2\varphi(d)\varphi(\delta_2^2)}\prod_{p\mid\delta_2}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}\\
&\ll (\log x)^{1-\Delta_0/2}\frac{1}{\varphi(d)}\frac{2^{\omega(\delta_2)}}{\delta_1^2\delta_2^2}.
\end{aligned}$$

On the other hand, the contribution of those $\widetilde{u_1}\le(\log x)^{\Delta_0}$ in $Y_d(x)$, as $\widetilde{u_2}$ runs up to $y_0 := \sqrt{x}/((\log x)^C\widetilde{u_1})$, and also using Lemma 15 so that

$$\sum_{\substack{\gcd(\widetilde{u_2},d\delta)=1\\ \widetilde{u_2}\le y_0}} \frac{1}{\varphi(\widetilde{u_2})} = \frac{1}{2}c_{d\delta}\log x + O(\log\log x),$$

we get

$$\begin{aligned}
Y_d(x) &= \frac{1}{2}c_{d\delta}(\log x+\log\log x)\frac{1}{\varphi(\delta_1^2 d)}\frac{1}{\varphi(\delta_2^2)}\sum_{\widetilde{u_1}\in G_{\delta_2}}\frac{1}{\widetilde{u_1}}\\
&\quad + O\left(\frac{2^{\omega(\delta_2)}}{\delta^2\varphi(d)}(\log x)^{1-C_0/2}\right).
\end{aligned} \tag{74}$$

Observing that $\varphi(\delta_1^2 d)=\delta_1^2\varphi(d)$ and that

$$\sum_{\widetilde{u}\in G_{\delta_2}}\frac{1}{\widetilde{u}} = \prod_{p\mid G_{\delta_2}}\left(1-\frac{1}{p}\right)^{-1} = \frac{\delta_2}{\varphi(\delta_2)},$$

(74) becomes

$$\begin{aligned}
Y_d(x) &= \frac{1}{2}\frac{c_{d\delta}}{\varphi(d)}\frac{1}{\delta_1^2\delta_2^2}\frac{\delta_2}{\varphi(\delta_2)}(\log x+\log\log x)\\
&\quad +O\left(\frac{2^{\omega(\delta_2)}}{\delta^2\varphi(d)}(\log x)^{1-C_0/2}\right).
\end{aligned} \tag{75}$$

Using (75) in (72), we obtain

$$\begin{aligned} B(x) &= (\mathrm{li}(x)-\mathrm{li}(x/2))\frac{c_d\delta}{\delta_1^2\delta_2\varphi(\delta_2)}(\log x+\log\log x) \\ &\quad + O\left(\frac{2^{\omega(\delta_2)}}{\delta^2\varphi(d)}(\log x)^{1-C_0/2}\right). \end{aligned}$$

Finally, summing over $d \le (\log x)^A$ and $\delta \le (\log x)^B$, the theorem follows.

7 Final remarks

It is interesting to inquire about the solutions to the equation

$$\widetilde{P}(n) = \widetilde{P}(n+1), \tag{76}$$

or equivalently

$$\frac{n}{\gamma(n)}\prod_{p|n}(2p-1) = \frac{n+1}{\gamma(n+1)}\prod_{q|n+1}(2q-1). \tag{77}$$

The first 13 solutions are 45, 225, 1125, 2025, 3645, 140 625, 164 025, 257 174, 703 125, 820 125, 1 265 625, 2 657 205 and 3 515 625.

One can easily check that if for some positive integers a and b, the number $p = \frac{3^a \cdot 5^b + 1}{2}$ is a prime number, then $n = 3^a \cdot 5^b$ is a solution of (76), with $\widetilde{P}(n) = 3^{a+1} \cdot 5^b$. If one could prove that there exist infinitely many primes of this form, it would follow that equation (76) has infinitely many solutions. Interestingly, a computer search establishes that (76) has 37 solutions $n < 10^{10}$ and that 21 of them are of the form $n = 3^a \cdot 5^b$.

Moreover, all 37 solutions n_0 are such that $3^3 \cdot 5 | \widetilde{P}(n_0)$. To see that $3|n_0$, proceed as follows. Suppose that $3 \not| \widetilde{P}(n_0)$, then, in light of (77), $3 \not| (2p-1)$ for each prime divisor p of n and $3 \not| (2q-1)$ for each prime divisor q of $n+1$. This would imply that $p \equiv 1 \pmod 3$ for all $p|n$ (implying that $n \equiv 1 \pmod 3$) and $q \equiv 1 \pmod 3$ for all $q|n+1$ (implying that $n+1 \equiv 1 \pmod 3$), a contradiction.

The fact that $3|\widetilde{P}(n_0)$ clearly implies that $5|\widetilde{P}(n_0)$.

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