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On the Extension of the Diophantine Pair $\{1, 3\}$ in $\mathbb{Z}[\sqrt{d}]$

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Abstract

In this paper, we consider Diophantine triples of the form $\{1, 3, c\}$ in the ring $\mathbb{Z}[\sqrt{d}]$. We prove that the Diophantine pair $\{1, 3\}$ cannot be extended to the Diophantine quintuple in $\mathbb{Z}[\sqrt{d}]$ with $d < 0$ and $d \neq -2$.

1 Introduction

A *Diophantine m-tuple* in a commutative ring R with the unit 1 is a set of m distinct non-zero elements with the property that the product of each two distinct elements increased by 1 is a perfect square in R. These sets are studied in many different rings: the ring of integers $\mathbb{Z}$, the ring of rationals $\mathbb{Q}$, the ring of Gaussian integers $\mathbb{Z}[i]$ ([2, 8]), the ring of integers of quadratic fields $\mathbb{Z}[\sqrt{d}]$ and $\mathbb{Z}[(1+\sqrt{d})/2]$ ([7, 9]), polynomial rings ([4, 5]). The most famous Diophantine m-tuples and historical examples are quadruples $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ (found by Diophant) and $\{1, 3, 8, 120\}$ (found by Fermat). More information about these sets can be found on Dujella's web page on Diophantine m-tuples [3].

In 1969, Baker and Davenport [1] showed that the Diophantine triple $\{1, 3, 8\}$ can be extended uniquely to the Diophantine quadruple $\{1, 3, 8, 120\}$ A030063 . Hence, the triple $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. Jones [10] proved that if the set $\{1, 3, c\}$ is a Diophantine triple then c must be of the form

$$c_k = \frac{1}{6}((2+\sqrt{3})(7+4\sqrt{3})^k + (2-\sqrt{3})(7-4\sqrt{3})^k - 4),$$

for some $k \in \mathbb{N}$ (A045899). It can be easily verified that sets $\{1,3,c_k,c_{k-1}\}$ and $\{1,3,c_k,c_{k+1}\}$ are Diophantine quadruples. Indeed, $c_kc_{k+1}+1=t_k^2$, where (t_k) is A051048. Moreover, the Diophantine triple $\{1,3,c_k\}$ can be extended to a quadruple only by the elements c_{k-1} and c_{k+1}. This assertion is proved in 1998. by Dujella and Pethö [6]. A direct consequence of their assertion is the fact that the pair $\{1,3\}$ cannot be extended to a Diophantine quintuple.

A natural question that may arise is can we extend the pair $\{1,3\}$ to a quintuple in a larger ring. In this paper, we have chosen the ring $\mathbb{Z}[\sqrt{d}]=\{a+b\sqrt{d}: a,b\in\mathbb{Z}\}$ where $d\in\mathbb{Z}$ and d is not a perfect square. So, we assume that $\{1,3,a+b\sqrt{d}\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{d}]$ and according to a definition it means that there exist integers $\xi_1,\eta_1,\xi_2,\eta_2$ such that

$$\begin{aligned} a+b\sqrt{d}+1 &= (\xi_1+\eta_1\sqrt{d})^2 & (1)\\ 3a+3b\sqrt{d}+1 &= (\xi_2+\eta_2\sqrt{d})^2. & (2)\end{aligned}$$

Although, it seams that we have much more 'freedom' in larger rings, the result for some rings stays the same.

Theorem 1. *Let d be a negative integer and $d\neq -2$. The Diophantine pair $\{1,3\}$ cannot be extended to a Diophantine quintuple in the ring $\mathbb{Z}[\sqrt{d}]$.*

In certain rings $\mathbb{Z}[\sqrt{d}]$ for positive d, Theorem 1 is not valid. For example $\{1,3,8,120,1680\}$ is the Diophantine quintuple in $\mathbb{Z}[\sqrt{8\cdot 1680+1}]$.

In last two sections, we describe the set of all Diophantine triples $\{1,3,c\}$ in $\mathbb{Z}[\sqrt{-2}]$ and in $\mathbb{Z}[\sqrt{d}]$ for some positive integer d, where c is an integer and such that $\{1,3,c\}$ is not a Diophantine triple in $\mathbb{Z}$. The existence of such triples is related to solvability of certain Pellian equations.

2 Main result

Proposition 2. *Let d be a negative integer. If $\{1,3,c\}$ is a Diophantine triple in the ring $\mathbb{Z}[\sqrt{d}]$, then c is an integer.*

Proof. Let $c=a+b\sqrt{d}$. There exist $\xi_1,\eta_1,\xi_2,\eta_2\in\mathbb{Z}$ such that

$$a+1=\xi_1^2+\eta_1^2d,\ b=2\xi_1\eta_1,\ 3a+1=\xi_2^2+\eta_2^2d,\ 3b=2\xi_2\eta_2. \tag{3}$$

Evidently, b must be an even number. By squaring and subtracting corresponding relations in (3), we get

$$(a+1)^2-db^2=x^2,\ (3a+1)^2-d(3b)^2=y^2, \tag{4}$$

where $x=\xi_1^2-\eta_1^2d$ and $y=\xi_2^2-\eta_2^2d$. By eliminating the quantities b and d from (4), we obtain

$$(3x)^2-y^2=(3a+3)^2-(3a+1)^2=4(3a+2), \tag{5}$$

It is clear, from (5), that $3x-y$ and $3x+y$ must be even. So, we have

$$3x-y=2^kc_1,\ 3x+y=2^lc_2,$$

where k, l are positive integers and c_1, c_2 are odd. Thus, we have

$$(3x - y)(3x + y) = 2^{k+l} n,$$

where and $c_1 c_2 = n$. Furthermore, it follows that

$$y = 2^{l-1} c_2 - 2^{k-1} c_1.$$

The expression $(3a+1)^2 - y^2$ can be given in terms of c_1 and c_2:

$$\begin{aligned}
(3a+1)^2 - y^2 &= (2^{k+l-2} n - 1)^2 - (2^{l-1} c_2 - 2^{k-1} c_1)^2 \\
&= 4^{k+l-2} n^2 - 2^{k+l-1} n + 1 - 4^{l-1} c_2^2 + 2^{k+l-1} c_1 c_2 - 4^{k-1} c_1^2 \\
&= 4^{k+l-2} n^2 + 1 - 4^{l-1} c_2^2 - 4^{k-1} c_1^2 \\
&= (4^{k-1} c_1^2 - 1)(4^{l-1} c_2^2 - 1).
\end{aligned}$$

On the other hand, $(3a+1)^2 - y^2 = d(3b)^2$ (by (4)). Since $|c_i| \geq 1$ and $k, l \geq 1$, we obtain that $d < 0$ implies $b = 0$. $\square$

According to the previous proposition, it may have sense to study Diophantine triples of the form $\{1, 3, c\}$ in $\mathbb{Z}[\sqrt{d}]$ such that c is an integer.

The triple $\{1, 3, c\}$, $c \in \mathbb{Z}$, will be called a *proper* triple in $\mathbb{Z}[\sqrt{d}]$ if it is a Diophantine triple $\mathbb{Z}[\sqrt{d}]$, but not a Diophantine triple in $\mathbb{Z}$. For instance, $\{1, 3, 161\}$ is a proper triple in $\mathbb{Z}[\sqrt{2}]$, $\{1, 3, -3\}$ is a proper triple in $\mathbb{Z}[\sqrt{-2}]$, but $\{1, 3, 120\}$ is not a proper triple in $\mathbb{Z}[\sqrt{d}]$ $(d \neq 1)$.

Let us, now, assume that $\{1, 3, c\}$ is proper triple in $\mathbb{Z}[\sqrt{d}]$. So, one of the following cases may occur

$$\begin{aligned} c + 1 &= \xi^2 \\ 3c + 1 &= d\eta^2, \end{aligned} \tag{6}$$

$$\begin{aligned} c + 1 &= d\eta^2 \\ 3c + 1 &= \xi^2, \end{aligned} \tag{7}$$

$$\begin{aligned} c + 1 &= d\eta_1^2 \\ 3c + 1 &= d\eta_2^2. \end{aligned} \tag{8}$$

Following proposition gives a connection between the extensibility of the Diophantine pair $\{1, 3\}$ and solvability of certain Pellian equations.

Proposition 3. *The Diophantine pair $\{1, 3\}$ can be extended to a proper Diophantine triple $\{1, 3, c\} \subset \mathbb{Z}$ in $\mathbb{Z}[\sqrt{d}]$ if and only if one of the following equations*

$$3x^2 - dy^2 = 2 \tag{9}$$

and

$$x^2 - 3dy^2 = -2 \tag{10}$$

is solvable in $\mathbb{Z}$.

Proof. $\boxed{\Rightarrow}$ If $\{1,3,c\}$ is a proper triple and (6) is fulfilled, then by eliminating c, we get $3\xi^2 - d\eta^2 = 2$. So, the equation (9) is solvable in $\mathbb{Z}$.

Analogously, by eliminating c from (7), we obtain that the equation (10) is solvable in $\mathbb{Z}$.

Finally, from (8) we have

$$d(\eta_2^2 - 3\eta_1^2) = -2,$$

and the only possibilities are:

(a) $d = -1$ and $\eta_2^2 - 3\eta_1^2 = 2$, which is impossible (because the $x^2 - 3y^2 = 2$ is not solvable in $\mathbb{Z}$) ,

(b) $d = 2$ and $\eta_2^2 - 3\eta_1^2 = -1$, which is also impossible for the same reason,

(c) $d = -2$ and $\eta_2^2 - 3\eta_1^2 = 1$, which is possible because the Pell's equation $x^2 - 3y^2 = 1$ has infinitely many solution. Note that the equation (9) is also solvable if $d = -2$, it has a unique solution $(0, 1)$.

$\boxed{\Leftarrow}$ Assume that $(\xi, \eta) \in \mathbb{Z}^2$ is a solution of the equation (9). Then the set $\{1, 3, \xi^2 - 1\}$ represents a proper triple in $\mathbb{Z}[\sqrt{d}]$. Indeed, $3(\xi^2 - 1) + 1 = d\eta^2 = (n\sqrt{d})^2$.

Similarly, if $(\xi, \eta) \in \mathbb{Z}^2$ is a solution of (10), then we get a proper triple $\{1, 3, d\eta^2 - 1\}$. □

In order to prove Theorem 1, we need a special case of Theorem 8 from [10].

Lemma 4 (Theorem 8, [10]). *If $\{1, 3, c\}$ is a Diophantine triple in $\mathbb{Z}$, then $c = c_k$ for some integer $k \geq 2$ where*

$$c_k = 14c_{k-1} - c_{k-2} + 6,\ c_1 = 8,\ c_0 = 0.$$

By solving the recursion from Lemma 4 we get

$$c_k = \frac{1}{6}((2 + \sqrt{3})(7 + 4\sqrt{3})^k + (2 - \sqrt{3})(7 - 4\sqrt{3})^k - 4),\ k \geq 0 \tag{11}$$

Proposition 5. *Let d be a negative integer and $d \neq -2$. If $\{1, 3, c\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{d}]$, then $c = c_k$ for some positive integer k, where c_k is given by (11).*

Proof. According to Proposition 2, c is an integer. Hence, $\{1, 3, c\}$ is a triple in $\mathbb{Z}$ or a proper triple in $\mathbb{Z}[\sqrt{d}]$. Since the equations (9) and (10) are not solvable in $\mathbb{Z}$ for $d < 0$ and $d \neq -2$, Proposition 3 implies that $\{1, 3, c\}$ is not a proper triple in $\mathbb{Z}[\sqrt{d}]$. Hence, it is a Diophantine triple in $\mathbb{Z}$. Finally, from Lemma 4 we get that $c = c_k$ for some k. □

Lemma 6 (Theorem 1, [6]). *If $\{1, 3, c_k, d\}$ is a Diophantine quadruple in $\mathbb{Z}$ where c_k is given by (11) and k is a positive integer, then $d \in \{c_{k-1}, c_{k+1}\}$.*

At this point we can conclude that Theorem 1 is proved. (It follows directly from Proposition 5 and Lemma 6.)

Since the extension of the pair $\{1, 3\}$ is closely related to the solvability of the Pellian equation $x^2 - 3y^2 = -2$, our result can be understood as the following statement.

Corollary 7. *Let d be a negative integer and $d \neq -2$. The equation*

$$x^2 - 3y^2 = -2 \tag{12}$$

in $\mathbb{Z}[\sqrt{d}]$ has only real solutions.

Proof. Suppose that $(\xi_1 + \eta_1\sqrt{d}, \xi_2 + \eta_2\sqrt{d})$ is a solution of the equation (12). For

$$a + b\sqrt{d} = (\xi_2 + \eta_2\sqrt{d})^2 - 1,$$

we obtain that $\{1, 3, a + b\sqrt{d}\}$ is a Diophantine triple u $\mathbb{Z}[\sqrt{d}]$. Indeed,

$$3(a + b\sqrt{d}) + 1 = 3(\xi_2 + \eta_2\sqrt{d})^2 - 2 = (\xi_1 + \eta_1\sqrt{d})^2.$$

According to Proposition 5, there exists some k such that $a + b\sqrt{d} = c_k$. Immediately, we conclude that $\eta_1 = \eta_2 = 0$. $\square$

3 The case $d = -2$

In this section, we will take a closer look at the Diophantine triples $\{1, 3, c\}$ in $\mathbb{Z}[\sqrt{-2}]$. In fact, we are able to describe all Diophantine triples in $\mathbb{Z}[\sqrt{-2}]$.

Proposition 8. *If $\{1, 3, c\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{-2}]$, then $c \in \{-1, c_k, d_k\}$ for some $k \in \mathbb{N}$, where c_k is given by (11) and*

$$d_k = -\frac{1}{6}\left((7 + 4\sqrt{3})^k + (7 - 4\sqrt{3})^k + 4\right) \tag{13}$$

Proof. By Proposition 2, we have that c is an integer. Since the equation (9) is solvable in $\mathbb{Z}$, Proposition 3 implies that $\{1, 3, c\}$ can be a triple in $\mathbb{Z}$ or a proper triple in $\mathbb{Z}[\sqrt{-2}]$. If $\{1, 3, c\}$ is a triple in $\mathbb{Z}$, then $c = c_k$ from Lemma 4. If $\{1, 3, c\}$ is a proper triple in $\mathbb{Z}[\sqrt{-2}]$, then one of the cases (6) and (8) may occur. The case (6) implies that $c = -1$, because $(0, 1)$ is the unique solution of (9).

In the case of (8), $c = (-2x^2 - 1)/3$ where x is a solution of the Pell's equation

$$x^2 - 3y^2 = 1. \tag{14}$$

All solutions of (14) are

$$x_{k+2} = 4x_{k+1} - x_k,\ x_1 = 2,\ x_0 = 1,$$

for $k \geq 0$. By solving the recursion we obtain that

$$x_k = \frac{1}{2}((2 - \sqrt{3})^k + (2 + \sqrt{3})^k),$$

and $d_k = (-2x_k^2 - 1)/3$. $\square$

Note that for $k = 0$, we have $d_0 = -1$. Also, it is interesting that $(-d_k)$ is A011922.

Remark 9. *It is known that* $\{1, 3, c_k, c_{k+1}\}$ *is a Diophantine quadruple in* $\mathbb{Z}$ *for* $k \geq 1$ *(Lemma 6). The same can be proven for the set* $\{1, 3, d_k, d_{k+1}\}$ *in* $\mathbb{Z}[\sqrt{-2}]$ *for* $k \geq 0$. *We have*

$$\begin{aligned}
36(d_k d_{k+1} + 1) &= 66 + (7 - 4\sqrt{3})^{2k+1} + (7 + 4\sqrt{3})^{2k+1} + \\
&\quad 16((7 - 4\sqrt{3})^k (2 - \sqrt{3}) + (2 + \sqrt{3})(7 + 4\sqrt{3})^k) \\
&= 64 + 16((2 + \sqrt{3})^{2k+1} + (2 - \sqrt{3})^{2k+1}) + \\
&\quad ((2 + \sqrt{3})^{2k+1} + (2 - \sqrt{3})^{2k+1})^2 \\
&= 4(x_{2k+1}^2 + 8x_{2k+1} + 16) = 4(x_{2k+1} + 4)^2,
\end{aligned}$$

where x_{2k+1} *is a solution of* $x^2 - 3y^2 = 1$. *It can be easily shown that* $x_{2k+1} \equiv 2 \pmod 3$ *(because* $x_{n+2} = 4x_{n+1} - x_n$, $x_0 = 1$, $x_1 = 2$*), so*

$$\sqrt{d_k d_{k+1} + 1} = \frac{1}{3}(x_{2k+1} + 4)$$

is an integer.

Further, the set $\{1, 3, c_k, d_l\}$ is not a Diophantine quadruple for $k \geq 1$ and $l \geq 0$. Indeed, it can be easily seen that $c_k d_l + 1$ is a negative odd number (from Lemma 4 it follows that c_k is even, and from (13) that $d_l < 0$). Hence, $\sqrt{c_k d_l + 1} \notin \mathbb{Z}[\sqrt{-2}]$.

4 Proper triples in $\mathbb{Z}[\sqrt{d}]$ for some $d > 0$

According to Proposition 3, the set $\{1, 3, c\} \subset \mathbb{Z}$ is a proper Diophantine triple in $\mathbb{Z}[\sqrt{d}]$ if and only if one of the equations (9), (10) is solvable in $\mathbb{Z}$.

Example 10. *Determine all proper Diophantine triples* $\{1, 3, c\}$ *in* $\mathbb{Z}[\sqrt{10}]$.

Note that $d = 10$ is the smallest positive integer, $d \neq 1$, such that the equation (9) is solvable in $\mathbb{Z}$. If (x_n, y_n) is a solution of (9), then $\{1, 3, x_n^2 - 1\}$ is a proper triple in $\mathbb{Z}[\sqrt{10}]$. All solutions of (9) are

$$x_{n+2} = 22x_{n+1} - x_n,\ x_1 = 42, x_0 = 2,$$

for $n \geq 0$. Hence,

$$x_n = \frac{1}{6}((6 + \sqrt{30})(11 + 2\sqrt{30})^n + (6 - \sqrt{30})(11 - 2\sqrt{30})^n),\ n \geq 0.$$

So, all proper triples $\{1, 3, c\}$ in $\mathbb{Z}[\sqrt{10}]$ are $\{1, 3, d_n\}$ where

$$d_n = x_n^2 - 1 = \frac{1}{6}((11 + 2\sqrt{30})^{2n+1} + (11 - 2\sqrt{30})^{2n+1} - 4),\ n \geq 0. \tag{15}$$

Besides these triples, sets $\{1, 3, 25 \pm 8\sqrt{10}\}$, $\{1, 3, 160 \pm 44\sqrt{10}\}$, $\{1, 3, 355 \pm 112\sqrt{10}\}$ are also Diophantine triples in $\mathbb{Z}[\sqrt{10}]$. The third element of these triples is related to the solution in $\mathbb{Z}[\sqrt{10}]$ of the equation $x^2 - 3y^2 = -2$. For instance, $(6 + 2\sqrt{10}, 4 + \sqrt{10})$ is solution of $x^2 - 3y^2 = -2$ and $25 + 8\sqrt{10} = (4 + \sqrt{10})^2 - 1$. More generally, if $X^2 - 3Y^2 = -2$ and $X, Y \in \mathbb{Z}[\sqrt{10}]$ then $\{1, 3, Y^2 - 1\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{10}]$.

Example 11. *Determine all proper Diophantine triples $\{1,3,c\}$ in $\mathbb{Z}[\sqrt{2}]$.*

Since the equation (10) is solvable in $\mathbb{Z}$, $\{1,3,2y_n^2-1\}$ is a proper triple in $\mathbb{Z}[\sqrt{2}]$ where (x_n,y_n) is a solution of (10). All solutions of (10) are

$$y_{n+2}=10y_{n+1}-y_n,\ y_1=9, y_0=1,$$

for $n\geq 0$, i.e.,

$$y_n=\frac{1}{6}((5-2\sqrt{6})^n(3-\sqrt{6})+(3+\sqrt{6})(5+2\sqrt{6})^n).$$

So, all proper triples $\{1,3,c\}$ in $\mathbb{Z}[\sqrt{2}]$ are $\{1,3,d_n\}$ where

$$d_n=2y_n^2-1=\frac{1}{6}((5+2\sqrt{6})^{2n+1}+(5-2\sqrt{6})^{2n+1}-4). \tag{16}$$

Also, because the equation (10) is solvable in $\mathbb{Z}[\sqrt{2}]$, there exist triples like $\{1,3,11\pm 8\sqrt{2}\}$, $\{1,3,32\pm 20\sqrt{2}\}$, $\{1,3,161\pm 112\sqrt{2}\}$. Like in Example 10, the third element of these triples is obtained from the solution $(X,Y)\in\mathbb{Z}[\sqrt{2}]^2$ of the equation $x^2-3y^2=-2$, i.e., $\{1,3,Y^2-1\}$ is a triple in $\mathbb{Z}[\sqrt{2}]$.

It is interesting to note that the expression (15) contains $11+2\sqrt{30}$ (i.e. $(11,2)$) which is a fundamental solution of the Pell's equation $x^2-30y^2=1$. Similarly, in (16), $5+2\sqrt{6}$ represents a fundamental solution of $x^2-6y^2=1$.

In what follows, we will give the complete set of proper Diophantine triples $\{1,3,c\}$ in $\mathbb{Z}[\sqrt{d}]$ for positive d which satisfies certain conditions. This problem is related to the solvability of the equation $x^2-3dy^2=6$. For this equation, we have the following Nagell's result.

Lemma 12 (Theorem 11, [11])**.** *Let D be a positive integer which is not a perfect square, and let $C\neq 1,-D$ be a square-free integer which divides $2D$. Then if (x_0,y_0) is the least positive (fundamental) solution of the equation*

$$x^2-Dy^2=C, \tag{17}$$

then

$$\frac{1}{|C|}(x_0+y_0\sqrt{D})^2=\frac{x_0^2+Dy_0^2}{|C|}+\frac{2x_0y_0}{|C|}\sqrt{D}$$

is a fundamental solution of the related Pell's equation $x^2-Dy^2=1$. All solutions of (17) are

$$\frac{x_n+y_n\sqrt{D}}{\sqrt{|C|}}=\left(\frac{x_0+y_0\sqrt{D}}{\sqrt{|C|}}\right)^n,$$

where n is a positive odd integer.

Theorem 13. *Let d be a positive integer such that neither d nor $3d$ are perfect squares and such that one of the equations (9), (10) is solvable in $\mathbb{Z}$. If $\{1,3,c\}$ is a proper Diophantine triple in $\mathbb{Z}[\sqrt{d}]$, $d \neq 2, 10$, then there exists nonnegative integer n such that*

$$c = d_n = \frac{1}{6}((\xi + \eta\sqrt{3d})^{2n+1} + (\xi - \eta\sqrt{3d})^{2n+1} - 4), \tag{18}$$

where (ξ, η) is a fundamental solution of the Pell's equation

$$x^2 - 3dy^2 = 1. \tag{19}$$

If $d \in \{2, 10\}$, then $c = d_n$ for some $n \geq 1$.

Proof. First, assume that the equation (9) is solvable in $\mathbb{Z}$. Hence, $c = x'^2 - 1$, where (x', y') is a solution of (9). By Lemma 12, we get that all solutions of (9) are

$$x_n = \frac{1}{6}((\alpha + \beta\sqrt{3d})(\xi + \eta\sqrt{3d})^n + (\alpha - \beta\sqrt{3d})(\xi - \eta\sqrt{3d})^n),\ n \geq 0,$$

where (α, β) is a fundamental solution of the equation

$$x^2 - 3dy^2 = 6 \tag{20}$$

and (ξ, η) is a fundamental solution of the related Pell's equation (19). (Note that (20) has exactly one class of solutions, since $6|x'x'' - 3dy'y''$ i $6|x'y'' - y'x''$ for each two solutions (x', y') and (x'', y'') of (20)). Hence,

$$d_n = \frac{1}{36}((\alpha + \beta\sqrt{3d})^2(\xi + \eta\sqrt{3d})^{2n} + (\alpha - \beta\sqrt{3d})^2(\xi - \eta\sqrt{3d})^{2n} + 12) - 1.$$

According to Lemma 12, $\frac{1}{6}(\alpha + \beta\sqrt{3d})^2$ (i.e., $(\frac{1}{6}(\alpha^2 + 3d\beta^2), \frac{1}{3}\alpha\beta)$ is a fundamental solution of $x^2 - 3dy^2 = 1$, and so, we get (18).

If (10) is solvable in $\mathbb{Z}$, we obtain (18) in similar manner. Indeed, if (x', y') is a solution of (10), then $c = dy'^2 - 1$. All solutions of (10) are

$$y_n = \frac{1}{2\sqrt{3d}}((x_0 + y_0\sqrt{3d})(\xi + \eta\sqrt{3d})^n + (-x_0 + y_0\sqrt{3d})(\xi - \eta\sqrt{3d})^n),\ n \geq 0,$$

where (x_0, y_0) and (ξ, η) are fundamental solutions of (10) and (19), respectively. Further, we have

$$d_n = \frac{1}{6}\left(\frac{1}{2}(x_0 + y_0\sqrt{3d})^2(\xi + \eta\sqrt{3d})^{2n} + \frac{1}{2}(x_0 - y_0\sqrt{3d})^2(\xi - \eta\sqrt{3d})^{2n} - 4\right).$$

From Lemma 12, we get that $\frac{1}{2}(x_0 \pm y_0\sqrt{3d})^2 = \xi \pm \eta\sqrt{3d}$ and again obtain (18).

Finally, we must determine in which cases $d_n \in \{0, 1, 3\}$, since a Diophantine triple consists of three nonzero distinct elements. Obviously, $d_n > 3$ for $n \geq 1$, so we have to see if $d_0 \in \{0, 1, 3\}$. Because $d_0 = (\xi - 2)/3$, it follows that $\xi \in \{2, 5, 11\}$ where ξ is a fundamental solution of (19). If

- $\xi = 2$, then $5 = 3d\eta^2$ which is not possible,
- $\xi = 5$, then $8 = d\eta^2$ which implies that $\eta = 1$, $d = 8$ or $\eta = 2$, $d = 2$,
- $\xi = 11$, then $40 = d\eta^2$ which implies that $\eta = 1$, $d = 40$ or $\eta = 2$, $d = 10$.

For $d = 8, 40$ equations (9) and (10) have no solution in $\mathbb{Z}$, but (9) is solvable for $d = 10$ and (10) is solvable for $d = 2$. That is the reason why the element d_0 in $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{10}]$ is omitted. $\square$

In Table 1 we give a list of $1 < d < 100$ such that the equation (9) is solvable. In the last column of Table 1, the smallest c of a proper triple $\{1, 3, c\}$ in $\mathbb{Z}[\sqrt{d}]$ is given.

d	fund. solution of (9)	fund. solution of (19)	c_{min}
10	(2,1)	(11,2)	1763
46	(4,1)	(47,4)	15
58	(22,5)	(1451,110)	483
73	(5,1)	(74,5)	24
94	(28,5)	(2351,140)	783

Table 1: A list of $1 < d < 100$ such that (9) is solvable

If the equation (10) is solvable, we obtain Table 2.

d	fund. solution of (9)	fund. solution of (19)	c_{min}
2	(2,1)	(5,2)	161
6	(4,1)	(17,4)	5
17	(7,1)	(50,7)	16
18	(22,3)	(485,66)	161
22	(8,1)	(65,8)	21
34	(10,1)	(101,10)	33
38	(32,3)	(1025,96)	341
41	(11,1)	(122,11)	40
54	(140,11)	(19601,1540)	6533
57	(13,1)	(170,13)	56
66	(14,1)	(197,14)	65
82	(298,19)	(88805,5662)	29601
86	(16,1)	(257,16)	85
89	(49,3)	(2402,147)	800
97	(17,1)	(290,17)	96

Table 2: A list of $1 < d < 100$ such that (10) is solvable

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