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Two Catalan-type Riordan Arrays and their Connections to the Chebyshev Polynomials of the First Kind

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Abstract

Riordan matrix methods and properties of generating functions are used to prove that the entries of two Catalan-type Riordan arrays are linked to the Chebyshev polynomials of the first kind. The connections are that the rows of the arrays are used to expand the monomials $(1/2)\,(2x)^n$ and $(1/2)\,(4x)^n$ in terms of certain Chebyshev polynomials of degree n. In addition, we find new integral representations of the central binomial coefficients and Catalan numbers.

1 Introduction

The Chebyshev polynomials have applications in numerical analysis and approximation theory, Fourier series, combinatorics, and other areas of mathematics. Riordan arrays are important for proving combinatorial identities and sums, and have applications in combinatorics and graph theory, combinatorial number theory, algebra, and special functions. Riordan array connections to the Chebyshev polynomials of the first kind have been given by Barry [4], Barry and Hennessy [5], Luzon and Moron [17], and others. In this paper, we establish that the entries of two special Riordan matrices, we call Catalan-type Riordan arrays, are linked to the Chebyshev polynomials of the first kind.

The following two infinite lower-triangular arrays, respectively denoted by $\mathbf{A}$ and $\mathbf{B}$, are proved by Riordan matrix multiplication to be linked to the Chebyshev polynomials of the first kind:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ 2 & 1 & 0 & \cdots & 0 & \cdots \\ 6 & 4 & 1 & \cdots & 0 & \cdots \\ 20 & 15 & 6 & \ddots & 0 & \cdots \\ 70 & 56 & 28 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \\ \binom{2n}{n} & \binom{2n}{n-1} & \binom{2n}{n-2} & \cdots & \binom{2n}{n-k} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix} \tag{1}$$

and

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ 3 & 1 & 0 & \cdots & 0 & \cdots \\ 10 & 5 & 1 & \cdots & 0 & \cdots \\ 35 & 21 & 7 & \ddots & 0 & \cdots \\ 126 & 84 & 36 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \\ \binom{2n+1}{n} & \binom{2n+1}{n-1} & \binom{2n+1}{n-2} & \cdots & \binom{2n+1}{n-k} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \end{pmatrix}. \tag{2}$$

The connections are that the rows of the arrays are used to expand the monomials $(1/2)\,(2x)^n$ and $(1/2)\,(4x)^n$ in terms of certain Chebyshev polynomials of degree n. Finding rows of other infinite lower-triangular arrays that have similar connections to the Chebyshev polynomials or other special functions would be of interest. Before proving the connections for $\mathbf{A}$ and $\mathbf{B}$, we mention some properties of the Chebyshev polynomials of the first kind.

The Chebyshev polynomials of the first kind, denoted by $T_n\,(x)$, are defined by the relation

$$T_n\,(x) = \cos n\theta \text{ where } x = \cos\theta, \ \ -1 \le x \le 1$$

and given by the recursion

$$T_{n+1}\,(x) = 2xT_n\,(x) - T_{n-1}\,(x) \ \ (n \ge 1)$$

with initial conditions $T_0\,(x) = 1$ and $T_1\,(x) = x$. The first few higher order polynomials are

$$\begin{aligned} T_2\,(x) &= 2x^2 - 1 \\ T_3\,(x) &= 4x^3 - 3x \\ T_4\,(x) &= 8x^4 - 8x^2 + 1. \end{aligned}$$

The bivariate generating function of $T_n\,(x)$, denoted by $T\,(x,z)$, is defined by

$$T\,(x,z) := \sum_{n\ge 0} T_n\,(x)\,z^n = \frac{1 - xz}{1 + z^2 - 2xz}.$$

The shifted Chebyshev polynomials of the first kind, denoted by $T_n^*\,(x)$, are defined by

$$T_n^*\,(x) := T_n\,(2x-1)\,, \ 0 \le x \le 1$$

and given by the recursion

$$T_{n+1}^{*}(x) = 2(2x-1)\,T_{n}^{*}(x) - T_{n-1}^{*}(x) \quad (n \geq 1)$$

with initial conditions $T_{0}^{*}(x) = 1$ and $T_{1}^{*}(x) = 2x-1$. The first few higher order shifted polynomials are

$$\begin{aligned} T_{2}^{*}(x) &= 8x^{2}-8x+1 \\ T_{3}^{*}(x) &= 32x^{3}-48x^{2}+18x-1 \\ T_{4}^{*}(x) &= 128x^{4}-256x^{3}+160x^{2}-32x+1. \end{aligned}$$

The bivariate generating function of $T_{n}^{*}(x)$, denoted by $T^{*}(x,z)$, is defined by

$$T^{*}(x,z) := \sum_{n\geq 0} T_{n}^{*}(x)\, z^{n} = \frac{1-2xz+z}{1+2z+z^{2}-4xz}.$$

Remark 1. By the property $T_{n}^{*}(x^{2}) = T_{2n}(x)$, one can show that

$$T^{*}\left(x^{2},z\right) = E(x,z)$$

where $E(x,z)$ denotes the generating function of the even coefficients of $T(x,z)$, see Equation (9).

More information on properties of the Chebyshev polynomials can be found in Mason and Handscomb [18], Lanczos [16], Rivlin [24], and Gradshteyn and Ryzhik [14].

We now show the connections for a few rows of each array. The arrays $\mathbf{A}$ and $\mathbf{B}$ are linked to the Chebyshev polynomials of the first kind as a result of the even and odd coefficients of $T(x,z)$ occurring in the expansion of $(1/2)(2x)^{n}$ (see Lanczos [16], Table VI). The connections are illustrated by, respectively, multiplying $\mathbf{A}$ and $\mathbf{B}$ by the column vectors containing the entries of the sequences of even coefficients $\{T_{0}(x), T_{2}(x), \ldots\}$ and odd coefficients $\{T_{1}(x), T_{3}(x), \ldots\}$ of $T(x,z)$. We denote $T_{2n}(x)$ by T_{2n} and make the adjustment $T_{0} = 1/2$, for the even coefficients. Then, for the first four rows of the product we get the following column vector

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 20 & 15 & 6 & 1 \end{pmatrix} \begin{pmatrix} T_{0} \\ T_{2} \\ T_{4} \\ T_{6} \end{pmatrix} = \begin{pmatrix} T_{0} \\ 2T_{0}+T_{2} \\ 6T_{0}+4T_{2}+T_{4} \\ 20T_{0}+15T_{2}+6T_{4}+T_{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2x^{2} \\ 8x^{4} \\ 32x^{6} \end{pmatrix}.$$

We write this vector as

$$(1/2)\left((2x)^{0} \quad (2x)^{2} \quad (2x)^{4} \quad (2x)^{6}\right)^{T}.$$

For the odd coefficients we denote $T_{2n+1}(x)$ by T_{2n+1}. Then, for the first four rows of the product in this case we get the following column vector

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 10 & 5 & 1 & 0 \\ 35 & 21 & 7 & 1 \end{pmatrix} \begin{pmatrix} T_{1} \\ T_{3} \\ T_{5} \\ T_{7} \end{pmatrix} = \begin{pmatrix} T_{1} \\ 3T_{1}+T_{3} \\ 10T_{1}+5T_{3}+T_{5} \\ 35T_{1}+21T_{3}+7T_{5}+T_{7} \end{pmatrix} = \begin{pmatrix} x \\ 4x^{3} \\ 16x^{5} \\ 64x^{7} \end{pmatrix}.$$

We write this vector as

$$(1/2)\left((2x)^1 \quad (2x)^3 \quad (2x)^5 \quad (2x)^7\right)^T.$$

Thus, it appears that the column vector whose entries are $(1/2)(2x)^n$ $(n \geq 0)$ can be expressed as linear combinations of the columns of $\mathbf{A}$ and $\mathbf{B}$ where $T_n(x)$ are the coefficients of the linear combinations.

The shifted Chebyshev polynomials of the first kind are linked to $\mathbf{A}$ as a result of the sequence of coefficients $\{T_0^*(x), T_1^*(x), \ldots\}$ of $T^*(x,z)$ occurring in the expansion of $(1/2)(4x)^n$ (see Lanczos [16], Table VIII). This connection is illustrated by multiplying the array by the column vector containing the coefficients of $T^*(x,z)$. Here we denote $T_n^*(x)$ by T_n^* and make the adjustment $T_0^* = 1/2$. Then, for the first four rows of the product we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 20 & 15 & 6 & 1 \end{pmatrix} \begin{pmatrix} T_0^* \\ T_1^* \\ T_2^* \\ T_3^* \end{pmatrix} = \begin{pmatrix} T_0^* \\ 2T_0^* + T_1^* \\ 6T_0^* + 4T_1^* + T_2^* \\ 20T_0^* + 15T_1^* + 6T_2^* + T_3^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2x \\ 8x^2 \\ 32x^3 \end{pmatrix}.$$

We write this vector as

$$(1/2)\left((4x)^0 \quad (4x)^1 \quad (4x)^2 \quad (4x)^3\right)^T.$$

Thus, it appears that the column vector whose entries are $(1/2)(4x)^n$ $(n \geq 0)$ can be expressed as a linear combination of the columns of $\mathbf{A}$ where $T_n^*(x)$ are the coefficients of the linear combination. Note, when using the shifted Chebyshev polynomials we do not distinguish between the even and odd coefficients of $T^*(x,z)$.

The above illustrations suggest that the Chebyshev connections can be established by matrix multiplication. The connections are mentioned by Lanczos [16] and a trigonometric proof is given by expanding x^n in terms of the shifted Chebyshev polynomials $T_k^*(x)$ for $0 \leq k \leq n$. We derive these expansions as well as the expansions of x^{2n} and x^{2n+1}. The monomial x^{2n} is expanded in terms of $T_k(x)$ for $0 \leq k \leq 2n$ and x^{2n+1} in terms of $T_k(x)$ for $0 \leq k \leq 2n+1$. The approach we follow to prove the connections makes use of Riordan matrix methods and properties of generating functions.

As a by-product of the connections, we derive integral representations of the central binomial coefficients A00984 [28] and Catalan numbers A000108 [28] (see Equations (15), (16), (17), (18), (20), (21), (22), and (23)). The integrals do not seem to explicitly appear in Gradshteyn and Ryzhik [14]. They do not appear in Stanley's textbook [32, 33] that contains a large number of combinatorial and analytical interpretations of the Catalan numbers, nor in Koshy's book [15] that contains a wide variety of applications of the Catalan numbers. In addition, the representations we give differ from the representations given by Penson and Sixdeniers [23], Sofo [29], Dana-Picard [9], [10], Aigner [1], and Yuan [35], [36]. Thus, it appears that the integral representations are new.

This paper is arranged as follows. The definition of a Riordan matrix is given in Section 2. We show that $\mathbf{A}$ and $\mathbf{B}$ are Catalan-type Riordan arrays in Section 3. The inverses of $\mathbf{A}$ and $\mathbf{B}$ are also mentioned in this section and are linked to certain families of monic orthogonal polynomials. In addition, a combinatorial interpretation of $\mathbf{B}$ is briefly mentioned. We prove that $\mathbf{A}$ and $\mathbf{B}$ are linked to the Chebyshev polynomials of the first kind in Section 4. Theorems 10 and 11 of this section are the main results of this paper. The integral representations of the central binomial coefficients and Catalan numbers are given in Section 5. We then conclude this paper by suggesting possible problems for future research in Section 6. The material in Sections 1, 2 and 3 is known in the areas of numerical approximation theory, enumerative combinatorics and Riordan array theory, and special functions. We make the paper self contained and the results in Sections 4 and 5 accessible to a wider audience by including this material. An interesting feature of this paper is that it brings together ideas from several areas of mathematics.

2 Riordan Matrix

Let $\mathbb{N}$ denote the natural numbers (including 0) and $\mathbb{C}$ the complex numbers. Then, an infinite matrix $L = (\ell_{n,k})_{n,k\in\mathbb{N}}$ with entries in $\mathbb{C}$ is called a *Riordan matrix* if the *kth* column satisfies

$$\sum_{n\geq 0} \ell_{n,k} z^n := g(z)(f(z))^k$$

where

$$g(z) = 1 + g_1 z + g_2 z^2 + \cdots, \text{ and } f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$$

belong to the ring of formal power series $\mathbb{C}[[z]]$ and $f_1 \neq 0$. Note that $g_0 = 1$ is for convenience and not a necessary condition for the definition. A formal power series of the form

$$b(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{n\geq 0} b_n z^n$$

where z is an indeterminate is called the *ordinary generating function* of the sequence $\{b_n\}$. Riordan matrices are typically denoted by pairs of generating functions as $L = (g(z), f(z))$. The matrices L can be defined by either ordinary or exponential generating functions. However, the matrices presented in this paper are defined by ordinary generating functions. We note here that the proper Riordan arrays given by Sprugnoli [30] are what we call Riordan arrays or matrices.

Two important results for multiplying Riordan matrices are now given.

Theorem 2. ([27, 22]) *If $L = (\ell_{n,k})_{n,k\in\mathbb{N}} = (g(z), f(z))$ is a Riordan matrix and $h(z)$ is the generating function of the sequence associated with the entries of the column vector $h = (h_k)_{k\in\mathbb{N}}$, then the product of L and $h(z)$, defined by $L \otimes h(z) = g(z)h(f(z))$, is the generating function of the sequence associated with the entries of the column vector $\left(\sum_{k=0}^{n} \ell_{n,k} h_k\right)_{n\in\mathbb{N}}$.*

Let $L * N$, or by simple juxtaposition LN, denote the row-by-column product of two Riordan matrices L and N.

Theorem 3. ([27, 30]) *If*

$$L = (\ell_{n,k})_{n,k\in\mathbb{N}} = (g(z), f(z)) \text{ and } N = (v_{n,k})_{n,k\in\mathbb{N}} = (h(z), l(z))$$

are Riordan matrices, then $L * N$ *is*

$$L * N = \left(\sum_{j=0}^{n} \ell_{n,j} v_{j,k} \right)_{n,k \in \mathbb{N}} = \left(g\left(z\right) h\left(f\left(z\right)\right), l\left(f\left(z\right)\right) \right),$$

and the set $\mathbf{R}$ *of all Riordan matrices is a group under the operation of matrix multiplication denoted by* $(\mathbf{R}, *)$.

The notation $(\mathbf{R}, *)$ denotes the *Riordan group*. The identity element of the group is $e = (1, z)$. This is the usual unit diagonal matrix. The inverse element of the group is

$$L^{-1} = \left(g\left(z\right), f\left(z\right)\right)^{-1} = \left(1/g\left(\overline{f}\left(z\right)\right), \overline{f}\left(z\right)\right) \tag{3}$$

where $\overline{f}\left(z\right)$ is the compositional inverse of $f\left(z\right)$.

See Shapiro et al. [27] and Sprugnoli [30] for more information on the group and Riordan matrices.

3 Catalan Type Riordan Matrices and Chebyshev Polynomials

Riordan matrices were first introduced by Shapiro et al. in 1991 [27].. It turns out that the arrays $\mathbf{A}$ and $\mathbf{B}$ given in 1956 by Lanczos [16] are the Riordan matrices (A094527 [28]) and (A111418 [28]), respectively. Array $\mathbf{B}$ can be obtained by multiplying the matrices

$$\mathbf{CE} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 5 & 4 & 1 & 0 & \cdots \\ 14 & 14 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

where $\mathbf{C}$ denotes the Shapiro-Catalan matrix (A039598 [28, 25], and $\mathbf{E}$ the matrix with all 1's on and below the main diagonal and 0's everywhere else (A000012 [28]). Note, the Chebyshev connection to $\mathbf{B}$ where $\mathbf{B}$ is a Riordan matrix is mentioned in Nkwanta [21] but not proved. Array $\mathbf{A}$ can be obtained by multiplying the matrices

$$\mathbf{PT} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & \cdots \\ 7 & 6 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5}$$

where $\mathbf{P}$ denotes the Pascal Triangle (A007318 [28]) written in lower triangular matrix form and $\mathbf{T}$ the matrix (A094531 [28]) whose leftmost column entries record the count of the number of king walks down a chessboard [13, 27]. The two matrix products are confirmed below by Proposition 6.

Definition 4. The *Catalan generating function, denoted by* $c(z)$, is defined as

$$c(z) := \left(1 - \sqrt{1-4z}\right)/2z = \sum_{n\geq 0} c_n z^n \tag{6}$$

where

$$c_n = (1/(n+1))\binom{2n}{n}$$

denotes the *nth Catalan number* (A000108 [28]).

The following lemma involving $c(z)$ is useful for proving Proposition 6, Theorems 10 and 11, and Propositions 8 and 13.

Lemma 5. *([11])*

$$\begin{aligned} &1.\ 1/\sqrt{1-4z} &&= c(z)/\left(1-zc^2(z)\right) &&= c(z)/(2-c(z)) \\ &2.\ zc^2(z) &&= c(z)-1. \end{aligned}$$

Proposition 6. *([19]), ([20]), ([28])*

$$\mathbf{P} * \mathbf{T} = \mathbf{A} \text{ and } \mathbf{C} * \mathbf{E} = \mathbf{B}.$$

Proof. (Sketch) The Riordan pair forms of $\mathbf{C}$ and $\mathbf{E}$ are $\mathbf{C} = \left(c^2(z), zc^2(z)\right)$ and $\mathbf{E} = (1/(1-z), z)$. Applying Theorem 3 and Lemma 5(1), then the Riordan pair form of $\mathbf{B}$ is

$$\mathbf{B} = \left(c(z)/\sqrt{1-4z}, zc^2(z)\right) = \mathbf{C} * \mathbf{E}. \tag{7}$$

The Riordan pair forms of $\mathbf{P}$ and $\mathbf{T}$ are $\mathbf{P} = (1/(1-z), z/(1-z))$ and

$$\mathbf{T} = \left(1/\sqrt{1-2z-3z^2}, \left(1-z-\sqrt{1-2z-3z^2}\right)/2z\right).$$

Applying Theorem 3, then the Riordan pair form of $\mathbf{A}$ is

$$\mathbf{A} = \left(1/\sqrt{1-4z}, c(z)-1\right) = \mathbf{P} * \mathbf{T}. \tag{8}$$

□

Thus, $\mathbf{A}$ and $\mathbf{B}$ are Catalan-type Riordan arrays since they involve $c(z)$. Details of the computations involved in the proofs are omitted and left for the reader as exercises and motivation for Theorem 3.

Remark 7. Arrays $\mathbf{A}$ and $\mathbf{B}$ can also be obtained from their formation rules (or dot diagrams). A formation rule, denoted by $[Z;A]$ where Z corresponds to the rule for obtaining the leftmost or zeroth column entries and A for obtaining all other column entries, is a recurrence relation that defines the way entries of a Riordan matrix are computed. The formation rules of $\mathbf{A}$ and $\mathbf{B}$ are, respectively, $[2,2;1,2,1]$ ([28]) and $[3,1;1,2,1]$ [19, 20]. See the cited references for more information on the formation rules and dot diagrams of Riordan matrices.

Arrays $\mathbf{A}$ and $\mathbf{B}$ are special cases of the generalized Riordan array of the form

$$\mathbf{L} = \left(\frac{1-\lambda z-\mu z^2}{1+az+bz^2}, \frac{z}{1+az+bz^2}\right)$$

given by Barry and Hennessy [5]. The array $\mathbf{L}$ is associated with a certain tridiagonal matrix called a Stieltjes matrix and has the special property that the entries of $\mathbf{L}^{-1}$ are monic orthogonal polynomials whose moments are the entries of the leftmost column of $\mathbf{L}$.

We now give the inverses of $\mathbf{A}$ and $\mathbf{B}$ and mention that the entries of $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ are monic orthogonal polynomials whose moments are certain binomial coefficients.

Proposition 8. *The inverses of* $\mathbf{A}$ *and* $\mathbf{B}$ *are, respectively,*

$$\mathbf{A}^{-1} = \left((1-z)/(1+z), z/(1+z)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & 0 & \dots \\ 2 & -4 & 1 & 0 & 0 & \dots \\ -2 & 9 & -6 & 1 & 0 & \dots \\ 2 & -16 & 20 & -8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathbf{B}^{-1} = \left((1-z)/(1+z)^2, z/(1+z)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -3 & 1 & 0 & 0 & 0 & \dots \\ 5 & -5 & 1 & 0 & 0 & \dots \\ -7 & 14 & -7 & 1 & 0 & \dots \\ 9 & -30 & 27 & -9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Details of the computations of the inverses are omitted and left for the reader as exercises and motivation for Equation (3).

Remark 9. Arrays $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ are, respectively, the coefficient arrays of certain families of monic orthogonal polynomials for which the central binomial coefficients

$$\left\{\binom{2n}{n}\right\}_{n\geq 0} = \{1, 2, 6, 20, 70, 252, \dots\} \quad (\text{A00984 [28]})$$

and binomial coefficients

$$\left\{\binom{2n+1}{n+1}\right\}_{n\geq 0} = \{1, 3, 10, 35, 126, 462, \dots\} \quad (\text{A001700 [28]})$$

are the moments associated with the polynomials.

The results mentioned by Remark 9 can be confirmed by Propositions 9-10, 16 and 20 of Barry and Hennessy [5].

Arrays $\mathbf{A}$ and $\mathbf{B}$ are of combinatorial interest. $\mathbf{A}$ is of interest given that: 1) the columns involve the binomial coefficients (see A000984, A001791, A002694 [28] for the first few columns) and 2) the first column contains the central binomial coefficients. **B** is of interest given that: 1) the columns here also involve the binomial coefficients (see A001700, A002054, A003516, and A030053 [28] for the first few columns), 2) the first column contains the sequence $\{1, 5, 21, 84, 330, \ldots\}$ (A002054 [28]) that goes back to Cayley [8], and 3) **B** has a lattice path interpretation.

The entries of **B** count certain unit-step lattice paths of length n and height k in $\mathbb{Z}^3$ [19]. The path step directions are illustrated below.

$(0,0,1) = \mathrm{N}$ (North)	$(0,0,-1) = \mathrm{S}$ (South)
$(0,1,0) = \mathrm{E}$ (East)	$(0,-1,0) = \mathrm{W}$ (West)
$(1,0,0) = \mathrm{F}$ (Forward)	

Fig. 1 Unit Step Directions

All of the paths begin at the origin $(0,0,0)$ and are considered to be in the three-dimensional Euclidean space, never passing below the (x, y) plane. The length of each path is the number of unitary steps, and the height corresponds to the z value of the path's end point (x, y, z). The paths counted by the entries of **B** are denoted as $NSEW\widetilde{F}$ paths where $\widetilde{F}$ denotes that all F steps are restricted to height zero. For example, $NESW\widetilde{F}\widetilde{F}NN$ denotes one of the $12,376 = \binom{17}{6}$ unit-step $NSEW\widetilde{F}$ paths of length $n = 8$ ending at height $k = 2$.

See Nkwanta [19] for more information on the lattice path interpretation of **B** and more general classes of other related higher dimensional lattice paths. See, respectively, Sloane [28] and Nkwanta [20] for information on the constructions of the arrays **A** and **B**. Also, see Sloane [28] for information on connections between **A** and topics from combinatorics. See Barry and Hennessy [5] for more information on moments, monic orthogonal polynomials, and Riordan arrays.

4 Riordan Matrix Proof of the Chebyshev Connection

Let $O(x, z)$ and $E(x, z)$ denote, respectively, the bivariate generating functions of the even and odd coefficients of $T(x, z)$. Then,

$$E(x, z) = \left(\frac{T(x, z) + T(x, -z)}{2}\right) = \frac{1 + z^2 - 2x^2 z^2}{1 + 2z^2 + z^4 - 4x^2 z^2} \tag{9}$$

and

$$O(x, z) = \left(\frac{T(x, z) - T(x, -z)}{2}\right) = \frac{xz\left(1 - z^2\right)}{1 + 2z^2 + z^4 - 4x^2 z^2}. \tag{10}$$

If $T_0 = 1/2$, then

$$E(x, z) - 1/2 = \frac{1 - z^4}{2\left(1 + 2z^2 + z^4 - 4x^2 z^2\right)}. \tag{11}$$

Let

$$L\left(x, \sqrt{z}\right) = E\left(x, \sqrt{z}\right) - 1/2$$

denote the generating function of the even coefficients where all the odd coefficients with zero are removed. That is, applying $\sqrt{z}$ removes the aerated form of the generating function.

This means that the sequence $\{T_0, 0, T_2, 0, T_4, \ldots\}$ associated with the generating function given by Equation (11) is converted to the sequence $\{T_0, T_2, T_4, \ldots\}$. Then,

$$L\left(x, \sqrt{z}\right) = (1/2) \sum_{n \geq 0} T_{2n}(x) z^n = \frac{1 - z^2}{2\left(1 + 2z + z^2 - 4x^2 z\right)} \tag{12}$$

is the generating function of the sequence $\{T_0, T_2, T_4, \ldots\}$. Let

$$M^*(x, z) = T^*(x, z) - 1/2$$

denote the generating function of the shifted polynomials with $T_0^* = 1/2$. Then,

$$M^*(x, z) = (1/2) \sum_{n \geq 0} T_n^*(x) z^n = \frac{1 - z^2}{2\left(1 + 2z + z^2 - 4xz\right)} \tag{13}$$

is the generating function of the sequence $\{T_0^*, T_1^*, T_2^*, \ldots\}$. Here we observe that Equations (12) and (13) differ only by the form of x. Let

$$N\left(x, \sqrt{z}\right) = O\left(x, \sqrt{z}\right) / \sqrt{z}$$

denote the generating function of the odd coefficients where all the even coefficients with zero are removed. Applying and dividing by $\sqrt{z}$ removes the aerated form of the generating function. In this case the sequence $\{0, T_1, 0, T_3, \ldots\}$ associated with the generating function given by Equation (10) is converted to the sequence $\{T_1, T_3, T_5, \ldots\}$. Then,

$$N\left(x, \sqrt{z}\right) = \sum_{n \geq 0} T_{2n+1}(x) z^n = \frac{x(1 - z)}{1 + 2z + z^2 - 4x^2 z} \tag{14}$$

is the generating function of the sequence $\{T_1, T_3, T_5, \ldots\}$. See Sprugnoli [31] for more information on the technique involving $\sqrt{z}$.

The generating functions given above lead to the following theorems.

Theorem 10. *Consider the Catalan-type Riordan arrays*

$$\mathbf{A} = \left(1/\sqrt{1 - 4z}, c(z) - 1\right) \text{ and } \mathbf{B} = \left(c(z)/\sqrt{1 - 4z}, zc^2(z)\right)$$

where $c(z)$ is the Catalan generating function. Let $L(x, \sqrt{z})$ be the generating function of the sequence of even coefficients $\{T_0, T_2, T_4, \ldots\}$ of $T(x, z)$. Let $N(x, \sqrt{z})$ be the generating function of the sequence of odd coefficients $\{T_1, T_3, T_5, \ldots\}$ of $T(x, z)$. Then,

1. $\mathbf{A} \otimes L(x, \sqrt{z}) = \frac{1}{2(1-4x^2 z)}$
2. $\mathbf{B} \otimes N(x, \sqrt{z}) = \frac{x}{1-4x^2 z}$.

Moreover, for $n \geq 0$, the column vector whose entries are $(1/2)(2x)^{2n}$ can be expressed as a linear combination of the columns of $\mathbf{A}$ where the even coefficients of $T(x, z)$ are the coefficients of the linear combination. The column vectors whose entries are $(1/2)(2x)^{2n+1}$ can be expressed as a linear combination of the columns of $\mathbf{B}$ where the odd coefficients of $T(x, z)$ are the coefficients of the linear combination.

Proof. The proof of part (1) of Theorem 10 is as follows. By Equations (8) and (12), and Theorem 2 we obtain

$$\begin{aligned}\mathbf{A}\otimes L\left(x,\sqrt{z}\right) &= \left(1/\sqrt{1-4z}, c\left(z\right)-1\right)\otimes\frac{1-z^2}{2\left(\left(1+z\right)^2-4x^2z\right)}\\ &= \left(1/\sqrt{1-4z}\right)\left(\frac{1-\left(c\left(z\right)-1\right)^2}{2\left(c^2\left(z\right)-4x^2\left(c\left(z\right)-1\right)\right)}\right).\end{aligned}$$

Applying Lemma 5(2) to the denominator and simplifying gives

$$\begin{aligned}\mathbf{A}\otimes L\left(x,\sqrt{z}\right) &= \left(1/\sqrt{1-4z}\right)\left(\frac{2c\left(z\right)-c^2\left(z\right)}{2c^2\left(z\right)-c^2\left(z\right)4x^2z}\right)\\ &= \left(1/\sqrt{1-4z}\right)\left(\frac{2-c\left(z\right)}{2c\left(z\right)\left(1-4x^2z\right)}\right).\end{aligned}$$

From Lemma 5(1) we have

$$\left(1/\sqrt{1-4z}\right)\left(2-c\left(z\right)\right)=c\left(z\right).$$

Making this substitution and simplifying we get

$$\mathbf{A}\otimes L\left(x,\sqrt{z}\right)=\frac{1}{2\left(1-4x^2z\right)}.$$

Now, expanding the resulting generating function gives

$$\frac{1}{2\left(1-4x^2z\right)}=(1/2)\left(1+4x^2z+16x^4z^2+64x^6z^3+\cdots\right).$$

Thus, the coefficients form the column vector with entries $(1/2)\left(2x\right)^{2n}, n\geq 0$.

The linear combination follows as a consequence of the matrix multiplication and rearrangement of the resulting column vector. Thus, for $n,k\geq 0$ we get

$$T_0\begin{pmatrix}1\\2\\6\\20\\\vdots\\\binom{2n}{n}\\\vdots\end{pmatrix}+T_2\begin{pmatrix}0\\1\\4\\15\\\vdots\\\binom{2n}{n-1}\\\vdots\end{pmatrix}+\cdots+T_{2k}\begin{pmatrix}\binom{0}{-k}\\\binom{2}{1-k}\\\binom{4}{2-k}\\\binom{6}{3-k}\\\vdots\\\binom{2n}{n-k}\\\vdots\end{pmatrix}+\cdots=\begin{pmatrix}1/2\\2x^2\\8x^4\\32x^6\\\vdots\\2^{2k-1}x^{2k}\\\vdots\end{pmatrix}.$$

This proves the first part of the theorem.

The proof of part (2) of Theorem 10 is as follows. By Lemma 5(1), Equations (7) and (14), and Theorem 2 we obtain

$$\begin{aligned}
\mathbf{B} \otimes N\left(x, \sqrt{z}\right) &= \left(c^2(z)/\left(1 - zc^2(z)\right), zc^2(z)\right) \otimes \frac{x(1-z)}{(1+z)^2 - 4x^2 z} \\
&= \left(c^2(z)/\left(1 - zc^2(z)\right)\right)\left(\frac{x\left(1 - zc^2(z)\right)}{\left(1 + zc^2(z)\right)^2 - 4x^2 zc^2(z)}\right) \\
&= \frac{xc^2(z)}{\left(1 + zc^2(z)\right)^2 - 4x^2 zc^2(z)}.
\end{aligned}$$

Applying Lemma 5(2) to the denominator here and simplifying we get

$$\begin{aligned}
\mathbf{B} \otimes N\left(x, \sqrt{z}\right) &= \left(\frac{xc^2(z)}{c^2(z) - c^2(z)\,4x^2 z}\right) \\
&= \frac{x}{1 - 4x^2 z}.
\end{aligned}$$

Now, expanding this generating function gives

$$\frac{x}{1-4x^2 z} = x\left(1 + 4x^2 z + 16x^4 z^2 + 64x^6 z^3 + \cdots\right).$$

The coefficients here form the column vector with entries $(1/2)\,(2x)^{2n+1}, n \geq 0$.

Similarly, the linear combination follows here as a consequence of the matrix multiplication and rearrangement of the resulting column vector. ☐

Theorem 11. *Consider the Catalan-type Riordan array*

$$\mathbf{A} = \left(1/\sqrt{1-4z}, c(z) - 1\right)$$

where $c(z)$ is the Catalan generating function. Let $M^*(x,z)$ be generating function of the sequence of coefficients $\{T_0^*, T_1^*, T_2^*, \ldots\}$ of $T^*(x,z)$. Then,

$$\mathbf{A} \otimes M^*(x,z) = \tfrac{1}{2(1-4xz)}.$$

Moreover, for $n \geq 0$, the column vector whose entries are $(1/2)\,(4x)^n$ can be expressed as a linear combination of the columns of $\mathbf{A}$ where the coefficients of $T^*(x,z)$ are the coefficients of the linear combination.

Proof. By Equations (8) and (13), Theorem 2, and simplifying we obtain

$$\begin{aligned}
\mathbf{A} \otimes M^*(x,z) &= \left(1/\sqrt{1-4z}, c(z) - 1\right) \otimes \frac{1 - z^2}{2\left((1+z)^2 - 4xz\right)} \\
&= \left(1/\sqrt{1-4z}\right)\left(\frac{2c(z) - c^2(z)}{2c^2(z) - c^2(z)\,4xz}\right) \\
&= \left(1/\sqrt{1-4z}\right)\left(\frac{2 - c(z)}{2c(z)\,(1-4xz)}\right).
\end{aligned}$$

Now, similar to the proof of Theorem 10(1) we use Lemma 5(1) and simplify to get

$$\mathbf{A} \otimes M^* (x, z) = \frac{1}{2(1-4xz)}..$$

Expanding this generating function gives

$$\frac{1}{2(1-4xz)} = (1/2)\left(1 + 4xz + 16x^2z^2 + 64x^3z^3 + \cdots\right).$$

The coefficients form the column vector with entries $(1/2)(4x)^n, n \geq 0$.

The linear combination follows here similar to that given in the proof of Theorem 10. □

Corollary 12. *The powers of x can be expressed in terms of $T_n(x)$ and $T_n^*(x)$ as follows:*

1. $x^{2n} = (2/4^n)\left(T_0\binom{2n}{n} + T_2\binom{2n}{n-1} + \cdots + T_{2n}\right)$
2. $x^n = (2/4^n)\left(T_0^*\binom{2n}{n} + T_1^*\binom{2n}{n-1} + \cdots + T_n^*\right)$
3. $x^{2n+1} = (1/4^n)\left(T_1\binom{2n+1}{n} + T_3\binom{2n+1}{n-1} + \cdots + T_{2n+1}\right).$

Another related connection involving $\mathbf{A}$ is now given.

Proposition 13. *Let*

$$\mathbf{D} = \left((1-z)/(1+z), 2z/(1+z)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 2 & 0 & 0 & 0 & \dots \\ 2 & -8 & 4 & 0 & 0 & \dots \\ -2 & 18 & -24 & 8 & 0 & \dots \\ 2 & -32 & 80 & -64 & 16 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then,

$$\mathbf{A} \otimes (\mathbf{D} \otimes (1/(1-xz))) = 1/(1-2xz).$$

Proof. (Sketch) By Theorem 2, Equation (8), and simplifying we get

$$\begin{aligned} \mathbf{A} \otimes (\mathbf{D} \otimes (1/(1-xz))) &= \mathbf{A} \otimes \left(((1-z)/(1+z)) \cdot \frac{1}{1-\left(2xz/(1+z)^2\right)}\right) \\ &= \mathbf{A} \otimes \frac{1-z^2}{1+(1-x)2z+z^2} \\ &= \left(1/\sqrt{1-4z}, c(z)-1\right) \otimes \frac{1-z^2}{1+(1-x)2z+z^2}. \end{aligned}$$

Now, again by Theorem 2, applying Lemma 5 multiple times to the numerator and denominator of the product, and simplifying gives the result

$$\begin{aligned} \left(1/\sqrt{1-4z}, c(z)-1\right) \otimes \frac{1-z^2}{1+(1-x)2z+z^2} &= \frac{c(z)}{c(z)-x2zc(z)} \\ &= \frac{1}{1-2xz}. \end{aligned}$$

Expanding the generating function gives

$$\frac{1}{1-2xz} = 1 + 2xz + 4x^2z^2 + 8x^3z^3 + \cdots .$$

The coefficients form the column vector with entries $(2x)^n, n \geq 0$. $\square$

As a consequence of this result, the rows of $\mathbf{A}$ can be used to expand the monomial $(2x)^n$ in terms of certain polynomials $q_n(x)$ of degree n where the entries of the Riordan matrix $\mathbf{D}$ are the coefficients of $q_n(x)$. For instance, for the first four rows

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 20 & 15 & 6 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 \\ 2q_0 + q_1 \\ 6q_0 + 4q_1 + q_2 \\ 20q_0 + 15q_1 + 6q_2 + q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2x \\ 4x^2 \\ 8x^3 \end{pmatrix}$$

where $q_0 = q_0(x) = 1$, $q_1 = q_1(x) = 2x - 2$,

$$q_2 = q_2(x) = 4x^2 - 8x + 2, \text{ and } q_3 = q_3(x) = 8x^3 - 24x^2 + 18x - 2.$$

We write the resulting vector as

$$\left((2x)^0 \;\; (2x)^1 \;\; (2x)^2 \;\; (2x)^3\right)^T.$$

Thus, the column vector whose entries are $(2x)^n$ $(n \geq 0)$ can be expressed as a linear combination of the columns of $\mathbf{A}$ where the polynomials $q_n(x)$ are the coefficients of the linear combination. Moreover, the powers of x can be expressed in terms of the polynomials.

5 Integral Representations

A fundamental problem in numerical analysis is to approximate a continuous function $f(x)$ by an *nth* degree polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

over the interval $[a, b]$. A continuous function $f(x)$ can be expanded over $[-1, 1]$ in terms of the Chebyshev polynomials to obtain a Fourier-Chebyshev expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_nT_n(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \cdots$$

where the coefficients a_n are given by

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx, \text{ and } a_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_n(x)}{\sqrt{1-x^2}}dx, \; n \geq 1 \text{ [24]}.$$

From a practical point of view, there still remains the challenge of accurately evaluating these integrals by using numerical procedures. On the other hand, the integrals can be easily evaluated for the functions given by Corollary 12. Moreover, the coefficients a_n are the entries of the Catalan-type arrays $\mathbf{A}$ and $\mathbf{B}$ and they can be read from the columns of the arrays.

We now use properties of Chebyshev polynomials and derive certain definite integrals. Consider the orthogonality condition

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \neq 0 \\ \pi, & \text{if } m = n = 0 \end{cases}$$

for the Chebyshev polynomials of the first kind [24, 14]. Recall that

$$T_0(x) = T_0 = 1/2.$$

Then, from part (1) of Corollary 12, multiplying both sides of

$$x^{2n} = (2/4^n)\left(T_0\binom{2n}{n} + T_2\binom{2n}{n-1} + \cdots + T_{2n}\right)$$

by $T_0(x)$ and the weight function $1/\sqrt{1-x^2}$, integrating the expression over the interval $[-1, 1]$ with respect to x, applying the orthogonality condition, and simplifying gives integral representations of the binomial coefficients and Catalan numbers. The representations, respectively, are

$$\frac{4^n}{\pi} \int_{-1}^{1} \frac{x^{2n}}{\sqrt{1-x^2}} dx = (n+1)\, c_n \tag{15}$$

and

$$\frac{4^n}{\pi(n+1)} \int_{-1}^{1} \frac{x^{2n}}{\sqrt{1-x^2}} dx = c_n. \tag{16}$$

From part (3) of Corollary 12, multiplying both sides of

$$x^{2n+1} = (1/4^n)\left(T_1\binom{2n+1}{n} + T_3\binom{2n+1}{n-1} + \cdots + T_{2n+1}\right)$$

by $T_1(x)$ and the weight function $1/\sqrt{1-x^2}$, integrating the expression over the interval $[-1, 1]$ with respect to x, applying the orthogonality condition, and simplifying we get two more representations. The representations are

$$\frac{(n+1)\, 2^{2n+1}}{\pi(2n+1)} \int_{-1}^{1} \frac{x^{2n+2}}{\sqrt{1-x^2}} dx = \frac{(n+1)}{2n+1}\binom{2n+1}{n} = (n+1)\, c_n \tag{17}$$

and

$$\frac{2^{2n+1}}{\pi(2n+1)} \int_{-1}^{1} \frac{x^{2n+2}}{\sqrt{1-x^2}} dx = \frac{1}{2n+1}\binom{2n+1}{n} = c_n. \tag{18}$$

Now, consider the orthogonality condition

$$\int_{0}^{1} \frac{T_j^*(x)\, T_k^*(x)}{\sqrt{x-x^2}} dx = \begin{cases} 0, & \text{if } j \neq k \\ \frac{\pi}{2}, & \text{if } j = k \neq 0 \\ \pi, & \text{if } j = k = 0 \end{cases}$$

for the shifted Chebyshev polynomials of the first kind [12]. Recall that

$$T_0^*(x) = T_0^* = 1/2.$$

Then, from part (2) of Corollary 12, multiplying both sides of

$$x^n = (2/4^n)\left(T_0^*\binom{2n}{n} + T_1^*\binom{2n}{n-1} + \cdots + T_n^*\right) \tag{19}$$

by $T_0^*(x)$ and the weight function $1/\sqrt{x-x^2}$, integrating the expression over the interval $[0,1]$ with respect to x, applying the orthogonality condition, and then simplifying also gives integral representations of the central binomial coefficients and Catalan numbers. The representations, respectively, are

$$\frac{4^n}{\pi}\int_0^1 \frac{x^n}{\sqrt{x-x^2}}dx = (n+1)\,c_n, \tag{20}$$

and

$$\frac{4^n}{\pi(n+1)}\int_0^1 \frac{x^n}{\sqrt{x-x^2}}dx = c_n. \tag{21}$$

Similarly, multiplying both sides of Equation (19) by $T_1^*(x)$ gives the following representations for $n \geq 1$,

$$\frac{4^n(n+1)}{\pi n}\int_0^1 \frac{2x^{n+1}-x^n}{\sqrt{x-x^2}}dx = \frac{(n+1)}{n}\binom{2n}{n-1} = (n+1)\,c_n, \tag{22}$$

and

$$\frac{4^n}{\pi n}\int_0^1 \frac{2x^{n+1}-x^n}{\sqrt{x-x^2}}dx = \frac{1}{n}\binom{2n}{n-1} = c_n. \tag{23}$$

The Wolfram Alpha Widget [34] was used to confirm all of the integrals that were derived above analytically. The Wolfram definite integral calculator is a computational tool useful for experimental mathematics. It is interesting that the calculator computes the integrals and gives the results in terms of the gamma function $\Gamma(n)$, $n > 0$ (integer). The exact formulae for the central binomial coefficients and Catalan numbers can be easily derived from certain properties of $\Gamma(n)$ for the results given by the calculator. See Koshy [15] and Dana-Picard [9] for interesting relationships between the Catalan numbers and $\Gamma(n)$. For more information on experimental mathematics, see Bailey and Borwein [2, 3].

6 Concluding Comments

A few problems for possible future research projects are suggested. Finding other Riordan arrays that are linked to the Chebyshev polynomials of the first kind as a result of the coefficients of $T(x,z)$ and $T^*(x,z)$ occurring in the expansion of a continuous function $f(x)$ is of interest. This may lead to finding new techniques for improving the numerical approximation of $f(x)$ by Chebyshev polynomials. Finding more algebraic properties as a result of the connection may also help with obtaining better approximations. See Rivlin [24] for more information on Chebyshev polynomials and methods for approximating continuous functions. Finding the inverse of other Riordan arrays that are linked to monic orthogonal polynomials and moments are also of interest. By Proposition 13 the array $\mathbf{A}$ is linked to a certain class of polynomials $q_n(x)$. So, finding other connections among other classes of polynomials, special functions, and $\mathbf{A}$ and $\mathbf{B}$ are of interest. Finding combinatorial interpretations of $\mathbf{A}$ and $\mathbf{B}$ in terms of Chebyshev polynomials is of interest as well as finding connections between the polynomials and the lattice paths given by Nkwanta [19]. Finding Chebyshev polynomial interpretations of the matrix multiplication of the arrays by column vectors made up of the coefficients of the generating functions associated with the Chebyshev and shifted Chebyshev polynomials is of interest. See Shapiro [26] and Benjamin et al. [6], [7] for combinatorial interpretations of the Chebyshev polynomials. Finding connections of $\mathbf{A}$ and $\mathbf{B}$ to the Chebyshev polynomials of the second kind is of interest. See Barry [4], Barry and Hennessy [5], and Luzon and Moron [17] for Riordan array connections to the Chebyshev polynomials of the second kind. Finding other Chebyshev, Riordan array connections may lead to integral representations of other important and interesting counting sequences.

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