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A Note on the Generating Function for the Stirling Numbers of the First Kind

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Abstract

In this short note, we present a simple constructive proof for the generating function for the unsigned Stirling numbers of the first kind using the equidistribution of pilots and cycles of permutations.

1 Introduction

There are many studies on different statistics of permutations in the literature, e.g., inversion number, excedance and descent [4]. In this note, we study another simple statistic of permutations which we call pilots (while they could be called right-to-left minima as well). For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ on $[n] = \{1, 2, \ldots n\}$, π_i is called a *pilot* of π if $\pi_i < \pi_j$ for all $j > i$. Note that π_n is always a pilot of π. We relate pilots to a representation of a permutation as a product of its disjoint cycles, that allows us to give a simple constructive proof for the generating function for the unsigned Stirling numbers of the first kind.

The unsigned Stirling number of the first kind $c(n, k)$ (see A132393 [3]) is the number of permutations on $[n]$ consisting of k disjoint cycles [2, 4]. Our main result is to prove the following theorem:

Theorem 1. *For $1 \leq k \leq n$, we have*

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1). \tag{1}$$

2 Proof of Theorem 1

There are four proofs of Eq. (1) in Stanley [4] and one in Callan [1]. In Stanley [4], when a permutation π is written as product of its disjoint cycles, a standard representation is defined as follows: each cycle is written with its largest element first, and all the cycles are written in increasing order of their largest element. By this standard representation, we can obtain a bijection between permutations with k cycles and permutations with k left-to-right maxima. However, to make use of pilots, we define a different representation as follows: we write $\pi = C_1C_2\cdots C_k$ so that $\min\{C_i\} < \min\{C_j\}$ for all $j > i$ and each cycle C_i ends with $\min\{C_i\}$ for all i. We call this new representation as the standard representation of type P. For example, $\pi = 76154832$ has three cycles: (173), (268) and (45). Then, in the standard representation of type P, we write $\pi = (731)(682)(54)$.

For a permutation π with k cycles written in the standard representation of type P, if we erase the parentheses of the cycles, we obtain a permutation as a word π'. For example, from $\pi = (731)(682)(54)$ we obtain $\pi' = 73168254$. Reversely, each pilot of π' induces a cycle of π, e.g., $1 \to (731)$, $2 \to (682)$, $4 \to (54)$. It is easy to observe that such a correspondence between permutations with k cycles and permutations with k pilots is a bijection, that is, we have

Lemma 2. *The number of permutations with k pilots equals to the number of permutations with k cycles.*

Let $pil(\pi)$ denote the number of pilots of π. Our idea to prove Eq. (1) is to show that

$$\sum_{\pi} x^{pil(\pi)} = x(x+1)(x+2)\cdots(x+n-1),$$

where the sum is over all permutations π on $[n]$.

Proof of Theorem 1. Note that π_1 is a pilot of $\pi = \pi_1\pi_2\cdots\pi_n$ if and only if $\pi_1 = 1$; the other $n-1$ cases will not make π_1 a pilot. The element π_2 is a pilot of π if and only if $\pi_2 = \min\{[n] \setminus \{\pi_1\}\}$; the remaining $n-2$ cases, i.e., $\pi_2 \in [n] \setminus \{\pi_1, \min\{[n] \setminus \{\pi_1\}\}\}$, will not make π_2 a pilot; and so on and so forth. In summary, to construct a permutation π starting from an empty word, suppose π_j has been determined for $1 \leq j \leq i-1$, then π_i has only one chance to be a pilot of π, i.e., $\pi_i = \min\{[n] \setminus \{\pi_1, \pi_2, \ldots \pi_{i-1}\}\}$, and the other $n-i$ cases not. Hence,

$$\begin{aligned}\sum_{\pi} x^{pil(\pi)} = &(x+n-1) && \text{by } \pi_1\\ &\times (x+n-2) && \text{by } \pi_2\\ &\quad\vdots \\ &\times (x+1) && \text{by } \pi_{n-1}\\ &\times x && \text{by } \pi_n.\end{aligned}$$

Therefore, Eq. (1) holds from Lemma 2. □

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