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Combinatorial Enumeration of Partitions of a Convex Polygon

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Abstract

We establish a class of polynomials on convex polygons, which provides a new counting formula to all partitions of a convex polygon by non-intersecting diagonals.

1 Introduction

Counting partitions of a convex polygon of a specified type by using its non-intersecting diagonals is a problem which can go back to Euler, Catalan, Cayley [1] and Przytycki and Sikora [2]. Recently, Floater and Lyche [3] showed a way to enumerate all partitions of a convex polygon of a certain type as follows.

Proposition 1 (Floater, Lyche [3]). *A partition of a convex $(n+1)$-gon is said to be of type* $\mathbf{b} = (b_2, b_3, \ldots, b_n)$ *if it contains b_2 triangles, b_3 quadrilaterals, and so on, and in general b_i $(i+1)$-gons. Then the number of partitions of a convex $(n+1)$-gon of type* $\mathbf{b} = (b_2, b_3, \ldots, b_n)$ *with $b_2 + b_3 + \cdots + b_n = k$ and $2b_2 + 3b_3 + \cdots + nb_n = n + k - 1$, is*

$$C(\mathbf{b}) = \frac{(n+k-1)(n+k-2)\cdots(n+1)}{b_2!b_3!\cdots b_n!}.$$

Inspired by Lee's result [4], Shephard [5] got an interesting equality on convex polygons with $n+2$ sides as follows.

Proposition 2 (Lee [4], Shephard [5]). *Given a $(n+2)$-gon, let d_1 be the number of diagonals, d_2 be the number of disjoint pairs of diagonals, and, in general, d_i be the number of sets of i diagonals of the polygon which are pairwise disjoint. Then we have*

$$d_1 - d_2 + d_3 - \cdots + (-1)^n d_{n-1} = 1 + (-1)^n$$

The original proof [4] of Proposition 2 is very complicated. We will provide a rather simple proof in the last part.

We organize this paper as follows. Section 2 shows the main result (Theorem 5) via the properties of a derivation acting on a special polynomial algebra. In Section 3, we prove Propositions 1 and 2 by our main result.

2 Main results

We call a vector space $\mathcal{A} := (\mathcal{A}, +)$ an algebra over the real field $\mathbb{R}$, if $\mathcal{A}$ possesses a bilinear product satisfying $(ab)c = a(bc)$, $(a+b)(c+d) = ac + bc + ad + bd$ and $(\lambda\mu)(ab) = (\lambda a)(\mu b)$, for all $\lambda, \mu \in \mathbb{R}$, $a, b, c, d \in \mathcal{A}$. Recall that a linear map D mapping $\mathcal{A}$ into itself is called a derivation if $D(xy) = (Dx)y + x(Dy)$ for all $x, y \in \mathcal{A}$.

Definition 3. Let $\mathcal{A}$ be a polynomial algebra generated by $\{x_i : i \in \mathbb{N}^+\}$, i.e., the collection of the polynomials with the form $\sum_{k=1}^{m} \sum_{i_1,\ldots,i_k \in \mathbb{N}^+} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k}$, where $a_{i_1,\ldots,i_k} \in \mathbb{R}$ and $m \in \mathbb{N}^+$. For given $y_i \in \mathcal{A}$, $i \in \mathbb{N}^+$, let $D' : \{x_i : i \in \mathbb{N}^+\} \to \mathcal{A}$ be such that $x_i \mapsto y_i$, $i \in \mathbb{N}^+$. The unique extension of D' to $\mathcal{A}$ via Leibniz's law determines a derivation D on $\mathcal{A}$, which is called the derivation defined by D'.

Lemma 4. *Let $\mathcal{A}$ be a polynomial algebra generated by $\{X_1, X_2, \ldots, X_n, \ldots\}$. Assume D is a derivation with action defined by*

$$DX_n = (an+b)\sum_{i=1}^{n-1} X_i X_{n-i},\ n \geq 2,\ DX_1 = 0, \tag{1}$$

where a and b are given real numbers. Then we have

$$D^m X_n = \frac{\prod_{k=1}^{m}(2an + (k+1)b)}{m+1} \sum_{i_1+i_2+\cdots+i_{m+1}=n} X_{i_1} X_{i_2} \cdots X_{i_{m+1}}. \tag{2}$$

Proof. Let $X(t) = \sum_{i\geq 1} X_i t^i$ be a generating function. It follows from

$$X(t)^2 = \sum_{n\geq 2} \left(\sum_{i=1}^{n-1} X_i X_{n-i} \right) t^n$$

that

$$\begin{aligned} \left(at\frac{d}{dt} + b \right) X(t)^2 &= \sum_{n\geq 2} \left(\sum_{i=1}^{n-1} X_i X_{n-i} \right) \left(at\frac{d}{dt} + b \right) t^n = \sum_{n\geq 2} \left(\sum_{i=1}^{n-1} X_i X_{n-i} \right) (an+b)t^n \\ &= \sum_{n\geq 1} (DX_n) t^n = DX(t). \end{aligned}$$

Similarly, the statement (2) becomes

$$D^m X = \frac{1}{m+1} \prod_{i=1}^{m} \left(2at\frac{d}{dt} + (i+1)b \right) X^{m+1}, \tag{3}$$

where $X := X(t)$. It is evident that (3) holds for $m = 1$. Assume that (3) holds for $m = k$. Now we show that (3) holds for $m = k+1$. In fact, together with (3) for $m = k$ and the fact

$$\begin{aligned} DX^m &= mX^{m-1}DX = mX^{m-1} \left(at\frac{d}{dt} + b \right) X^2 \\ &= 2amX^m t\frac{d}{dt}X + bmX^{m+1} \\ &= \frac{m}{m+1} \left(2at\frac{d}{dt} + (m+1)b \right) X^{m+1}, \end{aligned}$$

we immediately obtain

$$\begin{aligned} D^{k+1}X = D(D^k X) &= D \left(\frac{1}{k+1} \prod_{i=1}^{k} \left(2a\frac{d}{dt} + (i+1)b \right) X^{k+1} \right) \\ &= \frac{1}{k+1} \prod_{i=1}^{k} \left(2at\frac{d}{dt} + (i+1)b \right) DX^{k+1} \\ &= \frac{1}{k+1} \prod_{i=1}^{k} \left(2at\frac{d}{dt} + (i+1)b \right) \frac{k+1}{k+2} \left(2at\frac{d}{dt} + (k+2)b \right) X^{k+2} \\ &= \frac{1}{k+2} \prod_{i=1}^{k+1} \left(2at\frac{d}{dt} + (i+1)b \right) X^{k+2}. \end{aligned}$$

Therefore, by mathematical induction, we have completed the proof. $\square$

We call a strictly convex polygon with $n+2$ sides a $(n+2)$-gon, denoted by X_n, where $n \in \mathbb{N}^+$. Given an integer $n \geq 2$, we use Δ to denote a set of diagonals of X_n which are pairwise disjoint. It should be noted that a Δ with m elements divides X_n into $m+1$ convex polygons, $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$ and $X_{i_{m+1}}$ for some $i_1, i_2, \ldots, i_m$ and i_{m+1} in $\{1, 2, \ldots, n\}$. The set of such convex polygons is said to be a partition of the original convex polygon. We symbolically set $f(\Delta) = \prod_{j=1}^{m+1} X_{i_j}$ and Card $\Delta = m$. Figure 1 provides two examples of X_n for $n = 8$ and $n = 10$, respectively.

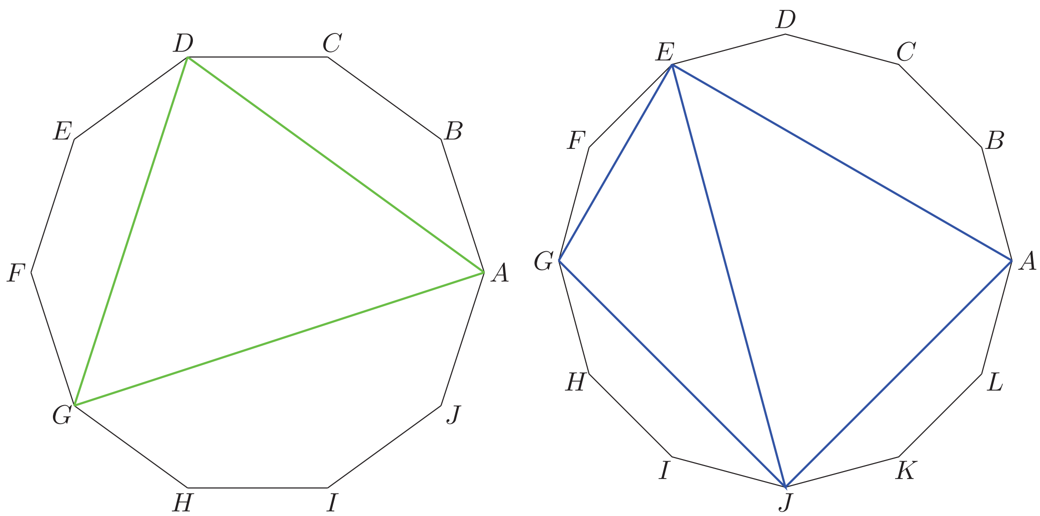

Figure 1: The left figure shows X_8 with $\Delta = \{AD, AG, DG\}$ and the corresponding partition $\{ABCD, DEFG, AGHIJ, ADG\}$, where Card $\Delta = 3$, $f(\Delta) = X_1X_2X_2X_3$. The right figure shows X_{10} with $\Delta = \{AE, AJ, EJ, EG, GJ\}$ and the corresponding partition $\{ABCDE, EFG, GHIJ, AJKL, EGJ, AEJ\}$, where Card $\Delta = 5$, $f(\Delta) = X_1X_1X_1X_2X_2X_3$.

Theorem 5. *Given $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$, we have*

$$\sum_{\text{Card } \Delta = m} f(\Delta) = \frac{1}{m+1} \binom{n+m+1}{m} \sum_{i_1+i_2+\cdots+i_{m+1}=n} X_{i_1} X_{i_2} \cdots X_{i_{m+1}}. \tag{4}$$

Proof. Consider partitions of X_n with m diagonals, in which the diagonals are labelled, say with integers $1, 2, \cdots, m$. Then the derivation D is an operator that acts as an analogue combinatorial device for splitting the polygon on a labelled diagonal; consequently, D^m is an operator that splits the polygon (with m diagonals) into $m+1$ polygons. We then divide by $m!$ to remove the effect of labelling the diagonals, so that $\frac{D^m}{m!}$ is the operator that produces the counting series for partitioning a polygon into $m+1$ parts.

Next, we calculate DX_n. Notice that there are $n-1$ diagonals starting from a vertex, and each diagonal divides X_n into two parts. So we have $n-1$ ways to divide X_n, which can be expressed as $X_1X_{n-1}+X_2X_{n-2}+\cdots+X_{n-1}X_1$ by using our notation. Since X_n has $n+2$ vertices, the whole set of partitions of X_n can be written as $(n+2)\sum_{i=1}^{n-1} X_iX_{n-i}$. However, each diagonal has two ends, and will be counted twice. Consequently, we should divide it by 2, and get $DX_n = \frac{n+2}{2}\sum_{i=1}^{n-1} X_iX_{n-i}$. Taking $a=\frac{1}{2}$ and $b=1$ in Lemma 4, we have

$$\sum_{\text{Card } \Delta=m} f(\Delta) = \frac{1}{m!}D^m X_n = \frac{1}{m+1}\binom{n+m+1}{m} \sum_{i_1+i_2+\cdots+i_{m+1}=n} X_{i_1}X_{i_2}\cdots X_{i_{m+1}}.$$

□

3 Applications

A result about partitioning polygons is as follows.

Corollary 6. *Given $i_1, i_2, \ldots, i_{m+1}$ with $i_1+i_2+\cdots+i_{m+1}=n$. Then the number of different ways of cutting X_n into sub-polygons $X_{i_1}, X_{i_2}, \ldots, X_{i_{m+1}}$ by diagonals is $\frac{S}{m+1}\binom{n+m+1}{m}$, where S is the number of permutations of $i_1, i_2, \ldots, i_{m+1}$.*

Proof. By Theorem 5, we obtain that there exist $\frac{S}{m+1}\binom{n+m+1}{m}$ ways to divide X_n into $X_{i_1}, X_{i_2}, \ldots, X_{i_{m+1}}$, where S is the number of permutations of $i_1, i_2, \ldots, i_{m+1}$. □

One can easily verify that Proposition 1 is equivalent to Corollary 6.

Example 7 (Catalan numbers)**.** Let $m=n-1$. Then $i_1=i_2=\cdots=i_{m+1}=1$ is the only positive integer solution of $i_1+i_2+\cdots+i_{m+1}=n$. Hence $S=1$, and we get the Catalan numbers $\frac{1}{n}\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}$.

Next we give a new proof for Proposition 2 by using Theorem 5 and the residue theorem.

Proof of Proposition 2. Consider X_n. Notice that the number of positive integer solutions of $i_1+i_2+\cdots+i_{m+1}=n$ is $\binom{n-1}{m}$. By Theorem 5, there are $\sum_{i_1+i_2+\cdots+i_{m+1}=n} \frac{1}{m+1}\binom{n+m+1}{m}$

monomials on the right-hand side of (4). Thus we get $d_m = \frac{1}{m+1}\binom{n-1}{m}\binom{n+m+1}{m}$, and then

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k d_k &= \sum_{k=1}^{n-1}(-1)^{k-1}\frac{\binom{n-1}{k}}{k+1}\binom{n+k+1}{k} \\
&= \sum_{k=1}^{n-1}(-1)^{k-1}\frac{\binom{n}{k+1}}{n}\mathrm{Res}\left(\frac{(1+u)^{n+k+1}}{u^{k+1}},0\right) \\
&= \frac{1}{n}\mathrm{Res}\left((1+u)^n\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k+1}\frac{(1+u)^{k+1}}{u^{k+1}},0\right) \\
&= \frac{1}{n}\mathrm{Res}\left((1+u)^n\left((1-\frac{u+1}{u})^n-(1-n\frac{u+1}{u})\right),0\right) \\
&= \frac{1}{n}\mathrm{Res}\left((1+u)^n\left((-\frac{1}{u})^n-1+n\frac{u+1}{u}\right),0\right) \\
&= \frac{1}{n}\left((-1)^n\mathrm{Res}\left(\frac{(1+u)^n}{u^n},0\right)+n\mathrm{Res}\left(\frac{(1+u)^{n+1}}{u},0\right)\right) \\
&= \frac{1}{n}\left((-1)^n\binom{n}{n-1}+n\cdot 1\right) \\
&= 1+(-1)^n,
\end{aligned}$$

where $\mathrm{Res}(f(u),0)$ means the residue of function $f(u)$ at $u=0$. □

Remark 8. Proposition 2 also follows immediately from the hypergeometric summation formula, by

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k d_k &= \sum_{k=1}^{n-1}(-1)^{k-1}\frac{\binom{n-1}{k}}{k+1}\binom{n+k+1}{k} = \frac{1}{n}\left(n-\sum_{i=1}^{n-1}\binom{n}{n-k-1}\binom{-n-2}{k}\right) \\
&= \frac{1}{n}\left(n-\binom{-2}{n-1}\right) = \frac{1}{n}(n+(-1)^n n) = 1+(-1)^n.
\end{aligned}$$

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