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Bivariate Extension of Bell Polynomials

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Abstract

In this paper, we study bivariate extensions of Bell polynomials and r-Bell polynomials. Some identities related to the r-Stirling numbers and Bell polynomials are presented as special cases.

1 Introduction and motivation

The Stirling number of the second kind is the number of ways to partition a set of n objects into k non-empty subsets (cf. Comtet [2, §5.1]) and denoted by $\left\{ {n \atop k} \right\}$ (see Graham et al. [4, §6.1]). The nth Bell number is the sum of the Stirling numbers of the second kind and denoted by

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\},\ n = 0, 1, \ldots.$$

The corresponding Bell polynomials are

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k,\ n = 0, 1, \ldots. \tag{1}$$

Recently, Spivey [6, Eq. 3] discovered the following remarkable formula

$$B_{m+n} = \sum_{k=0}^{m} \sum_{j=0}^{n} j^{n-k} \binom{n}{k} \left\{ {m \atop j} \right\} B_k \tag{2}$$

and gave a short combinatorial proof. Later, Gould and Quaintance [3] extended this Bell number to the following Bell polynomial

$$B_{m+n}(x) = \sum_{k=0}^{m} \sum_{j=0}^{n} j^{n-k} \binom{n}{k} \left\{ {m \atop j} \right\} B_k(x) x^j. \tag{3}$$

For any positive integer r, Carlitz [1] introduced the r-Stirling number $\{ {n \atop k} \}_r$, which is the number of ways to partition a set of $n+r$ objects into $k+r$ non-empty subsets such that the first r elements are in distinct subsets. He also found that the r-Stirling numbers and the Stirling numbers have the following relationship

$$\left\{ {n \atop m} \right\}_r = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} r^{n-k}. \tag{4}$$

Analogous to the definition of the r-Stirling number, Mező [5] extended the Bell number B_n to the r-Bell number

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r, \;\; n = 0, 1, \ldots$$

and derived the following identity

$$B_{m+n,r} = \sum_{k=0}^{m} \sum_{j=0}^{n} (j+r)^{n-k} \binom{n}{k} \left\{ {m \atop j} \right\}_r B_k.$$

The aim of this paper is to generalize further these identities to the bivariate case. In the next section, we give the bivariate extension of Bell polynomials. The bivariate extension of r-Bell polynomials and some examples will be shown in the third section.

2 Bivariate extension of Bell polynomials

Firstly, for any positive integer n, we define the bivariate Bell polynomials by

$$B_n(x, y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k y^k, \;\; n = 0, 1, \ldots,$$

where the falling factorial is defined by $(x)_0 = 1$ and

$$(x)_k = x(x-1)\cdots(x-k+1) \quad \text{for} \quad k = 1, 2, \ldots.$$

Letting $y \to \frac{y}{x}$ and then $x \to \infty$, we can see that the bivariate Bell polynomial $B_n(x, y)$ reduces to the univariate Bell polynomial in Eq. (1).

The exponential generating function of $B_n(x, y)$ is given by the following theorem.

Theorem 1.

$$\sum_{n=0}^{\infty} B_n(x,y)\frac{t^n}{n!} = \{1+y(e^t-1)\}^x.$$

Proof. Recalling the generating function [2, §5.2] (also see [4, §6.3])

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}, \tag{5}$$

we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(x,y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k y^k \frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty} (x)_k y^k \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty} (x)_k y^k \frac{(e^t-1)^k}{k!} \\
&= \sum_{k=0}^{\infty} \binom{x}{k} \left\{ y(e^t-1) \right\}^k.
\end{aligned}$$

By means of the binomial formula [2, §1.6]

$$(x+y)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k, \tag{6}$$

we have the desired result. □

If the exponential generating function $f(x)$ of the sequence $\{A_n\}$ is

$$\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = f(x), \tag{7}$$

then it is routine to get the generating function of the double sequence $\{A_{m+n}\}$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{i+j}\frac{x^i}{i!}\frac{y^j}{j!} = f(x+y). \tag{8}$$

From this, we can prove the following theorem.

Theorem 2. *Let $m, n \in \mathbb{N}$. We have*

$$B_{m+n}(x,y) = \sum_{k=0}^{m}\sum_{i=0}^{n} k^{n-i} \binom{n}{i} \left\{ {m \atop k} \right\} B_i(x-k,y)(x)_k y^k.$$

Proof. In view of Eqs. (7) and (8), we directly obtain the following bivariate exponential generating function of Bell polynomials $\{B_{m+n}(x,y)\}$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{m+n}(x,y)\frac{u^m}{m!}\frac{v^n}{n!} = \{1+y(e^{u+v}-1)\}^x. \tag{9}$$

The right-hand side of Eq. (9) can be reformulated to

$$\begin{aligned}\{1+y(e^{u+v}-1)\}^x &= \{1+y(e^v-1)+ye^v(e^u-1)\}^x\\ &= \sum_{k=0}^{\infty}\binom{x}{k}\{1+y(e^v-1)\}^{x-k}\{ye^v(e^u-1)\}^k,\end{aligned}$$

where we have employed the binomial theorem in Eq. (6). According to $\binom{x}{k} = \frac{(x)_k}{k!}$ and Theorem 1, the last expression can be transformed into

$$\begin{aligned}\{1+y(e^{u+v}-1)\}^x &= \sum_{k=0}^{\infty}\frac{(x)_k}{k!}\sum_{i=0}^{\infty}B_i(x-k,y)\frac{v^i}{i!}y^k e^{kv}(e^u-1)^k\\ &= \sum_{k=0}^{\infty}(x)_k y^k\frac{(e^u-1)^k}{k!}\sum_{i=0}^{\infty}B_i(x-k,y)\frac{v^i}{i!}\sum_{j=0}^{\infty}\frac{k^j v^j}{j!}.\end{aligned}$$

By means of Eq. (5), we have

$$\{1+y(e^{u+v}-1)\}^x = \sum_{k=0}^{\infty}(x)_k y^k\sum_{m=k}^{\infty}\left\{ {m \atop k} \right\}\frac{u^m}{m!}\sum_{i=0}^{\infty}B_i(x-k,y)\frac{v^i}{i!}\sum_{j=0}^{\infty}\frac{k^j v^j}{j!}.$$

Letting $i+j=n$ and then changing the summation order with respect to k and m, the last expression can be reformulated to

$$\{1+y(e^{u+v}-1)\}^x = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}\sum_{k=0}^{m}\sum_{i=0}^{n}k^{n-i}(x)_k y^k\binom{n}{i}\left\{ {m \atop k} \right\}B_i(x-k,y).$$

Now extracting the coefficient of $\frac{u^m}{m!}\frac{v^n}{n!}$ across Eq. (9), we get the desired result. □

Replacing y by $\frac{y}{x}$ and then letting $x \to \infty$ in Theorem 2, we recover Eq. (3) for the Bell polynomials of single variable.

3 Bivariate extension of r-Bell polynomials

In this section, we firstly define the following bivariate r-Bell polynomials

$$B_{n,r}(x,y) = \sum_{k=0}^{n}\left\{ {n \atop k} \right\}_r (x)_k y^k, \ n = 0,1,\ldots.$$

Theorem 3. *Let r be any positive integer. We have*

$$\sum_{n=0}^{\infty} B_{n,r}(x,y)\frac{t^n}{n!} = e^{rt}\{1+y(e^t-1)\}^x.$$

Proof. The univariate generating function of $B_{n,r}(x,y)$ can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,r}(x,y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r (x)_k y^k \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}(x)_k y^k \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_r \frac{t^n}{n!}.\end{aligned}$$

In view of Eq. (5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,r}(x,y)\frac{t^n}{n!} &= \sum_{k=0}^{\infty}(x)_k y^k \frac{e^{rt}(e^t-1)^k}{k!} \\ &= e^{rt}\sum_{k=0}^{\infty}\binom{x}{k}\{y(e^t-1)\}^k.\end{aligned}$$

Applying the binomial theorem (Eq. (6)) in the right-hand side of the above equation yields the desired result. □

Remark 4. The anonymous referee observed that the last identity is equivalent to

$$B_{n,r}(x,y) = \sum_{j=0}^{n}\binom{n}{j} B_{n-j}(x,y),$$

which is meaningful for all r, not just the nonnegative integers. It would be interesting to find combinatorial interpretations of $B_{n,r}(x,y)$ (or $B_{n,r}(x,1)$) when r is a negative integer.

Theorem 5. *For $\{r,m,n\} \in \mathbb{N}$, we have*

$$B_{m+n,r}(x,y) = \sum_{k=0}^{m}\sum_{i=0}^{n} k^{n-i}\binom{n}{i}\left\{ {m \atop k} \right\}_r B_{i,r}(x-k,y)(x)_k y^k.$$

Proof. According to Eqs. (7) and (8), we can write the following bivariate generating function directly:

$$\sum_{m,n\geq 0} B_{m+n,r}(x,y)\frac{u^m}{m!}\frac{v^n}{n!} = e^{r(u+v)}\{1+y(e^{(u+v)}-1)\}^x. \tag{10}$$

Reformulating the right-hand side of Eq. (10), we have

$$\begin{aligned}
e^{r(u+v)}\{1+y(e^{u+v}-1)\}^{x} &= e^{ru}e^{rv}\{1+y(e^{v}-1)+ye^{v}(e^{u}-1)\}^{x}\\
&= e^{ru}e^{rv}\sum_{k=0}^{\infty}\binom{x}{k}\{1+y(e^{v}-1)\}^{x-k}\{ye^{v}(e^{u}-1)\}^{k}\\
&= \sum_{k=0}^{\infty}e^{rv}\{1+y(e^{v}-1)\}^{x-k}e^{kv}e^{ru}\frac{(e^{u}-1)^{k}}{k!}(x)_{k}y^{k}.
\end{aligned}$$

With the help of Theorem 3 and Eq. (5), we get from the last expression

$$\begin{aligned}
e^{r(u+v)}\{1+y(e^{(u+v)}-1)\}^{x} = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{u^{j}}{j!}\frac{u^{s}}{s!}\frac{v^{l}}{l!}\frac{v^{i}}{i!}\\
\times\sum_{k=0}^{j}k^{l}r^{s}B_{i,r}(x-k,y)\left\{ {j \atop k} \right\}(x)_{k}y^{k}.
\end{aligned}$$

Letting $i+l=n$ and $s+j=m$, then extracting the coefficients of $\frac{u^m}{m!}\frac{v^n}{n!}$ across Eq. (10), we arrive, after having simplified the result by means of Eq. (4), at the desired identity. $\square$

Using the same method as that for Theorem 5, we get another expression for $B_{m+n,r}(x,y)$.

Theorem 6. *For $\{r,m,n\}\in\mathbb{N}$, we have*

$$B_{m+n,r}(x,y)=\sum_{i=0}^{n}\sum_{k=0}^{m}(k+r)^{n-i}\binom{n}{i}\left\{ {m \atop k} \right\}_{r}B_{i}(x-k,y)(x)_{k}y^{k}.$$

When $r=0$, the last two theorems reduce to Theorem 2.

Proposition 7.

$$B_{m+n,r}(y)=\sum_{i=0}^{n}\sum_{k=0}^{m}k^{n-i}\binom{n}{i}\left\{ {m \atop k} \right\}_{r}B_{i,r}(y)y^{k}.$$

Proof. Let $y\to\frac{y}{x}$, $x\to\infty$ in Theorem 5. $\square$

Proposition 8.

$$B_{m+n,r}(y)=\sum_{i=0}^{n}\sum_{k=0}^{m}(k+r)^{n-i}\binom{n}{i}\left\{ {m \atop k} \right\}_{r}B_{i}(y)y^{k}.$$

Proof. Let $y\to\frac{y}{x}$, $x\to\infty$ in Theorem 6. $\square$

Corollary 9.

$$B_{m+n,r} = \sum_{i=0}^{n} \sum_{k=0}^{m} k^{n-i} \binom{n}{i} \left\{ {m \atop k} \right\}_r B_{i,r}.$$

Proof. Let $y \to \frac{y}{x}$, $x \to \infty$ and then set $y = 1$ in Theorem 5. $\square$

Corollary 10. *[Mező [5, Theorem 2]]*

$$B_{m+n,r} = \sum_{i=0}^{n} \sum_{k=0}^{m} (k+r)^{n-i} \binom{n}{i} \left\{ {m \atop k} \right\}_r B_i.$$

Proof. Let $y \to \frac{y}{x}$, $x \to \infty$ and then set $y = 1$ in Theorem 6. $\square$

It should be pointed out that Corollary 9 and Corollary 10 can be obtained by setting $y = 1$ in Propositions 7 and 8, respectively. When $r = 0$, the last corollary reduces to Eq. (2).

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