Journal of Integer Sequences, Vol. 22 (2019),
Article 19.8.4

Square Matrices Generated by Sequences of Riordan Arrays

Ângela Mestre and José Agapito
Centro de Análise Funcional, Estruturas Lineares e Aplicações
Grupo de Estruturas Algébricas, Lineares e Combinatórias
Departamento de Matemática
Faculdade de Ciências, Universidade de Lisboa
1749-016 Lisboa
Portugal
amestre@fc.ul.pt
jaruiz@fc.ul.pt

Abstract

We consider sequences of images of Riordan arrays under some Riordan group automorphisms introduced by Bacher. We enclose their properties into several infinite square arrays which turn out to be of combinatorial interest. To illustrate our approach we consider Cameron and Nkwanta's sequence of generalized RNA arrays (whose first term is the well-known Nkwanta RNA array). Although this sequence of generalized RNA arrays was originally established as a sequence of pseudo-involutions, we show that it does not contain pseudo-involutions other than Nkwanta's array. We also show that these arrays are actually images of a new array under some of Bacher's automorphisms. We study the combinatorics of some square matrices related to the generalized RNA arrays and to sequences of genuine pseudo-involutions generated by Nkwanta's array.

1 Introduction

Let $d(t) = 1 + \sum_{k=1}^{\infty} d_k t^k$ and $h(t) = \sum_{k=1}^{\infty} h_k t^k$ be two generating functions over the field of complex numbers $\mathbb{C}$ with $h_1 \neq 0$. A *Riordan array* is an infinite lower triangular matrix

$D = (D_{k,n})_{k\geq n\geq 1}$, usually represented as a pair $D = (d(t), h(t))$, whose entries are given by

$$D_{k,n} = [t^{k-1}]d(t)h(t)^{n-1}, \tag{1}$$

where $[t^k]$ is the usual coefficient extraction operator. The set of Riordan arrays forms a group formally introduced by Shapiro, Getu, Woan, and Woodson [20], here denoted by $\mathfrak{Rio}$. The group multiplication is given by

$$(d_1(t), h_1(t))(d_2(t), h_2(t)) = (d_1(t)d_2(h_1(t)), h_2(h_1(t))), \tag{2}$$

while the group identity is $\mathrm{id}_{\mathfrak{Rio}} = (1, t)$. The inverse of any element $D = (d(t), h(t))$ is $D^{-1} = (1/d(\bar{h}(t)), \bar{h}(t))$, where $\bar{h}(t)$ denotes the compositional inverse of the generating function $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$. There exists an extensive literature on the Riordan group and related concepts as, for instance, [17, 19, 2, 7, 3, 9, 4, 5, 10, 12]. Throughout the years, Riordan arrays have been revealed to be very powerful tools in combinatorics, see, e.g., [22, 15, 16, 11, 13, 8].

This paper introduces several square arrays associated with sequences of Riordan arrays. The main intention is to encompass some collective properties of the arrays in the sequence into a properly constructed square array. Let $(D_n)_{n\geq 1}$ be a sequence of Riordan arrays. Fix integers $i, j \geq 1$. From each of the arrays in the sequence take the ith column (resp., the jth diagonal or the row-sum vector). Build a square matrix such that the nth column corresponds to the ith column (resp., the jth diagonal or the row-sum vector) of the matrix D_n. Here, the Riordan array sequences are generated by the images of a given Riordan element under a sequence of Riordan group automorphisms. We shall refer to the Riordan group element that generates the matrix sequence, as the *base array* of the sequence.

Given $\alpha, \gamma \in \mathbb{C}$ and $\beta \in \mathbb{C}^*$, with $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, Bacher [1] introduced the automorphisms $\varphi_{\alpha,\beta,\gamma} : \mathfrak{Rio} \to \mathfrak{Rio}$ as follows

$$\begin{aligned} \varphi_{\alpha,\beta,\gamma} : \mathfrak{Rio} &\to \mathfrak{Rio} \\ (d(t), h(t)) &\mapsto \left(\left(\frac{h(t)}{t}\right)^{\alpha} d(t)^{\beta} h'(t)^{\gamma}, h(t)\right), \end{aligned} \tag{3}$$

where $h'(t)$ denotes the derivative of $h(t)$. In particular, the automorphism $\varphi_{1,1,0}$ is usually known as the *diagonal translation operator* [1, 14], for $\varphi_{1,1,0}(D)$ is the matrix obtained from D by removing the first column and the first row (recall formula (1)).

This paper is organized as follows. Given a Riordan array D and fixed complex numbers $\beta \in \mathbb{C}^*$ and $\gamma \in \mathbb{C}$, Section 2.1 defines infinite square matrices whose columns are obtained by extracting a column, a diagonal slice, or the row sum sequence of each array in the sequences $(\varphi_{1,n,\gamma}(D))_{n\geq 1}$ and $(\varphi_{1,\beta,l}(D))_{l\geq 1}$. We give examples which show that these new matrices are compelling combinatorial devices enclosing the joint properties of the arrays in the sequences.

Section 3 expresses the A, Z-sequences [15, 9] and the $B(x)$-sequence [23] of the image of a Riordan array D under an automorphism $\varphi_{\alpha,\beta,0}$ in terms of the corresponding sequences of the base array. We define again infinite square matrices associated with the Z and $B(x)$-sequences of the arrays in the sequence $(\varphi_{1,n,\gamma}(D))_{n\geq 1}$. We illustrate our setting with the classical Pascal array.

Section 4 deals with pseudo-involutions of the Riordan group under the action of Bacher's automorphisms. We prove that they map pseudo-involutions to pseudo-involutions. We also show that the generalized RNA arrays of Cameron and Nkwanta [2] form an array sequence $(\varphi_{1,n,0}(\mathcal{D}))_{n\geq 1}$, where $\mathcal{D}$ is a new Riordan array. We show that contrary to what has been believed over the years, the arrays in this sequence are not pseudo-involutions. We define instead sequences of authentic pseudo-involutions by applying Bacher's automorphisms to Nkwanta's RNA triangle. We determine some of the infinite matrices of Section 2.1 for both the generalized RNA arrays and the new sequences of pseudo-involutions having Nkwanta's array as base.

Along the paper, all the closed forms for sequences that we give and the correspondence to sequences in the On-line Encyclopedia of Integer Sequences [21] were verified using Mathematica up to order $n = 100$. Note that often our sequences match the assigned sequence numbers only from a given element on.

2 Infinite matrices via Bacher's automorphisms with nonnegative parameters

This section introduces several infinite square matrices associated with Riordan array sequences through Bacher's automorphisms parametrized by $(1, n, \gamma)$ or $(1, \beta, l)$ with $n, l \geq 1$.

Let $D = (d(t), h(t))$ denote a Riordan array. We restrict the parameter α in the definition of Bacher's automorphisms (3) to nonnegative integers. Then, for fixed $\beta \in \mathbb{C}^*$ and $\gamma \in \mathbb{C}$ and nonnegative integers $m' \geq m$, we have $\varphi_{m',\beta,\gamma}(D) = (\varphi_{m'-m,1,0} \circ \varphi_{m,\beta,\gamma})(D)$. Thus, from a combinatorial perspective it is enough to examine what happens for $m = 1$, which we adopt in the sequel. To shorten notation, we define

$$\phi_{\beta,\gamma} = \varphi_{1,\beta,\gamma} \quad \text{and} \quad \phi_n = \varphi_{1,n,0}.$$

Also, as to the notation, note that throughout the paper we shall omit the index 0 associated with $\gamma = 0$.

2.1 The array sequence $(\phi_{n,\gamma}(D))_{n\geq 1}$

Among the sequences that can be associated with a Riordan array D, we focus on the columns, diagonal slices, and the row sum sequence. We construct infinite matrices by putting together columns, diagonal slices, or row sum sequences, extracted from each Riordan array in the sequence $(\phi_{n,\gamma}(D))_{n\geq 1}$.

Definition 1. Fix a number $\gamma \in \mathbb{C}$. For any Riordan array $D = (d(t), h(t))$ and any integer $i \geq 1$, define the following infinite matrices:

(i) Matrix $\mathfrak{Col}_i^{\gamma,D}$, whose nth column is the ith column of $\phi_{n,\gamma}(D)$:

$$(\mathfrak{Col}_i^{\gamma,D})_{k,n} = (\phi_{n,\gamma}(D))_{k,i} = [t^{k+i-1}]d(t)^n h'(t)^\gamma h(t)^i, \quad k,n \geq 1. \tag{4}$$

(ii) Matrix $\mathfrak{Diag}_i^{\gamma,D}$, whose nth column is the ith (descending) diagonal slice of $\phi_{n,\gamma}(D)$:

$$(\mathfrak{Diag}_i^{\gamma,D})_{k,n} = (\phi_{n,\gamma}(D))_{k+i,k} = [t^{k+i-1}]d(t)^n h'(t)^\gamma h(t)^k, \quad k,n \geq 1. \tag{5}$$

(iii) Let $(\mathfrak{r}_n^{\gamma,D})_k$, $k \geq 1$, be the sequence whose kth term is the sum of the entries of the kth row of $\phi_{n,\gamma}(D)$, for $n \geq 1$. In this context, we define the matrix $\mathfrak{RowS}^{\gamma,D}$, whose nth column is the sequence of the row sums of $\phi_{n,\gamma}(D)$:

$$(\mathfrak{RowS}^{\gamma,D})_{k,n} = (\mathfrak{r}_n^{\gamma,D})_k = [t^k]\frac{h(t)d(t)^n h'(t)^\gamma}{(1-h(t))}, \quad k,n \geq 1. \tag{6}$$

In the last equation, we used the fact that the generating function for the row sum of a Riordan array $(d(t), h(t))$ is given by $d(t)/(1-h(t))$ [19, 8].

Note that analogous matrix definitions hold for the sequences $(\phi_{-n,\gamma}(D))_{n\geq 1}$. In this context, the nth column of the similar matrices, denoted $\mathfrak{col}_i^{\gamma,D}$, $\mathfrak{diag}_i^{\gamma,D}$, and $\mathfrak{rowS}^{\gamma,D}$, corresponds to the ith column, the ith diagonal slice, or the row sum sequence of the matrix $\phi_{-n,\gamma}(D)$. Actually, given a Riordan array D, we may even extend the definitions to the sequences $\phi_{\pm 1/n,\gamma}(D))_{n\geq 1}$ so that the nth column of square matrices is associated with a column, diagonal slice or the row sum sequence of the matrix $\phi_{\pm 1/n,\gamma}(D)$.

Moreover, to avoid zero rows, in the definition of the square array $\mathfrak{Col}_i^{\gamma,D}$, we removed the first $i-1$ zero elements of the ith column of the matrices $\phi_{n,\gamma}(D)$ for all $i, n \geq 1$. When $\gamma = 0$ this definition may be seen as reciprocal to that of Riordan arrays, for the power of the second element of the pair $(d(t), h(t))$ remains fixed, while the power of the first element increases as the number of columns increases. Indeed, one may even transform the matrix $\mathfrak{Col}_i^D$ into a regular Riordan array by multiplying the entry (k,n) by t^{n-1}. The resulting array is defined in terms of the automorphism ϕ_i as given in the next remark.

Remark 2. Given any Riordan array $D = (d(t), h(t))$, let $D_S = (h(t)/t, td(t))$. Then, for any $i, n \geq 1$, the nth column of the following Riordan array

$$\phi_i(D_S) = \left(\left(\frac{h(t)}{t}\right)^i d(t), td(t)\right) \quad \text{with} \quad (\phi_i(D_S))_{k,n} = [t^{k+i-n}]d(t)^n h(t)^i, \quad k,n \geq 1,$$

from the nth row on, equals the nth column of the square array $\mathfrak{Col}_i^D$ (formula (4)).

Thus, both $(\phi_n(D))_{n\geq 1}$ and $(\phi_n(D_S))_{n\geq 1}$ carry the same combinatorial information regarding the columns.

Proposition 3. *Let $D = (d(t), h(t))$ be any Riordan array and let $D_S = (h(t)/t, td(t))$. Then, there is a 1-1 correspondence between the columns of the matrices in the array sequence $(\phi_n(D))_{n\geq 1}$ and those in the sequence $(\phi_n(D_S))_{n\geq 1}$ such that the ith column of $\phi_n(D)$ is associated with the nth column of $\phi_i(D_S)$ for any $i, n \geq 1$.*

Furthermore, some of the next examples show that the matrices of Definition 1 highlight the shared properties of the matrices in the sequence $(\phi_n(D))_{n\geq 1}$.

Example 4. We study the array sequence $(\phi_{-n}(P))_{n\geq 1}$, where $P = (1/(1-t), t/(1-t))$ is the classical Pascal array:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

The first arrays $\phi_{-n}(P)$ are

$$\phi_{-1}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_{-2}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_{-3}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_{-4}(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Clearly, from the $(n+1)$th column on we recover Pascal's array. We now consider the row sum matrix $\mathfrak{row}\mathfrak{S}^P$ associated with the sequence $(\phi_{-n}(P))_{n\geq 1}$, whose nth column yields the row sum sequence of the matrix $\phi_{-n}(P)$:

$$\mathfrak{row}\mathfrak{S}^P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & -1 & -2 & -3 & -4 & -5 & \cdots \\ 2 & 1 & 1 & 2 & 4 & 7 & 11 & \cdots \\ 4 & 2 & 1 & 0 & -2 & -6 & -13 & \cdots \\ 8 & 4 & 2 & 1 & 1 & 3 & 9 & \cdots \\ 16 & 8 & 4 & 2 & 1 & 0 & -3 & \cdots \\ 32 & 16 & 8 & 4 & 2 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We have $(\mathfrak{row}\mathfrak{S}^P)_{k,n} = \sum_{i=0}^{k-1}(-1)^i\binom{n-k+i}{i}$. Note that from the $(n+1)$th element on, the nth column is the row sum sequence of the Pascal array A000079 whose elements are powers of 2. Now, we examine the upper triangular matrix $\mathfrak{row}^P$ obtained by considering only the n first elements of the nth column of $\mathfrak{row}\mathfrak{S}^P$ for each $n \geq 1$:

$$\mathfrak{row}^P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & -1 & -2 & -3 & -4 & -5 & \cdots \\ 0 & 0 & 1 & 2 & 4 & 7 & 11 & \cdots \\ 0 & 0 & 0 & 0 & -2 & -6 & -13 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, the matrix $\mathfrak{row}^P$ is the sequence A220074 as an upper right triangle.

We now study the array sequences $(\phi_n(D))_{n\geq 1}$, where D denotes the triangles A025581 and A114284.

Example 5. We consider the Appell group element $T = (p(t)^2, t) = (1/(1-t)^2, t)$ which is the triangle $T' = (tp(t)^2, t)$ A025581 without the first row. Note that T' is not in the Riordan group, for the entry (1,1) is zero. The columns of T yield the second column of Pascal's triangle $P = (p(t), tp(t))$.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 3 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 4 & 3 & 2 & 1 & 0 & \cdots \\ 6 & 5 & 4 & 3 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Likewise, the images of T under the automorphisms ϕ_n yield the $(2n)$th columns of the Pascal array P. Thus, the columns of the matrices $\mathfrak{Col}_i^T$ (see formula (4)) are the even columns of Pascal's triangle for any $i \geq 1$:

$$\mathfrak{Col}_i^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & \cdots \\ 3 & 10 & 21 & 36 & 55 & 78 & \cdots \\ 4 & 20 & 56 & 120 & 220 & 364 & \cdots \\ 5 & 35 & 126 & 330 & 715 & 1365 & \cdots \\ 6 & 56 & 252 & 792 & 2002 & 4368 & \cdots \\ 7 & 84 & 462 & 1716 & 5005 & 12376 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Indeed, $(\mathfrak{Col}_i^T)_{k,n} = \binom{2n+k-2}{k-1}$, $k, n \geq 1$. The first sequence numbers are A000027, A000292, A000389, A000580, A000582, A001288, A010966, and A010968. From the second row on, the first sequence numbers are A005843, A014105, A002492, A053126, A053127, and A053128.

Example 6. We consider the array sequence $(\phi_n(A))_{n\geq 1}$, where $A = ((1-3t)/(1-t), t)$ A114284:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & -2 & 1 & 0 & 0 & 0 & \cdots \\ -2 & -2 & -2 & 1 & 0 & 0 & \cdots \\ -2 & -2 & -2 & -2 & 1 & 0 & \cdots \\ -2 & -2 & -2 & -2 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The base array A coincides with $\phi_1(A)$. The next arrays $\phi_n(A)$ are

$$\phi_2(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 0 & -4 & 1 & 0 & 0 & \cdots \\ 8 & 4 & 0 & -4 & 1 & 0 & \cdots \\ 12 & 8 & 4 & 0 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_3(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -6 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & -6 & 1 & 0 & 0 & 0 & \cdots \\ 10 & 6 & -6 & 1 & 0 & 0 & \cdots \\ 6 & 10 & 6 & -6 & 1 & 0 & \cdots \\ -6 & 6 & 10 & 6 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_4(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -8 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 16 & -8 & 1 & 0 & 0 & 0 & \cdots \\ 8 & 16 & -8 & 1 & 0 & 0 & \cdots \\ -16 & 8 & 16 & -8 & 1 & 0 & \cdots \\ -40 & -16 & 8 & 16 & -8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\phi_5(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -10 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 30 & -10 & 1 & 0 & 0 & 0 & \cdots \\ -10 & 30 & -10 & 1 & 0 & 0 & \cdots \\ -50 & -10 & 30 & -10 & 1 & 0 & \cdots \\ -42 & -50 & -10 & 30 & -10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By formula (6), the matrix for the row sums $\mathfrak{RowS}^A$ is

$$\mathfrak{RowS}^A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & -3 & -5 & -7 & -9 & -11 & -13 & -15 & -17 & \cdots \\ -3 & -3 & 1 & 9 & 21 & 37 & 57 & 81 & 109 & \cdots \\ -5 & 1 & 11 & 17 & 11 & -15 & -69 & -159 & -293 & \cdots \\ -7 & 9 & 17 & 1 & -39 & -87 & -111 & -63 & 121 & \cdots \\ -9 & 21 & 11 & -39 & -81 & -51 & 99 & 369 & 679 & \cdots \\ -11 & 37 & -15 & -87 & -51 & 141 & 393 & 465 & 37 & \cdots \\ -13 & 57 & -69 & -111 & 99 & 393 & 363 & -351 & -1709 & \cdots \\ -15 & 81 & -159 & -63 & 369 & 465 & -351 & -1791 & -2447 & \cdots \\ -17 & 109 & -293 & 121 & 679 & 37 & -1709 & -2447 & 479 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the removal of the first line of $\mathfrak{RowS}^A$ yields a symmetric matrix. Therefore, it follows from formula (3) and the formula for the generating function of the row sum sequence $d(t)/(1-h(t))$ [19, 8] that $\mathfrak{r}_n^A = (1-3t)^n/(1-t)^{n+1}$ is the generating function for both the nth column and the $(n+1)$th row of $\mathfrak{RowS}^A$, where $n \geq 1$. The first column (and the second row) of $\mathfrak{RowS}^A$ is the odd negative integers sequence A165747, i.e., $(\mathfrak{RowS}^A)_{k,1} = 1-2k$, $k \geq 1$. The second column (and the third row for $n \geq 3$) is the pinwheel numbers sequence A059993, i.e., $(\mathfrak{RowS}^A)_{k,2} = 2(k-4)^2 + 6(k-4) + 1$, $k \geq 4$.

2.2 The array sequences $(\phi_{\beta,l}(D))_{l\geq 1}$

We define infinite square arrays analogous to those of Section 2.1.

Definition 7. Fix a number $\beta \in \mathbb{C}^*$. For any Riordan array $D = (d(t), h(t))$ and any integer $i \geq 1$, define the following infinite matrices:

(i) Matrix $\mathfrak{BCol}_i^{\beta,D}$, whose lth column is the ith column of $\phi_{\beta,l}(D)$:

$$(\mathfrak{BCol}_i^{\beta,D})_{k,l} = (\phi_{\beta,l}(D))_{k,i} = [t^{k+i-1}]d(t)^\beta h'(t)^l h(t)^i, \quad k,l \geq 1.$$

(ii) Matrix $\mathfrak{BDiag}_i^{\beta,D}$, whose lth column is the ith (descending) diagonal slice of $\phi_{\beta,l}(D)$:

$$(\mathfrak{BDiag}_i^{\beta,D})_{k,l} = (\phi_{\beta,l}(D))_{k+i,k} = [t^{k+i-1}]d(t)^\beta h'(t)^l h(t)^k, \quad k,l \geq 1.$$

(iii) Let $(\mathfrak{r}_l^{\beta,D})_m$, $m \geq 0$, be the sequence whose mth term is the sum of the entries of the mth row of $\phi_{\beta,l}(D)$, for $l \geq 1$. In this context, we define the matrix $\mathfrak{BRowS}^{\beta,D}$, whose lth column is the sequence of the row sums of $\phi_{\beta,l}(D)$:

$$(\mathfrak{BRowS}^{\beta,D})_{k,l} = (\mathfrak{r}_l^{\beta,D})_k = [t^k]\frac{h(t)d(t)^\beta h'(t)^l}{(1-h(t))}, \quad k,l \geq 1.$$

Moreover, given a Riordan array D, analogous matrix definitions hold for the array sequences $(\phi_{\beta,-l}(D))_{l\geq 1}$ and $(\phi_{\beta,\pm 1/l}(D))_{l\geq 1}$.

Example 8. We now examine some of the square matrices of Definition 7 taking as base array the Pascal array $P = (p(t), tp(t))$, with $p(t) = 1/(1-t)$. First, we fix $\beta = 1$ and consider the matrix $\mathfrak{BCol}_i^{1,P}$. We have

$$(\mathfrak{BCol}_i^{1,P})_{k,l} = [t^{k-1}]\frac{1}{(1-t)^{2l+i+1}}, \quad k,l \geq 1.$$

Hence, the lth column of $\mathfrak{BCol}_i^{1,P}$ is generated by $1/(1-t)^{2l+i+1}$ for any $l \geq 1$. Therefore,

$$(\mathfrak{BCol}_i^{1,P})_{k,l} = \binom{2l+i+k-1}{2l+i} = \binom{2l+i+k-1}{k-1}.$$

For $i = 1$ we have

$$\mathfrak{BCol}_1^{1,P} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 4 & 6 & 8 & 10 & 12 & 14 & \cdots \\ 10 & 21 & 36 & 55 & 78 & 105 & \cdots \\ 20 & 56 & 120 & 220 & 364 & 560 & \cdots \\ 35 & 126 & 330 & 715 & 1365 & 2380 & \cdots \\ 56 & 252 & 792 & 2002 & 4368 & 8568 & \cdots \\ 84 & 462 & 1716 & 5005 & 12376 & 27132 & \cdots \\ 120 & 792 & 3432 & 11440 & 31824 & 77520 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequences for the first columns are A000292, A000389, A000580, A000582, A001288, and A010966. From the second row on the sequences for the listed rows are A005843, A014105, A002492, A053126, A053127, A053128, and A053129.

Also for $\beta = 1$, we have

$$(\mathfrak{BRowS}^{1,P})_{k,l} = [t^{k-1}]\frac{1}{(1-2t)(1-t)^{2l+1}}, \quad k,l \geq 1.$$

The lth column of $\mathfrak{BRowS}^{1,P}$ is thus generated by $1/((1-2t)(1-t)^{2l+1})$ for any $l \geq 1$. Therefore,

$$(\mathfrak{BRowS}^{1,P})_{k,l} = \sum_{i=0}^{k-1} 2^i \binom{2l+k-1-i}{k-1-i} = 2^{k+2l} - \sum_{i=0}^{2l} \binom{k+2l}{i}.$$

The matrix yields

$$\mathfrak{BRowS}^{1,P} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 5 & 7 & 9 & 11 & 13 & 15 & \cdots \\ 16 & 29 & 46 & 67 & 92 & 121 & \cdots \\ 42 & 93 & 176 & 299 & 470 & 697 & \cdots \\ 99 & 256 & 562 & 1093 & 1941 & 3214 & \cdots \\ 219 & 638 & 1586 & 3473 & 6885 & 12616 & \cdots \\ 466 & 1486 & 4096 & 9949 & 21778 & 43796 & \cdots \\ 968 & 3302 & 26333 & 63004 & 137980 & 280600 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequences for the first four columns are A002662, A002664, A035039, and A035041. The second and third rows are A005408 and A130883. From the fourth row on and from n such that $a(n) = 2^n - \sum_{i=0}^{2} \binom{n+2}{i}$, $n \geq 4$, the sequences for the listed rows yield the alternating elements of the following sequences $a(n)$: A000125, A000127, A006261, A008859, and A008860.

3 Automorphisms and the A and Z-sequences

Merlini, Sprugnoli, Rogers, and Verri [15], and He and Sprugnoli [9], showed that every Riordan array $D = (d(t), h(t))$ can be characterized by their A- and Z-sequences. We examine in this Section how the A- and Z-sequences of D are transformed by the automorphisms $\phi_{\alpha,\beta} = \varphi_{\alpha,\beta,0}$.

Lemma 9 ([15, 9, 23])**.** *Let $D = (d(t), h(t))$ be a Riordan array, and let $A(t)$ and $Z(t)$ be the generating functions for the corresponding A- and Z-sequences, respectively. Then,*

$$d(t) = \frac{1}{1 - tZ(h(t))}, \quad h(t) = tA(h(t)).$$

In terms of the inverse $D^{-1} = \big(1/d(\bar{h}(t)), \bar{h}(t)\big)$ we have

$$A(t) = \frac{t}{\bar{h}(t)} \quad \text{and} \quad Z(t) = \frac{A(t)}{t}\left(1 - \frac{1}{d(\bar{h}(t))}\right).$$

Moreover, let $(p_k(x))_{k\geq 1}$ be the polynomial sequence associated with a Riordan array D, which is given by $p_k(x) = \sum_{n=1}^{k} D_{k,n} x^{n-1}$. By the multiplication rule for Riordan arrays (2), the generating function of $(p_k(x))_{k\geq 1}$ is $p(t,x) = d(t)/(1 - xh(t))$. In this context, if $A_{\alpha,\beta}(t)$ and $Z_{\alpha,\beta}(t)$ denote the A- and Z-sequences of $\phi_{\alpha,\beta}(D)$, then a simple calculation yields

$$A_{\alpha,\beta}(t) = A(t) \quad \text{and} \quad Z_{\alpha,\beta}(t) = \frac{A(t)^{1-\alpha-\beta}}{t}\left(A(t)^{\alpha+\beta} - \big(A(t) - tZ(t)\big)^{\beta}\right). \tag{8}$$

Likewise, the generating function $p_{\alpha,\beta}(t,x)$ of the polynomial sequence of $\phi_{\alpha,\beta}(D)$ is given by $p_{\alpha,\beta}(t,x) = (h(t)/t)^{\alpha} d(t)^{\beta}/(1 - xh(t))$. Now, Yang [23] introduced the function

$$B(t;x) = \frac{A(t) - tZ(t)}{A(t) - xt},$$

which satisfies

$$(d(t), h(t))B(t;x) = \frac{1}{1 - xt}.$$

The function $B(t;x)$ turns out to be the generating function of the polynomial sequence associated with the Riordan array D^{-1}. Note that Yang's notation omits the variable x in the definition of $B(t;x)$, i.e., $B(t;x) = B(t)$ in [23]. Now, if we define $B_{\alpha,\beta}(t;x) = (A_{\alpha,\beta}(t) - tZ_{\alpha,\beta}(t))/(A_{\alpha,\beta}(t) - xt)$, this function reads in terms of $B(t;x)$ as follows

$$\begin{aligned} B_{\alpha,\beta}(t;x) &= A(t)^{1-\alpha-\beta}(A(t) - xt)^{\beta-1}B(t;x)^{\beta} \\ &= A(t)^{-\alpha}B(t;x)^{\beta}\sum_{i=0}^{\beta-1}\binom{\beta-1}{i}A(t)^{-i}(-xt)^{i}. \end{aligned} \tag{9}$$

In particular, for $\beta = 1$ we have $B_{\alpha,1}(t;x) = A(t)^{-\alpha}B(t;x)$.

3.1 Infinite matrices associated with the Z and $B(x)$-sequences

Let $D = (d(t), h(t))$ be any Riordan array. For each matrix in the sequence $(\phi_n(D))_{n\geq 1}$, we denote by Z_n^D and $B_n^D(x)$ the corresponding Z and $B(x)$-sequences. Recall that the A-sequence is invariant under Bacher's automorphisms. By analogy with Section 2, we define infinite matrices associated with the sequences $(Z_n^D(t))_{n\geq 1}$ and $(B_n^D(t;x))_{n\geq 1}$, where the sequences $Z_n^D(t)$ and $B_n^D(t;x)$ are defined in terms of the sequences A, Z, and $B(t;x)$ of the matrix D by formulas (8) and (9).

Definition 10. For any Riordan matrix D and any real variable x define the following two infinite matrices:

$$\mathfrak{Z}_{k,n}^{D} = [t^{k-1}]Z_n^D \quad \text{and} \quad \mathfrak{B}_{k,n}^{D}(x) = [t^{k-1}]B_n^D(x), \quad k, n \geq 1. \tag{10}$$

Hence, by definition the matrices $\mathfrak{Z}^D$ and $\mathfrak{B}^D(x)$ are those whose nth column is generated by the sequences Z and $B(x)$ of $\phi_n(D)$.

Again, analogous matrix definitions $\mathfrak{z}^D$ and $\mathfrak{b}^D(x)$ hold for the matrix sequence $(\phi_{-n}(D))_{n\geq 1}$.

Example 11. We study the matrices $\mathfrak{Z}^P$ and $\mathfrak{B}^P(x)$ defined by formula (10), for the Pascal array $P = (p(t), tp(t))$, where $p(t) = 1/(1-t)$ (see array (7)). The Pascal triangle has the following well-known A, Z, and $B(x)$-sequences:

$$A(t) = 1 + t, \quad Z(t) = 1, \quad \text{and} \quad B(t;x) = \sum_{i=0}^{\infty}(-1+x)^i t^i.$$

The matrix $\mathfrak{Z}^P$ yields

$$\mathfrak{Z}^P = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ -1 & -3 & -6 & -10 & -15 & -21 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ -1 & -5 & -15 & -35 & -70 & -126 & \cdots \\ 1 & 6 & 21 & 56 & 126 & 252 & \cdots \\ -1 & -7 & -28 & -84 & 210 & -462 & \cdots \\ 1 & 8 & 36 & 120 & 330 & 792 & \cdots \\ -1 & -9 & -45 & -165 & -495 & -1287 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, by taking absolute values for the entries, the rows of $\mathfrak{Z}^P$ are the columns of the matrix $\phi_1(P)$. More precisely,

$$\mathfrak{Z}^P_{k,n} = (-1)^{k-1}\binom{n+k-1}{k}, \quad n \geq 1, k > 1.$$

Next, we consider the matrix $\mathfrak{B}^P(x)$:

$$\mathfrak{B}^P(x) = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ -2+x & -3+x & -4+x & \cdots \\ 3-3x+x^2 & 6-4x+x^2 & 10-5x+x^2 & \cdots \\ -4+6x-4x^2+x^3 & -10+10x-5x^2+x^3 & -20+15x-6x^2+x^3 & \cdots \\ 5-10x+10x^2-5x^3+x^4 & 15-20x+15x^2-6x^3+x^4 & 35-35x+21x^2-7x^3+x^4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The entries $\mathfrak{B}^P(x)$ are thus monic polynomials of degree $k-1$:

$$\mathfrak{B}^P_{k,n}(x) = \sum_{i=0}^{k-1}(-1)^{k-1+i}\binom{n+k-1}{n+i}x^i, \quad k, n \geq 1.$$

For $x = 1$, the matrix $\mathfrak{B}^P(1)$ is Pascal's array (read as a square matrix) with alternating signs:

$$\mathfrak{B}^P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & -2 & -3 & -4 & -5 & -6 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\ -1 & -4 & -10 & -20 & -35 & -56 & \cdots \\ 1 & 5 & 15 & 35 & 70 & 126 & \cdots \\ -1 & -6 & -21 & -56 & -126 & -462 & \cdots \\ 1 & 7 & 28 & 84 & 330 & 792 & \cdots \\ -1 & -8 & -36 & -120 & -495 & -1287 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

4 Automorphisms and Riordan group pseudo-involutions

We study the action of Bacher's automorphisms on the pseudo-involutions of the Riordan group.

We recall that a Riordan array $D = (d(t), h(t))$ is an involution if and only if

$$\begin{aligned} d(t) &= \frac{1}{d(h(t))}, \\ h(h(t)) &= t. \end{aligned}$$

Note that for any involution I and any Riordan array D, the identity $(DI)^2 = \mathrm{id}_{\mathfrak{Rio}}$ is equivalent to $(ID)^2 = \mathrm{id}_{\mathfrak{Rio}}$. Therefore, since D has an inverse, we have $(DI)^2 = \mathrm{id}_{\mathfrak{Rio}}$ if and only if $(D^n I)^2 = \mathrm{id}_{\mathfrak{Rio}}$ for any integer n. In this context, Shapiro [18] and later Cameron and Nkwanta [2] introduced the following notion of pseudo-involution. An element D of the Riordan group has *pseudo-order 2* if the matrix DM has order 2, where $M = (1, -t)$ is an involution. Therefore, a Riordan array $D = (d(t), h(t))$ with pseudo-order 2 is known as a *pseudo-involution*. In this case, it readily follows that

$$d(t) = \frac{1}{d(-h(t))}, \tag{11}$$

$$h(-h(t)) = -t. \tag{12}$$

In particular, if $D = (1, h(t))$ is an element of the Lagrange subgroup such that

$$h(t) = \bar{h}(t) \quad \text{and} \quad h(-t) = -h(t),$$

then D has pseudo-order 2 [2]. We next give the following characterization of pseudo-involutions.

Proposition 12. *Let $\alpha : \mathfrak{Rio} \to \mathfrak{Rio}$ be any Riordan group automorphism. If $\alpha(M) = M$, then an element $D \in \mathfrak{Rio}$ has pseudo-order 2 if and only if $\alpha(D)$ has pseudo-order 2.*

Proof. Suppose that DM has order 2. Then, $(\alpha(D)M)^2 = \alpha((DM)^2) = \alpha(\mathrm{id}_{\mathfrak{Rio}}) = \mathrm{id}_{\mathfrak{Rio}}$. Suppose now that $\alpha(D)$ has pseudo-order 2. Then, $\mathrm{id}_{\mathfrak{Rio}} = (\alpha(D)M)^2 = \alpha((DM)^2)$. Therefore, $(DM)^2 = \mathrm{id}_{\mathfrak{Rio}}$, for α is injective. □

Note that the result above does not use the Riordan group structure. Therefore, the proposition remains valid for a generalization of the concept of pseudo-involution to any group. Furthermore, since $\varphi_{\alpha,\beta,\gamma}(M) = ((-1)^{\alpha+\gamma}, -t)$, not all the automorphisms $\varphi_{\alpha,\beta,\gamma}$ leave the Riordan array M invariant. Nevertheless, they all preserve pseudo-involutions as we show next.

Proposition 13. *For any $\alpha, \gamma \in \mathbb{C}$ and $\beta \in \mathbb{C}^*$, a Riordan matrix $D \in \mathfrak{Rio}$ has pseudo-order 2 if and only if $\varphi_{\alpha,\beta,\gamma}(D)$ has pseudo-order 2.*

Proof. Suppose that $D = (d(t), h(t))$ has pseudo-order 2. By equations (11) and (12) we have

$$\begin{aligned}(\varphi_{\alpha,\beta,\gamma}(DM)^2 &= \left(\left(\frac{h(t)}{t}\right)^{\alpha} d(t)^{\beta} h'(t)^{\gamma}, -h(t)\right)^2 \\ &= \left(\left(\frac{h(t)}{t}\right)^{\alpha} d(t)^{\beta} h'(t)^{\gamma} \left(\frac{h(-h(t))}{-h(t)}\right)^{\alpha} d(-h(t))^{\beta} h'(-h(t))^{\gamma}, t\right) \\ &= (1, t).\end{aligned}$$

Now, suppose that $\phi_{r,s}(D)$ has pseudo-order 2. From equation (11) we have

$$\left(\frac{h(t)}{t}\right)^{\alpha} d(t)^{\beta} h'(t)^{\gamma} = \left(\frac{-h(t)}{h(-h(t))}\right)^{\alpha} \frac{1}{d(-h(t))^{\beta} h'(-h(t))^{\gamma}}.$$

Using $h(-h(t)) = -t$ from equation (12) gives $d(t) = 1/d(-h(t))$. □

This proposition extends Cheon, Kim, Jin, and Shapiro's remark that says that if $(h(t)/t, h(t))$ is a pseudo-involution, then $((h(t)/t)^n, h(t))$ is also a pseudo-involution for all integers n [4].

Example 14. Consider the generalized RNA triangles introduced by Cameron and Nkwanta [2] for $n \geq 1$:

$$\mathcal{D}_{\pm n} = \left(g\left(t\right)\left(\frac{1-t}{1-tg\left(t\right)}\right)^{\pm n}, tg\left(t\right)\right),$$

where

$$g\left(t\right) = \frac{1 - t + t^2 - \sqrt{1 - 2t - t^2 - 2t^3 + t^4}}{2t^2}.$$

We recall that $\mathcal{D}_{\pm n} = A^{\mp n}\mathcal{D}_0 A^{\pm n}$, where $\mathcal{D}_0 = \left(g\left(t\right), tg\left(t\right)\right)$ is the Nkwanta RNA triangle, and $A^{\pm n} = \left(\frac{1}{(1-t)^{\pm n}}, t\right)$. We now give an alternative definition of the matrices $\mathcal{D}_{\pm n}$ in terms of the automorphisms $\phi_{\pm n}$ for $n \geq 1$. Let

$$\mathcal{D} = \left(\frac{1-t}{1-tg(t)}, tg(t)\right), \tag{13}$$

$$\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 4 & 3 & 1 & 0 & \cdots \\ 12 & 12 & 10 & 7 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The diagonal slices of the array $\mathcal{D}$ are of combinatorial interest. The sequence numbers for the first three nontrivial diagonals are A000124, A177787, and A116722. Moreover, the matrix $\mathcal{D}$ given by formula (13) is the base matrix for the sequences $(\mathcal{D}_{\pm n})_{n\geq 1}$. That is, for $n \geq 1$ we have

$$\mathcal{D}_{\pm n} = \phi_{\pm n}(\mathcal{D}).$$

Also, the matrices $\phi_{\alpha,\beta}(\mathcal{D}_{\pm n})$ can be expressed in terms of the base matrix $\mathcal{D}$ as follows

$$\phi_{\alpha,\beta}(\mathcal{D}_{\pm n}) = (\phi_{\alpha,\beta} \circ \phi_{\pm n})(\mathcal{D}) = \phi_{\alpha+\beta,(\pm n)\beta}(\mathcal{D}).$$

Note that the matrix $\mathcal{D}$ is not a pseudo-involution. Indeed,

$$(\mathcal{D}M)^2 = \left(\frac{1-t}{1-tg(t)}, -tg(t)\right)^2 = \left(\frac{1-t}{1-tg(t)}\frac{1+tg(t)}{1+t}, t\right) \neq (1,t),$$

since $g(t)$ satisfies the functional equation $g(t)g(-tg(t)) = 1$. Therefore, it follows directly from Proposition 13 that the matrices $\mathcal{D}_{\pm n}$ are not pseudo-involutions either, which contradicts Theorem 3 of [2]. Observe also that although the aforementioned definition of the generalized RNA matrices, namely,

$$\mathcal{D}_{\pm n} = A^{\mp n}\mathcal{D}_0 A^{\pm n} = \left((1-t)^{\pm n}, t\right)\left(g(t), tg(t)\right)\left(\frac{1}{(1-t)^n}, t\right)$$

given in [2] does not yield pseudo-involutions, we can easily define pseudo-involutions from Nkwanta's RNA matrix by setting

$$\mathcal{D}^{\pm}_{\pm n} = \left((1\pm t)^{\pm n}, t\right)\mathcal{D}_0\left(\frac{1}{(1\mp t)^{\pm n}}, t\right) = \left(g(t)\frac{1\pm t}{1\mp tg(t)}, tg(t)\right), \quad n \geq 1.$$

It is worth examining some of the matrices $\mathfrak{Col}_i^{\mathcal{D}}$, $\mathfrak{Diag}_i^{\mathcal{D}}$, and $\mathfrak{RowS}^{\mathcal{D}}$ (recall formulas (4), (5), and (6) of Section 2.1). First, we study the matrix $\mathfrak{Col}_1^{\mathcal{D}}$ whose columns are given by the first column of each of the matrices $\mathcal{D}_n$ for $n \geq 1$. We have

$$\mathfrak{Col}_1^{\mathcal{D}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 8 & 11 & 14 & 17 & 20 & \cdots \\ 12 & 21 & 31 & 42 & 54 & 67 & \cdots \\ 29 & 55 & 86 & 122 & 163 & 209 & \cdots \\ 71 & 144 & 237 & 351 & 487 & 646 & \cdots \\ 175 & 377 & 650 & 1001 & 1437 & 1965 & \cdots \\ 434 & 987 & 1775 & 2833 & 4197 & 5904 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The second column is the sequence A088305. Moreover, the sequences for the first nontrivial rows are

- $(\mathfrak{Col}_1^{\mathcal{D}})_{4,n} = 2 + 3n$, $n \geq 1$;
- $(\mathfrak{Col}_1^{\mathcal{D}})_{5,1} = 12$ and $(\mathfrak{Col}_1^{\mathcal{D}})_{5,n} = (\mathfrak{Col}_1^{\mathcal{D}})_{5,n-1} + 7 + n$, $n > 1$;
- $(\mathfrak{Col}_1^{\mathcal{D}})_{6,1} = 29$ and $(\mathfrak{Col}_1^{\mathcal{D}})_{6,n} = (\mathfrak{Col}_1^{\mathcal{D}})_{6,n-1} + 16 + 5n$, $n > 1$;
- $(\mathfrak{Col}_1^{\mathcal{D}})_{7,1} = 71$, $(\mathfrak{Col}_1^{\mathcal{D}})_{7,2} = 144$, and $(\mathfrak{Col}_1^{\mathcal{D}})_{7,n} = 2(\mathfrak{Col}_1^{\mathcal{D}})_{7,n-1} - (\mathfrak{Col}_1^{\mathcal{D}})_{7,n-2} + 17 + n$, $n > 2$;
- $(\mathfrak{Col}_1^{\mathcal{D}})_{8,1} = 175$, $(\mathfrak{Col}_1^{\mathcal{D}})_{8,2} = 377$, and $(\mathfrak{Col}_1^{\mathcal{D}})_{8,n} = 2(\mathfrak{Col}_1^{\mathcal{D}})_{8,n-1} - (\mathfrak{Col}_1^{\mathcal{D}})_{8,n-2} + 50 + 7n$, $n > 2$;
- $(\mathfrak{Col}_1^{\mathcal{D}})_{9,1} = 434$, $(\mathfrak{Col}_1^{\mathcal{D}})_{9,2} = 987$, and $(\mathfrak{Col}_1^{\mathcal{D}})_{9,n} = 2(\mathfrak{Col}_1^{\mathcal{D}})_{9,n-1} - (\mathfrak{Col}_1^{\mathcal{D}})_{9,n-2} + 136 + (n^2 + 63n)/2$, $n > 2$.

We now analyse the first nontrivial diagonal slices of the matrices $\mathcal{D}_n$. The matrix of the third diagonal slices reads as follows

$$\mathfrak{Diag}_3^{\mathcal{D}} = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\ 11 & 12 & 13 & 14 & 15 & 16 & \cdots \\ 16 & 17 & 18 & 19 & 20 & 21 & \cdots \\ 22 & 23 & 24 & 25 & 26 & 27 & \cdots \\ 29 & 30 & 31 & 32 & 33 & 34 & \cdots \\ 37 & 38 & 39 & 40 & 41 & 42 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The entries are given by

$$(\mathfrak{Diag}_3^{\mathcal{D}})_{k,n} = n + 1 + \frac{(k+2)^2 - 3(k+2)}{2}, \quad k, n \geq 1.$$

Also, the following recursive definition holds

$$\begin{aligned} (\mathfrak{Diag}_3^{\mathcal{D}})_{1,n} &= n + 1, \\ (\mathfrak{Diag}_3^{\mathcal{D}})_{k,n} &= (\mathfrak{Diag}_3^{\mathcal{D}})_{k-1,n} + k - 1, \quad k > 1. \end{aligned}$$

The first five sequence numbers are A000124, A152948, A152950, A145018, and A167499.

Furthermore, as for the fourth diagonal slices of the matrices $\mathcal{D}_n$ we have

$$\mathfrak{Diag}_4^{\mathcal{D}} = \begin{bmatrix} 5 & 8 & 11 & 14 & 17 & 20 & \cdots \\ 10 & 14 & 18 & 22 & 26 & 30 & \cdots \\ 18 & 23 & 28 & 33 & 38 & 43 & \cdots \\ 30 & 36 & 42 & 48 & 54 & 60 & \cdots \\ 47 & 54 & 61 & 68 & 75 & 82 & \cdots \\ 70 & 78 & 86 & 94 & 102 & 110 & \cdots \\ 100 & 109 & 118 & 127 & 136 & 145 & \cdots \\ 138 & 148 & 158 & 168 & 178 & 188 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The first column is of course the fourth diagonal slice of $\phi_1(\mathcal{D})$ (and of the base matrix $\mathcal{D}$), sequence A177787. The second column is the sequence A201347. The entries of $\mathfrak{Diag}_4^{\mathcal{D}}$ are given by

$$(\mathfrak{Diag}_4^{\mathcal{D}})_{k,n} = \frac{(k+1)(11+(k+1)^2)}{2} + (2+k)(n-1), \quad k,n \geq 1.$$

Equivalently, the sequence for the kth row, $k \geq 1$, yields

$$(\mathfrak{Diag}_4^{\mathcal{D}})_{k,n} \quad = \quad (\mathfrak{Diag}_4^{\mathcal{D}})_{k,n-1} + k + 2, \quad n > 1.$$

The recursion formula for the nth column, $n \geq 1$, is

$$\begin{aligned} (\mathfrak{Diag}_4^{\mathcal{D}})_{1,1} &= 5; \\ (\mathfrak{Diag}_4^{\mathcal{D}})_{1,n} &= (\mathfrak{Diag}_4^{\mathcal{D}})_{1,n-1} + 3, \quad n > 1, \\ (\mathfrak{Diag}_4^{\mathcal{D}})_{k,n} &= (\mathfrak{Diag}_4^{\mathcal{D}})_{k-1,n} + n + k + 6, \quad k > 1. \end{aligned}$$

We next study the matrix $\mathfrak{RowS}^{\mathcal{D}}$ for the row sum sequences of the matrices $\mathcal{D}_n$:

$$\mathfrak{RowS}^{\mathcal{D}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & 2 & 2 & 2 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ 13 & 17 & 21 & 25 & 29 & 33 & \cdots \\ 34 & 48 & 63 & 79 & 96 & 114 & \cdots \\ 89 & 134 & 185 & 242 & 305 & 374 & \cdots \\ 233 & 371 & 536 & 729 & 951 & 1203 & \cdots \\ 610 & 1021 & 1537 & 2166 & 2916 & 3795 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The first column is the sequence A001519:

$$\begin{aligned} (\mathfrak{RowS}^{\mathcal{D}})_{1,1} &= 1, \\ (\mathfrak{RowS}^{\mathcal{D}})_{2,1} &= 1, \\ (\mathfrak{RowS}^{\mathcal{D}})_{k,n} &= 3(\mathfrak{RowS}^{\mathcal{D}})_{k-1,1} - (\mathfrak{RowS}^{\mathcal{D}})_{k-2,1}, \quad k > 1. \end{aligned}$$

Also, the sequences for the first nontrivial rows are

- $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{4,n} = 9 + 4n$, $n \geq 1$;
- $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{5,1} = 34$ and $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{5,n} = (\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{5,n-1} + 12 + n$, $n > 1$;
- $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{6,1} = 89$ and $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{6,n} = (\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{6,n-1} + 33 + 6n$, $n > 1$;
- $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{7,1} = 233$, $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,2} = 371$, and
  $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{7,n} = 2(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{7,n-1} - (\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{7,n-2} + 24 + n$, $n > 2$;
- $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,1} = 610$, $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,2} = 1021$, and
  $(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,n} = 2(\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,n-1} - (\mathfrak{Row}\mathfrak{S}^{\mathcal{D}})_{8,n-2} + 81 + 8n$, $n > 2$.

Example 15. We study some of the matrices for the diagonal slices of Section 2.2 taking as base the matrix $\mathcal{D}_0 = (g(t), tg(t))$. For fixed $\beta \in \mathbb{C}^*$, the sequences $(D^{\beta}_{\pm l})_{l\geq 1} = (\phi_{\beta,\pm l}(\mathcal{D}_0))_{l\geq 1}$ are sequences of pseudo-involutions by Proposition 13. For the matrix $\mathcal{D}_0$, the second formula of Definition 7 yields

$$(\mathfrak{BDiag}_i^{\beta,\mathcal{D}_0})_{k,l} = [t^{i-1}]\big((tg)'(t)\big)^l g(t)^{k+\beta}, \quad k, l \geq 1.$$

Hence, restricting β to integers $n \neq 0$, the matrices $\mathfrak{BDiag}_i^{n+1,\mathcal{D}_0}$ are obtained from $\mathfrak{BDiag}_i^{n,\mathcal{D}_0}$ by removing the first row. However, the matrix for $n = 1$ is obtained from that for $n = -1$ by removing the first two rows. This is true for all the elements of the Bell subgroup. Thus, we now consider only $\beta = 1$. The value $i = 2$ defines the first nontrivial square matrix:

$$\mathfrak{BDiag}_2^{1,\mathcal{D}_0} = \begin{bmatrix} 4 & 6 & 8 & 10 & 12 & 14 & \cdots \\ 5 & 7 & 9 & 11 & 13 & 15 & \cdots \\ 6 & 8 & 10 & 12 & 14 & 16 & \cdots \\ 7 & 9 & 11 & 13 & 15 & 17 & \cdots \\ 8 & 10 & 12 & 14 & 16 & 18 & \cdots \\ 9 & 11 & 13 & 15 & 17 & 19 & \cdots \\ 10 & 12 & 14 & 16 & 18 & 20 & \cdots \\ 11 & 13 & 15 & 17 & 19 & 21 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence, we have

$$(\mathfrak{BDiag}_2^{1,\mathcal{D}_0})_{k,l} = k + 2l + 1, \quad k, l \geq 1.$$

Now, for $i = 3$ the matrix for the diagonal slices of $\mathcal{D}_l^1$ yields

$$\mathfrak{BDiag}_3^{1,\mathcal{D}_0} = \begin{bmatrix} 10 & 21 & 36 & 55 & 78 & 105 & \cdots \\ 15 & 28 & 45 & 66 & 91 & 120 & \cdots \\ 21 & 36 & 55 & 78 & 105 & 136 & \cdots \\ 28 & 45 & 66 & 91 & 120 & 153 & \cdots \\ 36 & 55 & 78 & 105 & 136 & 171 & \cdots \\ 45 & 66 & 91 & 120 & 153 & 190 & \cdots \\ 55 & 78 & 105 & 136 & 171 & 210 & \cdots \\ 66 & 91 & 120 & 153 & 190 & 231 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequence for the odd rows is A014105, while that for the even rows is A000384. As for the columns, the sequence is A000217. We have

$$(\mathfrak{BDiag}_3^{1,\mathcal{D}_0})_{k,l} = \binom{k+2l+2}{2}, \quad k,l \geq 1.$$

The next matrix for the diagonal slices of $\mathcal{D}_l^1$ is

$$\mathfrak{BDiag}_4^{1,\mathcal{D}_0} = \begin{bmatrix} 26 & 66 & 134 & 238 & 386 & 586 & \cdots \\ 42 & 95 & 180 & 305 & 478 & 707 & \cdots \\ 64 & 132 & 236 & 384 & 584 & 844 & \cdots \\ 93 & 178 & 303 & 476 & 705 & 998 & \cdots \\ 130 & 234 & 382 & 582 & 842 & 1170 & \cdots \\ 176 & 301 & 474 & 703 & 996 & 1361 & \cdots \\ 232 & 380 & 580 & 840 & 1168 & 1572 & \cdots \\ 299 & 472 & 701 & 994 & 1359 & 1804 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequences for the first two columns are A000125 and A060163. The equation for the entries is

$$(\mathfrak{BDiag}_4^{1,\mathcal{D}_0})_{k,l} = \frac{\big(k+2(l+1)\big)^3 + 5\big(k+2(l+1)\big) + 6(2l-1)}{6}, \quad k,l \geq 1.$$

5 Acknowledgments

We would like to thank the anonymous referee for his valuable suggestions. This research was made within the activities of the Group for Linear, Algebraic and Combinatorial Structures of the Center for Functional Analysis, Linear Structures and Applications (University of Lisbon, Portugal). The first and the second authors were supported by the research fellowships SFRH/BPD/100567/2014 and SFRH/BPD/111317/2015, respectively, provided by the Portuguese Foundation for Science and Technology (FCT).

References

[1] R. Bacher, Sur le groupe d'interpolation, preprint, 2006. Available at https://arxiv.org/abs/math/0609736.

[2] N. T. Cameron and A. Nkwanta, On some (pseudo) involutions in the Riordan group, *J. Integer Sequences* **8** (2005), Article 05.3.7.

[3] G.-S. Cheon, H. Kim, and L. W. Shapiro, Riordan group involutions, *Linear Algebra Appl.* **428** (2008) 941–952.

[4] G.-S. Cheon, S.-T. Jin, H. Kim, and L. W. Shapiro, Riordan group involutions and the Δ-sequence, *Discrete Appl. Math.* **157** (2009) 1696–1701.

[5] G.-S. Cheon, H. Kim, and L. W. Shapiro, The hitting time subgroup, Łukasiewicz paths and Faber polynomials, *European J. Combin.* **32** (2011) 82–91.

[6] G.-S. Cheon and S.-T. Jin, Structural properties of Riordan matrices and extending the matrices, *Linear Algebra Appl.* **435** (2011) 2019–2032.

[7] T.-X. He, L. C. Hsu, and P. J.-S. Shiue, The Sheffer group and the Riordan group, *Discrete Appl. Math.* **155** (2007) 1895–1909.

[8] T.-X. He and L. W. Shapiro, Row sums and alternating sums of Riordan arrays, *Linear Algebra Appl.* **507** (2016) 77–95.

[9] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.* **309** (2009) 3962–3974.

[10] C. Jean-Louis and A. Nkwanta, Some algebraic structure of the Riordan group, *Linear Algebra Appl.* **438** (2013) 2018–2035.

[11] A. Luzón and M. A. Morón, Recurrence relations for polynomial sequences via Riordan matrices, *Linear Algebra Appl.* **433** (2010) 1422–1446.

[12] A. Luzón, M. A. Morón, and L. F. Prieto-Martínez, A formula to construct all involutions in Riordan matrix groups, *Linear Algebra Appl.* **533** (2017) 397–417.

[13] A. Luzón, D. Merlini, M. A. Morón, and R. Sprugnoli, Identities induced by Riordan arrays, *Linear Algebra Appl.* **436** (2012) 631–647.

[14] A. Luzón, D. Merlini, M. A. Morón, and R. Sprugnoli, Complementary Riordan arrays, *Discrete Appl. Math.* **172** (2014) 75–87.

[15] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, On some alternative characterizations of Riordan arrays, *Canad. J. Math.* **49** (1997) 301–320.

[16] D. Merlini and M. C. Verri, Generating trees and proper Riordan arrays, *Discrete Math.* **218** (2000) 167–183

[17] P. Peart and W. Woan, A divisibility property for a subgroup of Riordan matrices, *Discrete Appl. Math.* **3** (2000) 121–128.

[18] L. Shapiro, Some open questions about random walks, involutions, limiting distributions, and generating functions, *Adv. in Appl. Math.* **27** (2001) 585–596.

[19] L. W. Shapiro, Bijections and the Riordan group, *Theoret. Comput. Sci.* **307** (2003) 403–413.

[20] L. W. Shapiro, S. Getu, W. Woan, and L. Woodson, The Riordan group, *Discrete Appl. Math.* **34** (1991) 229–239.

[21] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org, 2018.

[22] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994) 267–290.

[23] S.-L. Yang, Some inverse relations determined by Catalan matrices, *Int. J. Comb.* (2013) Art. ID 528584.

2010 *Mathematics Subject Classification*: Primary 05A15; Secondary 05B20, 05A19, 15B36.
Keywords: Riordan group, Riordan pseudo-involution, automorphism, infinite array, Z-sequence, $B(x)$-sequence.

(Concerned with sequences A000027, A000079, A000124, A000125, A000127, A000217, A000292, A000384, A000389, A000580, A000582, A001288, A001519, A002492, A002662, A002664, A005408, A005843, A006261, A008859, A008860, A010966, A010968, A014105, A025581, A035039, A035041, A053126, A053127, A053128, A053129, A059993, A060163, A088305, A114284, A116722, A130883, A145018, A152948, A152950, A165747, A167499, A177787, A201347, and A220074.)

Received April 2 2019; revised versions received November 4 2019; December 24 2019. Published in *Journal of Integer Sequences*, December 26 2019.

Return to Journal of Integer Sequences home page.