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Infinite Sets of b-Additive and b-Multiplicative Ramanujan-Hardy Numbers

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Abstract

Let b a numeration base. A b-additive Ramanujan-Hardy number N is an integer for which there exists at least one integer M, called the additive multiplier, such that the product of M and the sum of base-b digits of N, added to the reversal of the product, gives N. We show that for any b there exist infinitely many b-additive Ramanujan-Hardy numbers and infinitely many additive multipliers. A b-multiplicative Ramanujan-Hardy number N is an integer for which there exists at least an integer M, called the multiplicative multiplier, such that the product of M and the sum of base-b digits of N, multiplied by the reversal of the product, gives N. We show that for $b \equiv 4 \pmod 6$, and for $b = 2$, there exist infinitely many b-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers. If b even, $b \equiv 0 \pmod 3$ or $b \equiv 2 \pmod 3$, we show there exist infinitely many numeration bases for which there exist infinitely many b-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers.

These results completely answer two questions and partially answer two other questions asked in a previous paper of the author.

1 Introduction

Let $b \geq 2$ be a numeration base. In Niţică [6], motivated by some properties of the taxicab number, 1729, we introduce the classes of *b-additive Ramanujan-Hardy (or b-ARH) numbers* and *b-multiplicative Ramanujan-Hardy (or b-MRH) numbers*. The first class consists of numbers N for which there exists at least an integer M, called the *additive multiplier*, such that the product of M and the sum of base-b digits of N, added to the reversal of the product, gives N. The second class consists of numbers N for which there exists at least an integer M, called the *multiplicative multiplier*, such that the product of M and the sum of base-b digits of N, multiplied by the reversal of the product, gives N.

It is asked [6, Question 6] if the set of b-ARH numbers is infinite and it is asked [6, Question 8] if the set of additive multipliers is infinite. It is shown [6, Theorems 12 and 15] that the answer is positive if b is even. The case b odd is left open. It is asked [6, Question 7] if the set of b-MRH numbers is infinite and it is asked [6, Question 9] if the set of multiplicative multipliers is infinite. It is shown [6, Theorem 30] that the answer is positive if b is odd. The case b even is left open.

We recall that *Niven (or Harshad) numbers* are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied. See, for instance, Cai [1], Cooper and Kennedy [2], De Koninck and Doyon [3], and Grundman [4]. Of interest are also b-Niven numbers, which are numbers divisible by the sum of their base-b digits. See, for example, Fredricksen, Ionaşcu, Luca, and Stănică [5]. A b-MRH-number is a b-Niven number. High degree b-Niven numbers are introduced in [7].

The goal of this paper is to show that, for any numeration base, there exist infinitely many b-ARH numbers and infinitely many distinct additive multipliers. We also show that, for $b \equiv 4 \pmod 6$, and for $b = 2$, there exist infinitely many b-MRH numbers, and infinitely many distinct multiplicative multipliers. If b even, $b \equiv 0 \pmod 3$ or $b \equiv 2 \pmod 3$, we show there are infinitely many numeration bases for which there exist infinitely many b-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers. These results completely answer the first two questions from [6] revisited above, and partially answer the other two. We observe that a trivial example of infinitely many b-MRH numbers is given by the powers of 10. Our examples have at least two digits different from zero. Finding infinitely many b-MRH numbers with all digits different from zero remains an open question.

Our results about b-ARH numbers also give solutions to the Diophantine equation $N \cdot M = \text{reversal}(N \cdot M)$. Motivated by this link, we show that the Diophantine equation has solution for all integers N not divisible by the numeration base b. We do not know how to answer the following related question:

Question 1. Does there exist, for any integer N, an integer M such that $N \cdot M$ is a b-ARH number (or a b-MRH number, or a b-Niven number)?

Our final result shows that for any string of digits I there exist infinitely many b-Niven numbers that contain I in their base-b representation. We do not know a similar result for the classes of b-ARH and b-MRH numbers.

2 Statements of the main results

Let $s_b(N)$ denote the sum of base-b digits of integer N. If x is a string of digits, let $(x)^{\wedge k}$ denote the base-10 integer obtained by repeating x k-times. Let $[x]_b$ denote the value of the string x in base b.If N is an integer, let N^R denote the reversal of N, that is, the number obtained from N writing its digits in reverse order. The operation of taking the reversal is dependent on the base. In the definition of a b-ARH-number/b-MRH number N we take the reversal of the base-b representation of $s_b(N)M$.

Theorem 2. *Let $\alpha \geq 1$ integer, $b \geq \alpha+1$ integer, and $k = (1+\alpha)^\ell, \ell \geq 0$. Assume $b \equiv 2+\alpha \pmod{2+2\alpha}$. Define*

$$N_k = [(1\alpha)^{\wedge k}]_b.$$

Then there exists $M \geq 0$ integer such that

$$s_b(N_k) \cdot M = (s_b(N_k) \cdot M)^R = \frac{N_k}{2}.$$

In particular, the numbers $N_k, k \geq 1$, are b-ARH numbers and b-Niven numbers.

The proof of Theorem 2 is done in Section 3.

Remark 3. The particular case $b = 10, \alpha = 2$, of Theorem 2, which gives $N_k = (12)^{3^\ell}$, is covered by [6, Example 10]. Theorem 2 does not give any information if $b = 2$.

The following proposition gives positive answers to [6, Questions 5 and 6].

Proposition 4. *For any $b \geq 2$, there exist infinitely many b-ARH numbers and infinitely many additive multipliers. The b-ARH numbers are also b-Niven numbers.*

The proof of Proposition 4 is done in Section 4.

Remark 5. Note that [6, Theorems 12 and 15] show, for all even bases, infinitely many b-ARH numbers that are not b-Niven numbers. The case of odd base is open. The question of finding infinitely many b-Niven numbers that are not b-ARH numbers is also open. It is shown in [6, Theorem 28] that for any base there exist infinitely many numbers that are not b-ARH numbers.

The result in Theorem 2 gives many base-10 solutions for the equation:

$$N \cdot M = (N \cdot M)^R. \tag{1}$$

One can try to solve the equation (1) ,where $(N \cdot M)^R$ is the reversal of $N \cdot M$ written in base b, for any numeration base b.

Observe that if N is divisible by b, then $(N \cdot M)^R$ has less digits then $N \cdot M$, therefore N is not a solution of (1). Note also that if $N = N^R$ and N has k digits then (1) always has an infinite set of solutions with

$$M = [(1(0)^{\wedge\ell})^{\wedge p}1]_b, \ell \geq k-1, p \geq 0.$$

Consequently, if (N_0, M_0) is a solution of (1), then (1) has infinite sets of solutions of types (N_0, M) and (N, M_0).

Theorem 6. *Let $b \geq 2$ and $N \geq 1$ integer such that $b \nmid N$. Then N is a solution of* (1).

The proof of Theorem 6 is done in Section 5. For base 10, a proof belonging to David Radcliffe can be found at [8]. We learned about this reference from J. Shallit. We generalize the proof for an arbitrary numeration base. After our paper was written, we learned from J. Shallit [9] that he also has a proof of Theorem 6.

A *b-numeric palindrome* is a base-b integer N such that $N = N^R$.

Corollary 7. *All integers, not divisible by b, are factors of b-numeric palindromes.*

Definition 8. The *multiplicity* of a multiplicative multiplier M is the number of (N, M) solutions of (1).

It was observed above that for any solution (N, M) of (1), M has infinite multiplicity. The following theorem shows infinitely many solutions of (1) independent of above.

Theorem 9. *Let $b \geq 2$ a numeration base. Then, for all $k \geq 0$, we have*

$$[1(b-1)]_b \cdot [(b-1)^{\wedge k}]_b = [1(b-2)(b-1)^{\wedge k-2}(b-2)1]_b.$$

The proof of Theorem 9 is done in Section 6.

Our next results show, for b even, more examples of infinite sets of of b-ARH.

Theorem 10. *Let $b \geq 2$ even. Let $a \in \{1, 2, \ldots, b-1\}$ and let $k \geq 0$ be an integer.*
(a) Let

$$N_k = [a(0)^{\wedge k}a]_b.$$

Then N_k is a b-ARH number, but not a b-Niven number.

(b) Let

$$N_k = [\left(1(0)^{\wedge k}\right)^{\wedge b} 0 \left((0)^{\wedge k}1\right)^{\wedge b}]_b.$$

Then N_k is a b-ARH number, but not a b-Niven number.

(c) Let

$$N_k = [\left((0)^{\wedge k}1\right)^{\wedge b} 0 \left(1(0)^{\wedge k}\right)^{\wedge b}]_b.$$

Then N_k is a b-ARH number and a b-Niven number.

The proof of Theorem 10 is done in Section 7.

The following theorem gives partial answers to [6, Questions 7 and 8].

Theorem 11.
(a) Let $b \equiv 4 \pmod 6$. Let $k \geq 1$ integer such that $k \equiv 1 \pmod 3$. Define

$$\alpha_k = [1(0)^{\wedge k}(b-2)]_b.$$

Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a b-MRH number.

(b) Let $b = 2$ and let $k \geq 1$ be an even integer. Define

$$\alpha_k = [1(0)^{\wedge k}1]_2.$$

Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a b-MRH number.

In particular, for any numeration base b, $b \equiv 4 \pmod 6$, and for $b = 2$, there exist infinitely many b-MRH numbers and infinitely many multipliers.

The proof of Theorem 11 is done in Section 8.

Our next result lists several infinite sequences of 10-MRH-numbers.

Proposition 12. *Assume $k \geq 1$ integer and define $N_k = \alpha_k \cdot (\alpha_k)^R$, where α_k is one of the following numbers:*

- $[1(0)^{\wedge k}8]_{10}, k \equiv 1 \pmod 3$,
- $[7(0)^{\wedge k}2]_{10}$,
- $[5(0)^{\wedge k}4]_{10}$,
- $[4(0)^{\wedge k}5]_{10}$

Then N_k is a 10-MRH number.

The first item in Proposition 12 follows as a corollary of Theorem 11. The other items can be proved using the same approach as in the proof of Theorem 11.

If b even, $b \equiv 0 \pmod 3$ or $b \equiv 2 \pmod 3$, the next theorem shows there are infinitely many numeration bases for which there exist infinitely many b-MRH numbers and infinitely many multipliers.

Theorem 13.
(a) Let $b \geq 18$, $b = 6a$, and $a \equiv 1 \pmod{25}$. Let $\alpha_k = [1(0)^{\wedge k}4]_b$ with $k \equiv 4 \pmod 5$. Then $N_k = \alpha_k \cdot \alpha_k^R$ is a b-MRH number. The corresponding multipliers are distinct.

(b) Let $b \geq 18$, $b = 8a$, $a \equiv 1 \pmod{25}$, and $a \equiv 1 \pmod 3$. Let $\alpha_k = [1(0)^{\wedge k}4]_b$ with $k \equiv 4 \pmod{20}$. Then $N_k = \alpha_k \cdot \alpha_k^R$ is a b-MRH number. The corresponding multipliers are distinct.

The proof of Theorem 13 is done in section 9.

Theorem 14. *For any base b and for any string of base b digits I there exist infinitely many b-Niven numbers that contain the string I in their base-b representation.*

Proof. Let I be a string of base-b digits. There exist infinitely many base-b strings J such that $s_b([IJ]_b)$ is a power of b, say $b^k, k \geq 1$. Then the number $N_J = [IJ(0)^{\wedge k}]_b$ is a b-Niven number. □

3 Proof of Theorem 2

Proof. The condition $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$ implies that $b + \alpha$ is even. The base-b representation for $N_k/2$ is $N_k/2 = \left[\left(0\frac{b+\alpha}{2}\right)^{\wedge k}\right]_b$. One has that:

$$s_b(N_k) = k \cdot (1 + \alpha) = (1 + \alpha)^{\ell+1}. \tag{2}$$

The value of $N_k/2$ in base 10 is obtained summing a geometric series.

$$\begin{aligned} \frac{N_k}{2} &= \frac{b+\alpha}{2} \cdot b^{2k-2} + \frac{b+\alpha}{2} \cdot b^{2k-4} + \cdots + \frac{b+\alpha}{2} \cdot b^2 + \frac{b+\alpha}{2} = \frac{b+\alpha}{2} \cdot \frac{b^{2k}-1}{b^2-1} \\ &= \frac{b+\alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell}-1}{b^2-1}. \end{aligned} \tag{3}$$

Note that $N_k/2 = (N_k/2)^R$. We finish the proof of the theorem if we show that:

$$(1+\alpha)^{\ell+1} \Big| \frac{b+\alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell}-1}{b^2-1}. \tag{4}$$

We prove (4) by induction on ℓ. For $\ell = 0$ equation (4) becomes $1 + \alpha \Big| \frac{b+\alpha}{2}$, which is true because $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$.

Now we assume that (4) is true for ℓ and show that it is true for $\ell + 1$.

$$\begin{aligned} \frac{b+\alpha}{2} \cdot \frac{b^{2(1+\alpha)^{\ell+1}}-1}{b^2-1} &= \frac{b+\alpha}{2} \cdot \frac{\left(b^{2(1+\alpha)^\ell}\right)^{1+\alpha}-1}{b^2-1} \\ = \frac{b+\alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell}-1}{b^2-1} &\left(B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1\right), \end{aligned} \tag{5}$$

where

$$B = b^{2(1+\alpha)^\ell}. \tag{6}$$

The congruence $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$ implies that

$$b^2 \equiv (2+\alpha)^2 \equiv \alpha^2 + 4\alpha + 4 \equiv \alpha^2 \equiv 1 \pmod{1+\alpha},$$

which implies that

$$b^m \equiv 1 \pmod{1+\alpha},\ m \text{ even}. \tag{7}$$

From (6) and (7) follows that $B^p \equiv 1 \pmod{1+\alpha}, 1 \le p \le \alpha$, so

$$1 + \alpha | B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1. \tag{8}$$

Combining (4) (for ℓ) and (8), and using (5), it follows that (4) is true for $\ell + 1$. ◻

4 Proof of Proposition 4

Proof. The case $b = 2$ is covered by [6, Theorem 12]. If $b \geq 3$, choose $\alpha = b - 2$ and apply Theorem 2. We show that, for a fixed b, the multipliers appearing in the proof of Theorem 2 are all distinct. It follows from (2) and (3) that the multiplier for N_k is given by:

$$M = \frac{\frac{N_k}{2}}{s_b(N_k)} = \frac{\frac{b+\alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell}-1}{b^2-1}}{(1+\alpha)^{\ell+1}}. \tag{9}$$

Note that $\alpha = b - 2$. After algebraic manipulations, equation (9) becomes

$$M = \frac{b^{2(1+\alpha)^\ell} - 1}{(b-1)^\ell(b^2-1)}.$$

In order to show that the multipliers are distinct it is enough to show that the sequence of multipliers is strictly increasing as a function of ℓ That is, we need to show that:

$$\frac{b^{2(1+\alpha)^\ell} - 1}{(b-1)^\ell(b^2-1)} < \frac{b^{2(1+\alpha)^{\ell+1}} - 1}{(b-1)^{\ell+1}(b^2-1)}. \tag{10}$$

After algebraic manipulations (10) becomes

$$(b-1)(b^{2(1+\alpha)^\ell} - 1) < b^{2(1+\alpha)^{\ell+1}} - 1. \tag{11}$$

After denoting

$$B = b^{2(1+\alpha)^\ell} = b^{2(b-1)^\ell},$$

right hand side of (11) factors as follows:

$$b^{2(1+\alpha)^{\ell+1}} - 1 = (b^{2(1+\alpha)^\ell} - 1)(B^\alpha + B^{\alpha-1} + \cdots + B + 1). \tag{12}$$

Now (11) follows from (12) and the following inequality:

$$b - 1 < b^{2(b-1)^\ell}, \ell \geq 0, \ell \geq 0, b \geq 3.$$

□

5 Proof of Theorem 6

Proof. Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \alpha_i \geq 1$, p_i prime, $1 \leq i \leq k$. We recall that a base-b integer N is divisible by p_i^γ if the last γ digits of N form a base-b integer divisible by p_i^γ. Let $N = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} w$, where $\gcd(w, b) = 1$. Let $m = \max(\beta_1, \beta_2, \cdots, \beta_k)$. Let L be the base-b integer equal to $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$. As $b \not| N$, the last digit of L is not 0. Let ℓ be the length of

L. Consider the base-b palindrome $P = [L^R(0)^{\wedge m-\ell}L]_b$, where L^R is the reversal of base-b representation of L. As P is divisible by $p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$, this is the end of the proof if $w = 1$.

Assume $w > 1$. Let ϕ be Euler's totient function which counts the positive integers up to a given integer n that are relatively prime to n. As $\gcd(w, b) = 1$ Euler's theorem implies that $b^{\phi(w)} - 1 \equiv 0 \pmod{w}$.

Let r be an multiple of $\phi(w)$ which is greater than $l + m$, the length of P. Let $q \geq 1$ a multiple of $b^{\phi(w)} - 1$. Consider the infinite family of integers given by

$$\begin{aligned} Q_{r,q} &= \left[1\left((0)^{\wedge r-1}1\right)^{\wedge q}\right]_b = 1 + b^r + b^{2r} + \cdots + b^{qr} \\ &= 1 + b^r + b^{2r} + \cdots + b^{qr} + q - q \\ &= \left(b^r - 1\right) + \left(b^{2r} - 1\right) + \left(b^{3r} - 1\right) + \cdots + \left(b^{qr} - 1\right) + q. \end{aligned} \tag{13}$$

All terms in the last part of (13) are divisible by $b^{\phi(w)} - 1$, so $Q_{r,q}$ is divisible by $b^{\phi(w)} - 1$ and by w. We finish the proof observing that $P \cdot Q_{r,q}$ is a base-b palindrome divisible by N. □

6 Proof of Theorem 9

Proof. Observe that:

$$\begin{aligned} (b-1)\cdot(b-1) = b(b-2) + 1 = [(b-2)1]_b \\ (b-1)b^k + (b-1)b^k = b^k + (b-2)b^{k-1} = [1(b-2)0^{\wedge k}]_b. \end{aligned} \tag{14}$$

Using (14) we get

$$\begin{aligned} [1(b-1)]_b \cdot [(b-1)^{\wedge k}]_b &= (b + b - 1)\cdot\left(\sum_{i=0}^{k-1}(b-1)b^i\right) \\ &= \sum_{i=0}^{k-1}\left((b-1)b^{i+1} + (b(b-2)+1)\,b^i\right) \\ &= \sum_{i=1}^{k}(b-1)b^i + \sum_{i=0}^{k-1}(b(b-2)+1)\,b^i \\ &= (b-1)b^k + \sum_{i=1}^{k-1}\Big((b-1) + b(b-2) + 1\Big)b^i + b(b-2) + 1 \\ &= (b-1)b^k + \sum_{i=1}^{k-1}(b-1)b^{i+1} + b(b-2) + 1 \\ &= (b-1)b^k + (b-1)b^k + \sum_{i=1}^{k-2}(b-1)b^{i+1} + b(b-2) + 1 \end{aligned}$$

$$= b^k + (b-2)b^{k-1} + \sum_{i=1}^{k-2}(b-1)b^{i+1} + b(b-2) + 1$$
$$= [1(b-2)(b-1)^{\wedge k-2}(b-2)1]_b.$$

□

7 Proof of Theorem 10

Proof. (a) Note that $s_b(N_k) = 2a$. As b is even, there exists an integer M such that:

$$2a \cdot M = [a(0)^{\wedge k+1}]_b.$$

The following computation shows that N_k is a b-ARH number:

$$s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R = [a(0)^{\wedge k+1}]_b + [a]_b = [a(0)^{\wedge k}a]_b = N_k.$$

To show that N_k is not b-Niven observe that $N_k/a = [1(0)^{\wedge k}1]_b$ is odd.

(b) Note that $s_b(N_k) = 2b$. As b is even, the multiplier $M = [(1(0)^{\wedge k})^{\wedge b}(0)^{\wedge kb+b-1}]_b/2$ is an integer.

The following computation shows that N_k is a b-ARH number:

$$s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R$$
$$= [(1(0)^{\wedge k})^{\wedge b}(0)^{\wedge kb+b}]_b + [((0)^{\wedge k}1)^{\wedge b}]_b = [(1(0)^{\wedge k})^{\wedge b}0((0)^{\wedge k}1)^{\wedge b}]_b = N_k.$$

To show that N_k is not b-Niven observe that N_k is not divisible by b.

(c) The proof is similar to that of b). □

8 Proof of Theorem 11

Proof. (a) Using the fact that

$$(b-2)^2 = b^2 - 4b + 4 = b(b-4) + 4 = [(b-4)4]_b,$$

an equivalent base-b representation for N_k is given by

$$N_k = \begin{cases} [(b-2)(0)^{\wedge k-1}(b-4)5(0)^{\wedge k}(b-2)]_b, & \text{if } b \neq 4; \\ [2(0)^{\wedge k-1}11(0)^{\wedge k}2]_4, & \text{if } b = 4. \end{cases} \tag{15}$$

If $b \neq 4$ one has $s_b(N_k) = 3(b-1)$ and if $b = 4$ one has $s_4(N_k) = 6$. To finish the proof of case a) it is enough to show that α_k is divisible by $s_b(N_k)$.

If $b \neq 4$ we get

$$\alpha_k = b^{k+1} + b - 2 = b^{k+1} - 1 + b - 1 = (b-1)\big(b^k + b^{k-1} + \cdots + b^2 + b + 2\big)$$

and

$$b^k + b^{k-1} + \cdots + b^2 + b + 2 \equiv k + 2 \equiv 0 \pmod 3.$$

For the first congruence we used $b \equiv 1 \pmod 3$ and for the second we used $k \equiv 1 \pmod 3$.

If $b = 4$, then clearly α_k is divisible by 2. Moreover

$$\alpha_k = 4^{k+1} + 2 = (3+1)^{k+1} + 2 \equiv 0 \pmod 3,$$

which shows that α_k is divisible by 6.

(b) Now assume that $b = 2$. Then an equivalent base-2 representation for N_k is given by

$$N_k = [1(0)^{\wedge k-1}10(0)^{\wedge k}1]_2,$$

so $s_2(N_k) = 3$. To finish the proof, we use the fact that k is even to show that α_k is divisible by 3:

$$\alpha_k = 2^{k+1} + 1 = (3-1)^{k+1} + 1 \equiv 0 \pmod 3.$$

To prove the last claim in the theorem, we show that the multipliers corresponding to various values of k are distinct. This follows from the explicit formulas below. All sequences of multipliers are strictly increasing as functions of k.

If $b = 2$ the sequence of multipliers is given by $M_k = \frac{2^{k+1}+1}{3}$.

If $b = 4$ the sequence of multipliers is given by $M_k = \frac{4^{k+1}+2}{6}$.

If $b > 4$ the sequence of multipliers is given by $M_k = \frac{b^{k+1}+b-2}{3(b-1)}$.

□

9 Proof of Theorem 13

Proof. (a) The base-b representation for N_k is

$$N_k = [4(0)^{\wedge k-1}(17)(0)^{\wedge k}4]_b.$$

Therefore $s_b(N_k) = 25$. If $k = 5\ell + 4$. one has that:

$$\alpha_k = 6^k a^k + 4 \equiv (6^5)^\ell 6^4 + 4 \equiv (7776)^\ell \cdot 296 + 4 \equiv 0 \pmod{25}.$$

Hence N_k is a b-MRH number with multiplier $\frac{\alpha_k}{25} = \frac{(6a)^k+4}{25}$.

(b) As above, $s_b(N_k) = 25$. If $k = 20\ell + 4$, one has that:

$$\alpha_k = 8^k a^k + 4 \equiv (8^{20})^\ell 8^4 + 4 \equiv (76)^\ell \cdot 96 + 4 \equiv 0 \pmod{25}.$$

Hence N_k is a b-MRH number with multiplier $\frac{\alpha_k}{25} = \frac{(8a)^k+4}{25}$. □

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References

[1] T. Cai, On 2-Niven numbers and 3-Niven numbers, *Fibonacci Quart.*, **34** (1996), 118–120.

[2] C. N. Cooper and R. E. Kennedy, On consecutive Niven numbers, *Fibonacci Quart.*, **21** (1993), 146–151.

[3] J. M. De Koninck and N. Doyon, Large and small gaps between consecutive Niven numbers, *J. Integer Sequences* **6** (2003), Article 03.2.5.

[4] H. G. Grundman, Sequences of consecutive Niven numbers, *Fibonacci Quart.*, **32** (1994), 174–175.

[5] H. Fredricksen, E. J. Ionaşcu, F. Luca, and P. Stănică, Remarks on a sequence of minimal Niven numbers, in S. W. Golomb, G. Gong, T. Helleseth and H.-Y. Song, eds., *Sequences, Subsequences, and Consequences*, Lecture Notes in Comp. Sci., Vol. 4893, Springer, 2007, pp. 162–168.

[6] V. Niţică, About some relatives of the taxicab number, *J. Integer Sequences*, **21** (2018), Article 18.9.4.

[7] V. Niţică, High degree b-Niven numbers, to appear in *Integers*. Preprint available at `http://arxiv.org/abs/1807.02573`, July 6 2018.

[8] D. Radcliffe, World of numbers, `http://www.worldofnumbers.com/em36.htm`.

[9] J. Shallit, Palindromic multiples, preprint, 2018.

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