Journal of Integer Sequences, Vol. 7 (2004),
Article 04.1.6

# Partial Complements and Transposable Dispersions

Clark Kimberling and John E. Brown
Department of Mathematics
University of Evansville
1800 Lincoln Avenue
Evansville, IN 47722
USA
ck6@evansville.edu

**Abstract**

Suppose $A = \{a(i, j)\}$, for $i \geq 0$ and $j \geq 0$, is the dispersion of a strictly increasing sequence $r = (r(0), r(1), r(2), \ldots)$ of integers, where $r(0) = 1$ and infinitely many postive integers are not terms of $r$. Let $R$ be the set of such sequences, and define $t$ on $R$ by $tr(n) = a(0, n)$ for $n \geq 0$. Let $F$ be the subset of $R$ consisting of sequences $r$ satisfying $ttr = r$. The set $F$ is characterized in terms of ordered arrangements of numbers $i + j\theta$. For fixed $i \geq 0$, the sequence $a(i, j)$, for $j \geq 1$, is the $(i + 1)$st partial complement of $r$. Central to the characterization of $F$ is the role of the families of figurate (or polygonal) number sequences and the centered polygonal number sequences. Finally, it is conjectured that for every $r$ in $R$, the iterates $t^{(2m)}r$ converge to a sequence in $F$.

## 1 Introduction

Let $N$ denote the set of positive integers. Suppose $r = (r(0), r(1), r(2), \ldots)$ is a strictly increasing sequence of integers with $r(0) = 1$ and infinite complement in $N$. Let $r^*$ be the sequence obtained by arranging in increasing order the complement of $r$. Let

$$a(0, 0) = 1, \ a(0, 1) = r^*(1),$$

and inductively,

$$a(0, j) = r^*(a(0, j - 1))$$

for $j \geq 1$. The sequence $a(0, j)$, for $j \geq 1$, is the *1st partial complement* of $r$. For arbitrary $i \geq 0$, suppose that the sequence $a(i, j)$ for $j \geq 0$ is defined, and let $a(i+1, 0)$ be the least number in $N$ that is not $a(h, j)$ for any $(h, j)$ satisfying $0 \leq h \leq i$ and $j \geq 0$. Define

$$a(i+1, j) = r^*(a(i+1, j-1)) \tag{1}$$

for $j \geq 1$. In this inductive manner, an array $A = \{a(i, j)\}$, for $i \geq 0,\ j \geq 0$, is defined. It is called the *dispersion* of $r^*$. (In [3], where the terms dispersion and interspersion are introduced, the indexing is by $i \geq 1, j \geq 1$ instead of $i \geq 0,\ j \geq 0$.) For fixed $i \geq 0$ and variable $j \geq 1$, the sequence $a(i, j)$ is the $(i+1)$*st partial complement* of $r$.

To summarize, the dispersion $A$ of the complement of $r$, henceforth denoted by $A(r)$, consists of first column $r$ together with the terms of $r^*$ dispersed into partial complements; row $i$ of $A(r)$ consists of first term $r(i)$ followed by the $(i+1)$st partial complement of $r$. It is sometimes desirable to use a recurrence for $a(i+1, j)$ that refers directly to $r$. To develop such a recurrence, for any strictly increasing sequence $\rho$ in $N$, let $\#(\rho(n) \leq m)$ denote the number of $n \geq 1$ satisfying $\rho(n) \leq m$. Clearly, $\rho(h)$ is the *least* $m$ satisfying $\#(\rho(n) \leq m) = h$. Now take $\rho = r^*$ and $h = a(i, j)$ to see that

$$a(i, j+1) = \text{ least } m \text{ satisfying } \Big(\#(r^*(n) \leq m) = a(i, j)\Big).$$

As $\#(r(n) \leq m) + \#(r^*(n) \leq m) = m$, we obtain

$$\begin{aligned} a(i, j+1) &= \text{ least } m \text{ satisfying } \Big(m - \#(r(n) \leq m) = a(i, j)\Big) \\ &= \min \Big\{ m : \max\{n : r(n) \leq m\} = a(i, j) - m + 1 \Big\}. \end{aligned} \tag{2}$$

This recurrence is especially useful in coding computer programs that generate dispersions.

Let $R$ be the set of sequences $r$ described in the first sentence. Let $tr$ denote the first partial complement of $r$, and let $F$ be the family of sequences $r$ for which $ttr = t$. A main objective of this paper is to characterize $F$ in terms of sequences associated with multisets of the form

$$S_\theta = \{i + j\theta : i \geq 0,\ j \geq 0\}, \tag{3}$$

where $\theta$ is a positive real number. The numbers in $S_\theta$ are distinct if $\theta$ is irrational; otherwise, write $\theta = c/d$, where $c$ and $d$ are relatively prime positive integers. Then a number $i + j\theta$ occurs more than once in $S_\theta$ if and only if $i \geq c$, and in this case the representations of $i + j\theta$ are these:

$$i,\ i - c + d\theta,\ i - 2c + 2d\theta, \cdots,\ i - \lfloor i/c \rfloor c + \lfloor i/c \rfloor d\theta.$$

In order to treat these as distinct objects, we represent each $h + k\theta$ as an ordered pair $(h, k)$, so that the representations of $i + j\theta$ become

$$(i, 0),\ (i - c, d),\ (i - 2c, 2d), \cdots,\ (i - \lfloor i/c \rfloor c, \lfloor i/c \rfloor d).$$

Define an order relation $\prec$ on the set $\{(i, j) :\ i \geq 0,\ j \geq 0\}$ by

$$\begin{aligned} &(i_1, j_1) \prec (i_2, j_2) \text{ if } i_1 + j_1\theta < i_2 + j_2\theta \\ &(i_1, j_1) \prec (i_2, j_2) \text{ if } i_1 + j_1\theta = i_2 + j_2\theta \text{ and } j_1 < j_2. \end{aligned}$$

We extend this definition to the case that $\theta$ is irrational, noting that the condition $i_1 + j_1\theta = i_2 + j_2\theta$ does not occur. Now for any real $\theta > 0$, the *rank array of* $\theta$ is defined by

$$A_\theta = \{a_\theta(i,j) : i \geq 0,\ j \geq 0\},$$

where

$$a_\theta(i,j) = \text{ rank of } (j,i) \text{ under } \prec .$$

(Defining $a_\theta(i,j)$ as the rank of $(j,i)$ rather than that of $(i,j)$ facilitates later developments, such as equation (13) and connections with Farey sequences. For irrational $\theta$, the condition "rank of $(j,i)$ under $\prec$" can be replaced by "rank of $j + i\theta$ under [ordinary] $<$"; in this case, $r_\theta(n)$ is simply the rank of $n\theta$ among all the numbers $h + k\theta$.)

As an example, let $c = 4$ and $d = 3$. Then

$$\begin{aligned}
&(0,0) \prec (1,0) \prec (0,1) \prec (2,0) \prec (1,1) \prec (0,2) \prec (3,0) \prec (2,1) \prec (1,2) \prec \\
&(4,0) \prec (0,3) \prec (3,1) \prec (2,2) \prec (5,0) \prec (1,3) \prec (4,1) \prec (0,4) \prec (3,2) \prec \\
&(6,0) \prec (2,3) \prec (5,1) \prec (1,4) \prec (4,2) \prec (0,5) \prec (7,0) \prec (3,3) \prec (6,1) \prec \\
&(2,4) \prec (5,2) \prec (1,5) \prec (8,0) \prec (4,3) \prec (0,6) \prec \cdots
\end{aligned}$$

Numbering these from 1 to 33 shows that the rank array $A_{4/3}$ starts like this:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 10 | 14 | 19 | 25 | 31 |
| 3 | 5 | 8 | 12 | 16 | 21 | 27 | | |
| 6 | 9 | 13 | 18 | 23 | 29 | | | |
| 11 | 15 | 20 | 26 | 32 | | | | |
| 17 | 22 | 28 | | | | | | |
| 24 | | | | | | | | |
| 33 | | | | | | | | |

**Theorem 1.1** *Suppose $\theta > 0$. Then $A_\theta$ is an interspersion.*

**Proof:** We shall show that each of the propositions (I1)-(I4) that define an interspersion in [3] is satisfied:

(I1) The rows of $A_\theta$ comprise a partition of $N$, as $N$ is the set of ranks of numbers in $S_\theta$, and these ranks are partitioned by the rows of $A_\theta$.

(I2) Every row of $A_\theta$ is an increasing sequence, as $(j,i) \prec (j+1,i)$ for all $i \geq 0$, $j \geq 0$.

(I3) Every column of $A_\theta$ is an increasing sequence, as $(j,i) \prec (j,i+1)$ for all $i \geq 0$, $j \geq 0$.

(I4) If $(u_j)$ and $(v_j)$ are distinct rows of $A_\theta$ and if $p$ and $q$ are indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$. That is, in the present context, if $i \neq i'$ and $a(i,p) < a(i',q) < a(i,p+1)$, or equivalently,

$$(p,i) \prec (q,i') \prec (p+1,i),$$

then

$$(p+1,i) \prec (q+1,i') \prec (p+2,i),$$

or equivalently, $a(i,p+1) < a(i',q+1) < a(i,p+2)$. $\square$

**Corollary 1.1** *For $i \geq 0$, let $s$ be the sequence obtained by deleting the initial term of row $i$ of $A_\theta$. Then $s$ is the $(i+1)$st partial complement of column 0 of $A_\theta$.*

**Proof:** By Theorem 1 of [3], the array $A_\theta$ is a dispersion, and as shown in the proof in [3],

$$a(i, j+1) = t(a(i,j)),$$

where $(t(k))$ denotes the sequence of numbers, arranged in increasing order, in the set $N \backslash \{a(0,j) : j \geq 0\}$, this set being the complement of column 0 of $A_\theta$. $\square$

Let $r_\theta$ denote column 0 of $A_\theta$. By Corollary 1, row 0 of $A_\theta$ is the sequence $tr_\theta$, a property to be used in the next proof.

**Corollary 1.2** *Suppose $\theta$ is a positive irrational number. Then $tr_\theta = r_{1/\theta}$ and $ttr_\theta = r_\theta$.*

**Proof:** Referring to the array $A_{1/\theta}$, we have, for $m \geq 0$ and $n \geq 0$,

$$\begin{aligned} a_{1/\theta}(m,n) &= (\text{rank of } m + n/\theta \text{ in } S_{1/\theta}) \\ &= (\text{rank of } m\theta + n \text{ in } S_\theta) \\ &= a_\theta(n,m). \end{aligned}$$

That is, $A_{1/\theta}$ is the transpose of $A_\theta$. Consequently, $A_{1/(1/\theta)} = A_\theta$, so that $ttr_\theta = r_\theta$. $\square$

A dispersion $\{a(i,j)\}$ is *transposable* if its transpose, $\{a(j,i)\}$, is a dispersion. According to Corollary 3, the dispersions of $A_{\sqrt{2}}$ and $A_{1/\sqrt{2}}$ are transposable; northwest corners of these arrays are shown in Example 1 We shall see in section 3 that there are transposable dispersions other than those given by the proof of Corollary 3.

**Example 1:** The arrays $A_\theta$ and $A_{1/\theta}$ for $\theta = \sqrt{2}$

$$\begin{array}{cccccc} 1 & 2 & 4 & 7 & 10 & \ldots \\ 3 & 5 & 8 & 12 & 16 & \ldots \\ 6 & 9 & 13 & 18 & 23 & \ldots \\ 11 & 15 & 20 & 26 & 32 & \ldots \\ 17 & 22 & 28 & 35 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{cccccc} 1 & 3 & 6 & 11 & 17 & \ldots \\ 2 & 5 & 9 & 15 & 22 & \ldots \\ 4 & 8 & 13 & 20 & 28 & \ldots \\ 7 & 12 & 18 & 26 & 35 & \ldots \\ 10 & 16 & 23 & 32 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Initial terms of $tr$ can be written out from initial terms of $r$, for any $r$ in $R$, by means of a handly little algorithm. We use the sequence $r = (1, 2, 4, 7, 10, \ldots)$ of column 0 of $A_{\sqrt{2}}$ to exemplify:

**Step 1.** Write initial terms of the complement, $r^*$, with counting numbers above:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \hline 3 & 5 & 6 & 8 & 9 & 11 & 12 & 13 & 15 & 16 & 17 & 18 & 20 & 21 \\ \hline \end{array}$$

**Step 2.** Determine the chain $1 \to 3 \to 6 \to 11 \to 17 \to \cdots$, as indicated by the paired rectangles. These numbers form the sequence $tr$, alias row 0 of $A_{\sqrt{2}}$, alias (after the initial 1) the 1st partial complement of $r$. (Note that repeating the procedure starting with the pair $(2,5)$ yields the 2nd partial complement of $r$, and so on.)

If $\theta \in N$, then the sequences $r_\theta$ and $r_{1/\theta}$, we shall prove in section 2, are closely related to certain well-known sequences. We continue this introduction with an overview of those sequences, with reference to indexing systems in [5] and [6]. Sequences of polygonal (or figurate) numbers, typified by

$$\begin{aligned} P_1 &= (1,3,6,10,15,\ldots) = \text{triangular numbers (A000217 in [6], 253 in [5])} \\ P_2 &= (1,4,9,16,25,\ldots) = \text{square numbers (A000290 in [6], 338 in [5])} \\ P_3 &= (1,5,12,22,35,\ldots) = \text{pentagonal numbers (A000326 in [6], 339 in [5])}, \end{aligned}$$

are given for $k \geq 1$ by

$$P_k(n) = k\binom{n+1}{2} + n + 1 \tag{4}$$

for $n \geq 0$. Sequences of central polygonal numbers,

$$\begin{aligned} Q_1 &= (1,3,7,13,21,\ldots) = \text{central polygonal numbers (A002061 in [6])} \\ Q_2 &= (1,4,10,19,31,\ldots) = \text{central triangular numbers (A005448 in [6])} \\ Q_3 &= (1,5,13,25,41,\ldots) = \text{central square numbers (A001844 in [6])}, \end{aligned}$$

are given for $k \geq 1$ by

$$Q_k(n) = (k+1)\binom{n+1}{2} + 1 \tag{5}$$

for $n \geq 0$, so that

$$Q_k(n) = P_{k+1}(n) - n. \tag{6}$$

The sequences $P_k$ and $Q_k$ are easily expressed in terms of $P_1$ and $Q_1$:

$$\begin{aligned} P_k(n) &= P_1(n) + (k-1)P_1(n-1), \\ Q_k(n) &= Q_1(n) + (k-1)P_1(n-1), \end{aligned}$$

for $k \geq 1$, $n \geq 1$. For a colorful introduction to the geometry associated with the numbers $P_k$ and $Q_k$, see [1, pp. 38-42]. Consider next the sequences

$$\begin{aligned} \widehat{P}_1 &= (1,2,4,7,11,\ldots) = \text{lazy caterer sequence (A000124 in [6], 386 in [5])} \\ \widehat{P}_2 &= (1,2,3,5,7,\ldots) = \text{(A001401 in [6], 354 in [5])} \\ \widehat{P}_3 &= (1,2,3,4,6,\ldots) = \text{(A008748 in [6])} \end{aligned}$$

and

$$\begin{aligned}
\widehat{Q}_1 &= (1,2,4,6,9,\ldots) = \text{quarter-squares sequence (A002620 in [6], 105 in [5])}\\
\widehat{Q}_2 &= (1,2,3,5,7,\ldots) = \text{(A001840 in [6], 207 in [5])}\\
\widehat{Q}_3 &= (1,2,3,4,6,\ldots) = \text{(A001972 in [6], 208 on [5]).}
\end{aligned}$$

These typify families defined for $k \geq 1$ by

$$\widehat{P}_k(n) = \widehat{P}_k(n-1) + \left\lfloor \frac{n+k-1}{k} \right\rfloor, \tag{7}$$

$$\widehat{Q}_k(n) = \widehat{Q}_k(n-1) + \left\lfloor \frac{n+k+1}{k+1} \right\rfloor, \tag{8}$$

for $n \geq 1$, where $\widehat{P}_k(0) = \widehat{Q}_k(0) = 1$. For $k = 1$, the recurrence (7) gives $\widehat{P}_1(n) = \widehat{P}_1(n-1)+n$, whence by induction,

$$\widehat{P}_1(n) = P_1(n) - n. \tag{9}$$

Also, equation (7) implies that $\widehat{P}_k$ is the sequence obtained by arranging in increasing order the numbers in the set

$$\{1\} \cup \{P_k(n) + i(n+1) + 1 : 0 \leq i \leq k-1, n \geq 1\}. \tag{10}$$

Likewise, equation (8), or alternatively the identity

$$\widehat{Q}_k(n) = \widehat{P}_{k+1}(n+1) - 1, \tag{11}$$

can be used to prove that the sequence $\widehat{Q}_k$ results by ordering the numbers in the set

$$\{1\} \cup \{Q_k(n) + in + i - 1 : 0 \leq i \leq k, n \geq 1\}. \tag{12}$$

The sequences $\widehat{Q}_k$, for $k \geq 2$, count certain restricted partitions called *denumerants* in [2, pp. 108-124] and [5]. References listed in [6] and [5] lead to extensive literature on the sequences $P_k$ and $Q_k$. However, observations of relationships (e.g., Theorem 4) between $P_k$ and $\widehat{P}_k$ (and between $Q_k$ and $\widehat{Q}_k$) to be proved in section 2 may be new.

As a final introductory note, interspersions are closely related to fractal sequences; see [7] for a list of references. Thus, results proved below for interspersions (hence dispersions) could to be stated in terms of fractal sequences.

# 2 Rank Sequences

In this section, the word *rank* refers to the ordinary less-than-or-equal relation, $\leq$ (not the $\theta$-dependent relation $\prec$ defined in section 1, which will be considered further in section 4.) Recall from section 1 that for rational $\theta$, the multiset $S_\theta$ in (3) contains repeated elements, so that an element may have more than one rank.

Suppose $\theta > 0$ and $n \geq 0$. The *minrank sequence of* $\theta$, denoted by $m_\theta$, is defined by

$$m_\theta(n) = \text{ least } h \text{ such that } n \text{ has rank } h \text{ in } S_\theta,$$

and the *maxrank sequence of* $\theta$, denoted by $M_\theta$, is defined by

$$M_\theta(n) = \text{ greatest } h \text{ such that } n \text{ has rank } h \text{ in } S_\theta.$$

If $\theta$ is a positive integer, then the *lower self-rank sequence of* $\theta$ is defined by

$$\ell_\theta(n) = m_\theta(\theta n),$$

and the *upper self-rank sequence of* $\theta$, by

$$\mathcal{L}_\theta(n) = M_\theta(\theta n).$$

As $\mathcal{L}_\theta$ is clearly the sequence $r_\theta$ of section 1, we shall henceforth write it as $r_\theta$. Of course, $m_\theta = M_\theta$ (and $\ell_\theta = r_\theta$) if and only if $\theta$ is irrational.

For any real $\theta > 0$,

$$r_\theta(n) = \#\{(i,j) : \ i + j\theta \leq n\theta\}.$$

Consider the set of $(i,j)$ satisfying $(n-1)\theta < i + j\theta \leq n\theta$, or equivalently, $(n-j-1)\theta < i \leq (n-j)\theta$. There is one such $i$ for $j = n$, and for $j = 0, 1, \ldots, n-1$, the number of such $i$ is $\lfloor (n-j)\theta \rfloor - \lfloor (n-j-1)\theta \rfloor$. Summing gives a simple recurrence

$$r_\theta(n) = r_\theta(n-1) + \lfloor n\theta \rfloor + 1 \tag{13}$$

for $n \geq 1$, from which follows

$$r_\theta(n) = n + 1 + \sum_{i=1}^{n} \lfloor i\theta \rfloor . \tag{14}$$

Alternatively, we may write

$$\begin{aligned} r_\theta(n) &= \#\{(i,j) : \ i + j\theta \leq n\theta\} \\ &= \#\{(i,j) : \ j \leq n - i/\theta\}, \end{aligned}$$

so that the numbers $j$ to be counted are $0, 1, 2, \ldots, \lfloor n - i/\theta \rfloor$, for $i = 0, 1, \ldots, \lfloor n\theta \rfloor$, and

$$\begin{aligned} r_\theta(n) &= \sum_{i=0}^{\lfloor n\theta \rfloor} (1 + \lfloor n - i/\theta \rfloor) \\ &= \lfloor n\theta \rfloor + 1 + \sum_{i=0}^{\lfloor n\theta \rfloor} \lfloor n - i/\theta \rfloor . \end{aligned} \tag{15}$$

(Equations much like (14) and (15) appear in [4]; the sequence $r_{3/2}$ is A077043 in [6].)

Clearly,

$$M_\theta(n) = M_{1/\theta}(n/\theta) = r_{1/\theta}(n). \tag{16}$$

If $\theta$ is rational, write $\theta = c/d$, where $c$ and $d$ are relatively prime numbers in $N$. Then there are a total of $\lfloor n/c \rfloor + 1$ pairs $(i, j)$ satisfying $i + j\theta = n$, so that there are $M_\theta(n) - (\lfloor n/c \rfloor + 1)$ pairs $(i, j)$ satisfying $i + j\theta < n$. Consequently,

$$m_{c/d}(n) = M_{c/d}(n) - \lfloor n/c \rfloor = r_{d/c}(n) - \lfloor n/c \rfloor \,. \tag{17}$$

**Table 1. Sequences associated with $\theta = 2$**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_\theta(n)$ | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 |
| $m_\theta(n)$ | 1 | 2 | 3 | 5 | 7 | 10 | 13 | 17 | 21 | 26 | 31 | 37 |
| $M_\theta(n)$ | 1 | 2 | 4 | 6 | 9 | 12 | 16 | 20 | 25 | 30 | 36 | 42 |
| $\ell_\theta(n)$ | 1 | 3 | 7 | 13 | 21 | 31 | 43 | 57 | 73 | 91 | 111 | 133 |
| $r_\theta(n)$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 |

As suggested by Table 1, the self-rank sequences of a positive integer, $k$, are, loosely speaking, the sequences $P_k$ and $Q_k$ in section 1. A more precise statement of this connection takes the form of Theorem 2.

**Theorem 2.1** *Suppose $\theta \in N$. The upper self-rank sequence of $\theta$ is given by*

$$r_\theta(n) = P_\theta(n), \;\; n \geq 0.$$

*The lower self-rank sequence of $\theta$ is given by*

$$\begin{aligned} \ell_\theta(n) &= Q_{\theta-1}(n), \;\; n \geq 0, \;\; \text{if } \theta \geq 2, \\ \ell_1(n) &= \widehat{P}_1(n), \;\; n \geq 0. \end{aligned}$$

**Proof:** Suppose $n \geq 0$. Putting $\lfloor i\theta \rfloor = i\theta$ in (14) gives

$$\begin{aligned} r_\theta(n) &= n + 1 + \theta \sum_{i=0}^{n} i. \\ &= P_\theta(n), \;\; \text{by (4)}. \end{aligned}$$

As the rank of the first occurrence of $\theta n$ in $S_\theta$, the number $\ell_\theta(n)$ is easily obtained from $r_\theta(n)$, the total number of occurrences of $\theta n$ being $n + 1$; thus

$$\ell_\theta(n) = r_\theta(n) - n = P_\theta(n) - n.$$

In case $\theta \geq 2$, we therefore have $\ell_\theta(n) = Q_{\theta-1}(n)$, by (6), and if $\theta = 1$, then $\ell_\theta(n) = \widehat{P}_1(n)$, by (9). $\square$

**Theorem 2.2** *Suppose $\theta$ is an integer $\geq 2$. If $n \geq 0$, then*

$$\begin{aligned} M_\theta(n) &= r_{1/\theta}(n) = \widehat{Q}_{\theta-1}(n), && (18)\\ m_{1/\theta}(n) &= Q_{\theta-1}(n), && (19)\\ M_{1/\theta}(n) &= P_\theta(n), && (20)\\ m_\theta(n) &= \widehat{P}_\theta(n). && (21) \end{aligned}$$

**Proof:** That $M_\theta(n) = r_{1/\theta}(n)$ has already been established. Equations (14) and (8) give

$$r_{1/\theta}(n) = r_{1/\theta}(n-1) + \left\lfloor \frac{n}{\theta} \right\rfloor + 1 \quad \text{and} \quad \widehat{Q}_{\theta-1}(n) = \widehat{Q}_{\theta-1}(n-1) + \left\lfloor \frac{n}{\theta} \right\rfloor + 1,$$

showing that the sequences $r_{1/\theta}$ and $\widehat{Q}_{\theta+1}$ have a common recurrence relation. As they also have identical initial terms, (18) follows.

Next,

$$\begin{aligned} m_{1/\theta}(n) &= r_\theta(n) - n, \text{ by (17)}\\ &= P_\theta(n) - n, \text{ by Theorem 2}\\ &= Q_{\theta-1}(n) \text{ by (6), and (18) is proved.} \end{aligned}$$

Further,

$$\begin{aligned} M_{1/\theta}(n) &= r_\theta(n), \text{ by (12)}\\ &= P_\theta(n), \text{ by Theorem 2, so that (19) holds,} \end{aligned}$$

and

$$\begin{aligned} m_\theta(n) &= r_{1/\theta}(n) - \lfloor n/\theta \rfloor, \text{ by (17)}\\ &= \widehat{Q}_{\theta-1}(n) - \lfloor n/\theta \rfloor, \text{ by (18)}\\ &= \widehat{P}_\theta(n+1) - 1 - \lfloor n/\theta \rfloor, \text{ by (11)}\\ &= \widehat{P}_\theta(n), \text{ by (7).} \qquad \square \end{aligned}$$

Theorems 4 and 5 show the manner in which the sequences $P_k, Q_k, \widehat{P}_k, \widehat{Q}_k$ are related to minrank and maxrank sequences. We turn next toward Theorem 6, which establishes that two of the sequences are partial complements of the other two. A preliminary example may be helpful. We begin by writing the complement $P_2^*$ of $P_2$ in labeled rows of consecutive integers:

$$\begin{array}{lllllllll} \mathcal{S}_0: & \boxed{2} & \boxed{3} & & & & & & \\ \mathcal{S}_1: & \boxed{5} & 6 & \boxed{7} & 8 & & & & \\ \mathcal{S}_2: & \boxed{10} & 11 & 12 & \boxed{13} & 14 & 15 & & \\ \mathcal{S}_3: & \boxed{17} & 18 & 19 & 20 & \boxed{21} & 22 & 23 & 24 \\ \vdots & & & & & & & & \end{array}$$

Two equally spaced numbers from $\mathcal{S}_n$ are boxed. These numbers, one can easily check, taken together and preceded by 1, form the first partial complement of $P_2$. By equation (7), they also form the sequence $\widehat{P}_2$. We generalize this method in the next proof.

**Theorem 2.3** *Suppose $k \geq 1$. The 1st partial complement of $P_k$ is $\widehat{P}_k$, and the 1st partial complement of $Q_k$ is $\widehat{Q}_k$.*

**Proof:** The complement $P_k^*$ of $P_k$ consists of segments $\mathcal{S}_n$ of $k(n+1)$ consecutive integers, given by

$$\mathcal{S}_n = \{P_k(n)+1, P_k(n)+2, \ldots, P_k(n+1)-1\}$$

for $n \geq 0$. Thus, the initial segment $\mathcal{S}_0$ consists of the $k$ numbers $2, 3, \ldots, k+1$, for which we have

$$tP_k(h) = h + 1 = \widehat{P}_k(h) \text{ for } 1 \leq h \leq k.$$

(Recall that $tr$ denotes the first partial complement of a sequence $r$; the notation $tr(h)$ abbreviates $(tr)(h)$.) As the number $k+2$ is not in $P_k^*$, we have from $\mathcal{S}_1$ the $k+1$ numbers

$$k+3,\ k+5,\ \ldots,\ 3k+1$$

satisfying

$$\begin{aligned} tP_k(k+1) &= k+3, \\ tP_k(k+3) &= k+5, \\ &\vdots \\ tP_k(3k-1) &= 3k+1 = P_k(2)-2. \end{aligned}$$

In order to extend these patterns to represent the appropriate numbers in each segment $\mathcal{S}_n$, let

$$s_{n,j} = P_k(n) + 1 + (j-1)(n+1), \quad \text{for } 1 \leq j \leq k,\ n \geq 0,$$

and arrange these in segments of length $k+1$ :

**Table 2. The numbers $s_{n,j}$**

| | | | |
|---|---|---|---|
| $s_{0,1} = 2$ | $s_{0,2} = 3$ | $\ldots$ | $s_{0,k} = k+1$ |
| $s_{1,1} = k+3$ | $s_{1,2} = k+5$ | $\ldots$ | $s_{1,k} = 3k+1$ $= P_k(2)-2$ |
| $s_{2,1} = P_k(2)+1$ | $s_{2,2} = P_k(2)+4$ | $\ldots$ | $s_{2,k} = P_k(2)+3k-2$ $= P_k(3)-3$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_{n,1} = P_k(n)+1$ | $s_{n,2} = P_k(n)+n+2$ | $\ldots$ | $s_{n,k} = P_k(n)+(k-1)n+k+1$ $= P_k(n+1)-n-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

The numbers in the following tableau satisfy the chain of equations defining first partial complement:

| | | | |
|---|---|---|---|
| $tP_k(1) = s_{0,1}$ | $tP_k(s_{0,1}) = s_{0,2}$ | $\ldots$ | $tP_k(s_{0,k-1}) = s_{0,k}$ |
| $tP_k(s_{0,k}) = s_{1,1}$ | $tP_k(s_{1,1}) = s_{1,2}$ | $\ldots$ | $tP_k(s_{1,k-1}) = s_{1,k}$ |
| $tP_k(s_{1,k}) = s_{2,1}$ | $\ldots$ | $\ldots$ | $\ldots$ |

(22)

In order to confirm that the sequence $tP_k$ has the values indicated by (22), these equations must be proved:

(i) $tP_k(s_{n,k}) = s_{n+1,1}$ for $n \geq 0$, as in column 1;
(ii) $tP_k(s_{n,j}) = s_{n,j+1}$ for $1 \leq j \leq k-1$, $n \geq 0$, as in columns 2 to $k$.

We begin with (ii). Within $\mathcal{S}_n$ lie the consecutive integers

$$s_{n,j}, s_{n,j}+1, \ldots, s_{n,j+1}.$$

Inductively, the first of these is $tP_k(s_{n,j-1})$, so that in $P_k^*$, these integers are indexed as indicated by the top row of the next array:

| | | | | |
|---|---|---|---|---|
| $s_{n,j-1}$ | $s_{n,j-1}+1$ | $s_{n,j-1}+2$ | $\ldots$ | $s_{n,j-1}+n+1 = s_{n,j}$ |
| $s_{n,j}$ | $s_{n,j}+1$ | $s_{n,j}+2$ | $\ldots$ | $s_{n,j}+n+1 = s_{n,j+1}$ |

In other words, because the integers in both rows are consecutive, the inductively assumed relation $tP_k(s_{n,j-1}) = s_{n,j}$ implies $tP_k(s_{n,j}) = s_{n,j+1}$.

Now regarding (i), we modify (22) to obtain

| | | | |
|---|---|---|---|
| $s_{n,k-1}$ | $\ldots$ | $s_{n,k-1}+n$ | $s_{n,k-1}+n+1 = s_{n,k}$ |
| $s_{n,k}$ | $\ldots$ | $s_{n,k}+n$ | $s_{n,k}+n+2 = s_{n+1,1}$ |

The point here is that the integers in row 1 are consecutive, but that those in row 2, consecutive up to the penultimate term, skip over the number $s_{n,k}+n+1 = P_k(n)$.

As indicated by Table 2, $tP_k$ is the sequence having initial term 1 and difference sequence consisting of $k$ 1s followed by $k$ 2s followed by $k$ 3s, and so on. The same characterization holds, by (7), for the sequence $\widehat{P}_k$. Therefore, $\widehat{P}_k = tP_k$.

We turn now to the proposition that the 1st partial complement of $Q_k$ is $\widehat{Q}_k$. The complement $Q_k^*$ of $Q_k$ consists of segments $\mathcal{T}_n$ of $(k+1)(n+1)-2$ consecutive integers, given by

$$\mathcal{T}_n = \{Q_k(n)+1, Q_k(n)+2, \ldots, Q_k(n+1)-1\}$$

for $n \geq 0$. The method of proof used above for $P_k$ is $\widehat{P}_k$ applies to these segments, leading to the conclusion, via (8), that $\widehat{Q}_k = tQ_k$. $\square$

# 3 Farey trees

In section 1, it is proved that if $\theta$ is a positive irrational number, then $ttr_\theta = r_\theta$, or equivalently, that the dispersion $A_\theta$ is transposable. In case $\theta \in N$, the sequences $r_\theta$ and $r_{1/\theta}$ are those discussed in section 2. Here in section 3, we consider $r_\theta$ when $\theta$ is a rational number and introduce certain limiting sequences, to be denoted by $r_{\theta^-}$. In section 4, we outline a possible proof that the only sequences $r$ satisfying the equation $ttr = r$ are of the forms $r_\theta$ and $r_{\theta^-}$.

Suppose $n \geq 1$. The set $\mathcal{F}_n$ of Farey fractions of order $n$ consists of the rational numbers $c/d$ for which $0 \leq c \leq d$, $1 \leq d \leq n$, and $c$ and $d$ are relatively prime. For example, the Farey fractions of order 5 are

$$0,\ \frac{1}{5},\ \frac{1}{4},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ 1.$$

The numbers in $\mathcal{F}_n$, taken consecutively as endpoints, determine a partition of the interval $[0, 1)$; for example, for $n = 5$, the subintervals are

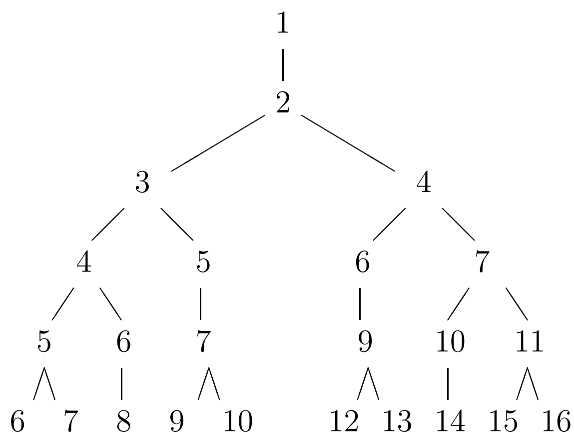
$$[0, \frac{1}{5}), [\frac{1}{5}, \frac{1}{4}), [\frac{1}{4}, \frac{1}{3}), [\frac{1}{3}, \frac{2}{5}), [\frac{2}{5}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{5}), [\frac{3}{5}, \frac{2}{3}), [\frac{2}{3}, \frac{3}{4}), [\frac{3}{4}, \frac{4}{5}), [\frac{4}{5}, 1). \tag{23}$$

The connection between Farey fractions and sequences $r_\theta$ is indicated by the following example: for $0 \leq \theta < 1$, the number $r_\theta(5)$ has one of the ten values

$$6, 7, 8, 9, 10, 12, 13, 14, 15, 16$$

according to which of the intervals in (23) contains $\theta$.

Corresponding to successive sets $\mathcal{F}_n$, we show six levels of the 0-*Farey tree*, which represents all the sequences $r_\theta$ for which $\theta \in [0, 1)$ :




Level $n$ consists of $|\mathcal{F}_n|$ numbers, ranging from $n + 1$ to $\widehat{P}_1(n)$. When progressing from level $n$ to level $n+1$, a branching occurs at a number if and only if the corresponding $n$-level interval contains a fraction (in reduced form) having denominator $n + 1$. For example, if $\theta \in [1/5, 1/4)$ then $r_\theta(8) = 13$, in level 8 of the Farey tree. The fraction $2/9$ lies in $[1/5, 1/4)$, so that a branching occurs at 13:

$$r_\theta(9) = \begin{cases} 15 & \text{if } \theta \in [1/5, 2/9) \\ 16 & \text{if } \theta \in [2/9, 1/4). \end{cases} .$$

It is easy to adapt the Farey tree for numbers $\theta'$ in an interval $\lfloor h, h + 1)$, where $h$ is a positive integer; viz., equation (14) yields

$$r_{\theta'}(n) = r_\theta(n) + h \binom{n+1}{2}, \tag{24}$$

where $\theta = \theta' - h$, so that the desired tree, which we call the $h$-Farey tree, results by adding $n(n+1)/2$ to the $n$-level nodes of the Farey tree, for all $n \geq 0$. It follows from (24) that in the *$h$-Farey tree* at level $n$ the numbers range from $P_h(n)$ up to $Q_h(n)$.

The set of infinite paths from 1 down through a Farey tree are of three types:

(i) paths that eventually stay left
(ii) paths that eventually stay right
(iii) all other paths.

The description, "paths that eventually stay left," applies to any path such that, after *some* level in a Farey tree, *every* time a branching occurs, the path takes the left branch, and likewise for "paths that eventually stay right".

In order to interpret these paths as representations of three kinds of sequences $r_\theta$, recall that each node $x$ of a Farey tree corresponds to an interval $\lfloor u, w)$. If $x$ connects to only one successor, $y$, in the next level, then $y$ corresponds to the same interval, $\lfloor u, w)$. The only other possibility is that $x$ is a branching node, meaning that $x$ has two successors, $y$ and $z$, in the next level, and that there is a rational number $v$ such that $y \in \lfloor u, v)$ and $z \in \lfloor v, w)$.

Clearly then, paths of type (i) represent $r_\theta$ when $\theta$ is a rational number, and the property, "eventually stay left" corresponds to $\theta$ lying in the interval $\lfloor \theta, w)$ for every $w$.

Next, consider an eventually-stay-right path. From some node on, the corresponding intervals are of the form $\lfloor v_m, w)$ where $(v_m)$ is a nondecreasing sequence of rationals with rational limit $w$. The sequence corresponding to these intervals, which we denote by $r_{(w-)}$, is therefore given by

$$r_{(w-)} = \lim_{m\to\infty} r_{v_m}. \tag{25}$$

For example, $r_{(2-)} = Q_1$ (whereas $r_2 = P_2$).

Paths of type (iii) represent $r_\theta$ when $\theta$ is an irrational number, the limit point of a nest of intervals $\lfloor u_m, w_m)$ having rational endpoints.

**Theorem 3.1** *Suppose $\theta = c/d$, where $c$ and $d$ in $N$ are relatively prime. Then*

$$r_{(c/d-)}(n) = r_{d/c}(n) - \lfloor n/c \rfloor . \tag{26}$$

**Proof:** Clearly (26) holds for $n = 0$. Suppose $n \geq 1$, and let

$$\mu = \min\{kd/c - \lfloor kd/c \rfloor : \quad 1 \leq k \leq n,\ c \nmid k\}. \tag{27}$$

Let $p, q$ in $N$ be relatively prime satisfying $d/c - p/q < \mu$. (That is, the interval $[p/q, c/d)$ is of the form $[v_m, w)$ used for (25) to define $r_{(c/d-)}$.) Then

$$\begin{aligned} r_{(c/d-)}(n) &= r_{p/q}(n) \\ &= n + 1 + \lfloor p/q \rfloor + \lfloor 2p/q \rfloor + \cdots + \lfloor np/q \rfloor . \end{aligned} \tag{28}$$

By (27),

$$\lfloor kp/q \rfloor = \begin{cases} \lfloor kd/c \rfloor & \text{if } c \nmid kd; \\ \lfloor kd/c \rfloor - 1 & \text{if } c \mid kd \end{cases}$$

for $1 \leq k \leq n$. As $c$ and $d$ are relatively prime, the values of $k$ for which $c \mid kd$ are $c, 2c, \ldots, \lfloor n/c \rfloor c$, so that (28) gives

$$\begin{aligned} r_{(c/d-)}(n) &= n + 1 + \lfloor d/c \rfloor + \lfloor 2d/c \rfloor + \cdots + \lfloor nd/c \rfloor - \lfloor n/c \rfloor \\ &= r_{d/c}(n) - \lfloor n/c \rfloor, \text{ by (14).} \qquad \square \end{aligned}$$

# 4 Rank Arrays and the Family $F$

Throughout section 4, the word *rank* refers to the order relation $\prec$ except where otherwise indicated. (The relation $\prec$, dependent on rational $\theta$, is defined in section 1 on the set of ordered pairs $(i, j)$, $i \geq 0$, $j \geq 0$.)

**Theorem 4.1** *Suppose $\theta = c/d$, where $c$ and $d$ in $N$ are relatively prime. Then the 1st partial complement of $r_\theta$ is given by*

$$tr_\theta(n) = m_\theta(n) = r_{1/\theta}(n) - \lfloor n/c \rfloor. \tag{29}$$

**Proof:** The sequence $tr_\theta$ is row 0 of $A_\theta$, so that $tr_\theta(n)$ is the rank of $(n, 0)$, for all $n \geq 0$. This means that $tr_\theta(n)$ is the rank of the first occurrence of the number $n$ when the numbers in $S_\theta$ are ranked under $\leq$ . In other words, $tr_\theta(n) = m_\theta(n)$, where $m_\theta$ is the minrank sequence of section 2. Thus, (29) follows from (17). $\square$

**Corollary 4.1** *Suppose $\theta = c/d$, where $c$ and $d$ in $N$ are relatively prime. Then the 1st partial complement of $r_\theta$ is given by*

$$tr_\theta(n) = r_{(\theta-)}(n) \tag{30}$$

*for $n \geq 0$.*

**Proof:** This is an obvious consequence of Theorems 5 and 6. $\square$

**Lemma 4.1** *Suppose $r \in R$ and $n_1 \in N$. Then there exists $n_0$ in $N$ such that if $s$ in $R$ satisfies $s(n) = r(n)$ for $n = 0, 1, \ldots, n_0$, then $s^*(n) = r^*(n)$ for $n = 1, 2, \ldots, n_1$.*

**Proof:** Let $n_0$ be the least $n$ for which $r(n) > r^*(n_1)$. The numbers $r^*(1), r^*(2), \ldots, r^*(n_1)$ are then uniquely determined by the numbers $r(0), r(1), \ldots, r(n_0)$, as the former simply occupy in increasing order the positions not occupied by the latter in the list $1, 2, 3, \ldots, r(n_0)$ of consecutive integers. $\square$

**Lemma 4.2** *Suppose $r \in R$ and $n' \in N$. Then there exists $n_0$ in $N$ such that if $s$ in $R$ satisfies $s(n) = r(n)$ for $n = 0, 1, \ldots, n_0$, then $ts(n) = tr(n)$ for $n = 0, 1, \ldots, n'$.*

**Proof:** Let $n_1$ be the number such that $tr(n') = r^*(n_1)$. The numbers $tr(n)$ for $n = 0, 1, \ldots, n'$ are among, and are uniquely determined by, the numbers $r^*(n)$ for $n = 1, 2, \ldots, n_1$. Let $n_0$ be as in Lemma 10, so that the numbers $tr(n)$ for $n = 0, 1, \ldots, n'$ are uniquely determined by the numbers $r(n)$ for $n = 0, 1 \ldots, n_0$. $\square$

**Corollary 4.2** *Suppose $\theta = c/d$, where $c$ and $d$ in $N$ are relatively prime. Then*

$$ttr_\theta = r_\theta. \tag{31}$$

**Proof:** Suppose $n_0$ and $n_1$ are positive integers. By Corollary 9, there exist $c'$ and $d'$, relatively prime in $N$, such that $d'/c' < d/c$ and

$$tr_{c/d}(n) = r_{d'/c'}(n) \text{ for } n = 0, 1, \ldots, n_0. \tag{32}$$

It might be tempting to say that we apply $t$ to (32) to get

$$ttr_{c/d}(n) = tr_{d'/c'}(n) \text{ for } n = 0, 1, \ldots, n_0,$$

but this mistake overlooks the meaning of the notation $tr(n)$ as $(tr)(n)$, not $t(r(n))$. Instead, we can, in accord with Lemma 11, and do, take $n_0$ large enough that (32) implies

$$ttr_{c/d}(n) = tr_{d'/c'}(n) \text{ for } n = 0, 1, \ldots, n_1. \tag{33}$$

Let $\gamma$ and $\delta$, relatively prime in $N$, satisfy $\gamma/\delta > c/d$ and

$$r_{\gamma/\delta}(n) = r_{c/d}(n) \text{ for } n = 0, 1, \ldots, n_1. \tag{34}$$

By Corollary 9, there exist $c''$ and $d''$, relatively prime in $N$, such that $c''/d'' < c'/d'$ and

$$tr_{d'/c'}(n) = r_{c''/d''}(n) \text{ for } n = 0, 1, \ldots, n_1. \tag{35}$$

As $c'/d' > c/d$, we can and do choose $c''$ and $d''$ so that $c''/d'' > c/d$ and $c''/d'' > \gamma/\delta$. Then by (33)-(35),

$$\begin{aligned} ttr_{c/d}(n) &= r_{c''/d''}(n) \\ &= r_{c/d}(n) \text{ for } n = 0, 1, \ldots, n_1. \end{aligned}$$

As $n_1$ is arbitrary, (31) follows. $\square$

Corollaries 9 and 12 extend Corollary 3 from the case that $\theta$ is irrational to the case that $\theta$ is any positive real number. In other words, $r_\theta$ is in the family $F$ of fixed points under the operator $tt$, or, in yet other words, the 1st partial complement of the 1st partial complement of any $r_\theta$ is $r_\theta$ itself. The next corollary identifies additional members of $F$, and in section 5, we conjecture that there are no others.

**Corollary 4.3** *Suppose $\theta = c/d$, where $c$ and $d$ in $N$ are relatively prime. Then*

$$ttr_{(\theta-)} = r_{(\theta-)}. \tag{36}$$

**Proof:** Suppose $n_0$, $n_1$, and $n_2$ are positive integers. Let $c'$ and $d'$, relatively prime in $N$, satisfy

$$\begin{aligned} r_{c'/d'}(n) &= r_{(\theta-)}(n) \text{ for } n = 0, 1, \ldots, n_0 \text{ and} \\ ttr_{c'/d'}(n) &= ttr_{(\theta-)}(n) \text{ for } n = 0, 1, \ldots, n_1. \end{aligned}$$

By Corollary 12, we can, and do, choose $n_0$ and $n_1$ large enough that

$$ttr_{(\theta-)}(n) = r_{c'/d'}(n) \text{ for } n = 0, 1, \ldots, \max\{n_0, n_1, n_2\}.$$

As $n_0, n_1, n_2$ are arbitrarily large, (36) follows. $\square$

For any array $A = \{a(i,j)\}$, let $TA$ denote the transpose, $\{a(j,i)\}$, of $A$. Let $A_{(\theta-)}$ denote the interspersion whose column 0 is the sequence $r_{(\theta-)}$. Membership in the family $F$ can now be stated in terms of transposable interspersions:

**Corollary 4.4** *Suppose $\theta$ is a positive real number. Then $TA_{(\theta-)} = A$, $TTA_\theta = A_\theta$, and $TTA_{(\theta-)} = A_{(\theta-)}$.*

**Proof:** If $\theta$ is a positive integer, the three asserted equations follow from Theorem 4, as in this case, $r_\theta$ and $r_{(\theta-)}$ are $P_\theta$ and $\widehat{P}_\theta$, (and $r_{1/\theta}$ and $r_{(1/\theta-)}$ are $Q_\theta$ and $\widehat{Q}_\theta$). For irrational, $\theta$, we have $A_{(\theta-)} = A_{1/\theta}$, and the proposition follows from Corollary 12. For all other positive $\theta$, the asserted equations follow immediately from Corollaries 9, 12, and 13.

**Example 2:** The interspersions $A_{3/2}$ and $A_{(3/2-)}$

$$\begin{array}{cccccc} 1 & 2 & 4 & 6 & 9 & \ldots \\ 3 & 5 & 8 & 11 & 15 & \ldots \\ 7 & 10 & 14 & 18 & 23 & \ldots \\ 12 & 16 & 21 & 26 & 32 & \ldots \\ 19 & 24 & 30 & 36 & 43 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{cccccc} 1 & 3 & 7 & 12 & 19 & \ldots \\ 2 & 5 & 10 & 16 & 24 & \ldots \\ 4 & 8 & 14 & 21 & 30 & \ldots \\ 6 & 11 & 18 & 26 & 36 & \ldots \\ 9 & 15 & 23 & 32 & 43 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

**Example 3:** The interspersions $A_{2/3}$ and $A_{(2/3-)}$

$$\begin{array}{cccccc} 1 & 3 & 6 & 11 & 17 & \ldots \\ 2 & 5 & 9 & 15 & 22 & \ldots \\ 4 & 8 & 13 & 20 & 28 & \ldots \\ 7 & 12 & 18 & 26 & 35 & \ldots \\ 10 & 16 & 23 & 32 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 4 & 7 & 10 & \ldots \\ 3 & 5 & 8 & 12 & 16 & \ldots \\ 6 & 9 & 13 & 18 & 23 & \ldots \\ 11 & 15 & 20 & 26 & 32 & \ldots \\ 17 & 22 & 28 & 35 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

**Theorem 4.2** *Suppose $\theta$ is a positive real number. Neighboring terms in column $j$ of the rank array $A_\theta$ satisfy the equation*

$$a(i,j) - a(i-1,j) = \lfloor i\theta \rfloor + j + 1 \tag{37}$$

*for $i \geq 1,\ j \geq 0$.*

**Proof:** Suppose $i \geq 1$. By (13),

$$a(i,0) - a(i-1,0) = r_\theta(i) - r_\theta(i-1) = \lfloor i\theta \rfloor + 1.$$

Thus, equation (37) is equivalent to

$$\Re(j,i) - \Re(j,i-1) = \Re(0,i) - \Re(0,i-1) + j, \tag{38}$$

where $\Re(h,k)$ denotes the rank of $(h,k)$ under $\prec$ . Let $v = \lfloor i\theta \rfloor$ . The $v$ ordered pairs $(h,k)$ counted by $\Re(0,i) - \Re(0,i-1)$ we represent as

$$(0,i-1) \prec (h_1,k_1) \prec (h_2,k_2) \prec \cdots \prec (h_v,k_v) = (0,i). \tag{39}$$

Clearly, (39) implies

$$(1,i-1) \prec (h_1+1,k_1) \prec (h_2+1,k_2) \prec \cdots \prec (h_v+1,k_v) = (1,i). \tag{40}$$

It is easy to check that there is exactly one integer $q$ satisfying

$$(1,i-1) \prec (0,i+q) \prec (1,i), \tag{41}$$

namely

$$q = \begin{cases} 0 & \text{if } \theta = 1; \\ \lfloor 1/\theta \rfloor & \text{otherwise.} \end{cases}$$

Thus, by (40) and (41), there are $v+1$ ordered pairs $(h,k)$, satisfying

$$(1,i-1) \prec (h,k) \prec (1,i). \tag{42}$$

Now if $(h,k)$ is an ordered pair satisfying (42), then either $h = 0$, so that $k = q$, or else $(h-1,k)$ is one of the ordered pairs in (39); thus every $(h,k)$ satisfying (40) is one of the $v+1$, so that (37) holds for $j = 1$.

The method used to get from (39) to (42), namely, adding 1 to all first coordinates and then inserting the sole ordered pair having first coordinate 0, applies inductively, so that (38) and (37) hold for all $j \geq 0$. $\square$

**Corollary 4.5** *Suppose $\theta$ is a positive real number. Neighboring terms in column $j$ of the array $A_{(\theta-)} = \{b(i,j)\}$ satisfy the equation*

$$b(i,j) - b(i-1,j) = \lceil i/\theta \rceil + j, \tag{43}$$

*where $\lceil\ \rceil$ denotes the ceiling function, for $i \geq 1,\ j \geq 0$.*

**Proof:** Suppose $n_0 \in N$. Let $c'$ and $d'$, relatively prime in $N$, satisfy

$$r_{c'/d'}(n) = r_{(\theta-)}(n)$$

for $n = 0, 1, \ldots, n_0$. In the rank array $A_{c'/d'} = \{a'(i,j)\}$, each $a'(i,j)$ that is $\leq r_{c'/d'}(n_0)$ has the same position $(i,j)$ that the same number has in $A_{(\theta-)}$; that is, $b(i,j) = a'(i,j)$ for $a'(i,j)$ ranging from 1 up to $r_{c'/d'}(n_0)$. By (37),

$$a'(i,j) = a'(i-1,j) = \lfloor ic'/d' \rfloor + j + 1,$$

so that $b(i,j) - b(i-1,j) - j = \lfloor ic'/d' \rfloor + 1$, showing that

$$b(i,j) - b(i-1,j) - j$$

is invariant of $j$. Now $b(i,0) = r_{d/c} - \lfloor i/c \rfloor$, by (26), so that

$$\begin{aligned} b(i,0) - b(i-1,0) &= r_{d/c}(i) - r_{d/c}(i-1) - (\lfloor i/c \rfloor - \lfloor (i-1)/c \rfloor \\ &= \lfloor id/c \rfloor + 1 - \lfloor i/c \rfloor + \lfloor (i-1)/c \rfloor. \end{aligned}$$

Consequently,

$$b(i,j) - b(i-1,j) = \lfloor id/c \rfloor - \lfloor i/c \rfloor + \lfloor (i-1)/c \rfloor + j + 1,$$

and it is easy to verify that the right-hand side simplifies as in (43). $\square$

**Corollary 4.6** *Suppose $\theta$ is a positive real number. The terms of the dispersion $A_\theta$ satisfy the equation*

$$a(i,j) = a(i,0) + a(0,j) + ij - 1$$

*for $i \geq 0,\ j \geq 0$.*

**Proof:** By (37),

$$a(h,j) - a(h-1,j) = \lfloor h\theta \rfloor + j + 1$$

for $h = 1, 2, \ldots, i$. Summing over those values of $h$ gives

$$\begin{aligned} a(i,j) &= a(0,j) + \lfloor \theta \rfloor + \lfloor 2\theta \rfloor + \cdots + \lfloor i\theta \rfloor + i(j+1) \\ &= a(0,j) + r_\theta(i) - i - 1 + i(j+1), \text{ by (13)} \\ &= a(i,0) + a(0,j) + ij - 1. \quad \square \end{aligned}$$

**Corollary 4.7** *Suppose $\theta = c/d$, where $c$ and $c$ are relatively prime in $N$. The terms of the dispersion $A_{(\theta-)} = \{b(i,j)\}$ satisfy the equation*

$$b(i,j) = b(i,0) + b(0,j) + ij + \lfloor i/c \rfloor$$

*for $i \geq 0,\ j \geq 0$.*

**Proof:** By (43),

$$b(h,j) - b(h-1,j) = \lceil h/\theta \rceil + j$$

for $h = 1, 2, \ldots, i$. Summing over those values of $h$ as in the proof of Corollary 7.2 and simplfying via the identify

$$\lceil x \rceil = \begin{cases} x & \text{if } x \text{ is an integer;} \\ \lfloor x \rfloor + 1 & \text{otherwise} \end{cases}$$

yield the asserted equation. $\square$

## 5 Convergence

It appears likely that if $r \in R$, then the sequence of sequences,

$$r, \ ttr, \ tt(ttr), \ \ldots,$$

whose general term we abbreviate as $t^{(2n)}r$, converges, and that the limiting sequence is one of the sequences $r_\theta$ or $r_{(\theta-)}$ for some positive real number $\theta$. If so, then the family $F$ of sequences satisfying $ttr = r$ clearly contains no sequences other than those already accounted for.

Proof that $(t^{(2n)}r)$ converges seems elusive. This section offers lemmas which may someday be found useful in a proof but are of independent interest in any case. The items for which proof is sought are then presented as Conjectures 22-26.

Suppose that $u$ and $v$ are sequences in $R$ and that their initial segments of some length are identical Then some initial segments of the complements, $u^*$ and $v^*$, must be identical. The following lemma provides some insight.

**Lemma 5.1** *Suppose $u$ and $v$ are sequences in $R$ such that $u(i) = v(i)$ for $i = 0, 1, \ldots, n$, and $u(n+1) \geq v(n+1)$. Then*

$$u^*(j) = v^*(j) \quad \textit{for } j = 1, 2, \ldots, v(n+1) - n - 2.$$

**Proof:** For $h = 0, 1, \ldots, n$, the $v(h+1) - v(h) - 1$ numbers $v(h)+1$ to $v(h+1) - 1$, taken in order, comprise an initial segment of both $v^*$ and $u^*$. The length of this common segment is

$$\sum_{h=0}^{n}[v(h+1) - v(h) - 1] = v(n+1) - n - 2. \qquad \square \tag{44}$$

**Lemma 5.2** *Suppose $c$ and $d$ in $N$ are relatively prime. Suppose $r \in R$ and that there exists $n \geq 2$ such that $r(i) = r_{c/d}(i)$ and $r(n+1) \geq r_{c/d}(n+1)$. Then $r^*(j) = r^*_{c/d}(j)$ for $j = 1, 2, \ldots, J$, where*

$$J = r_{c/d}(n) + \lfloor (n+1)c/d \rfloor - n - 1. \tag{45}$$

**Proof:** In Lemma 19, take $u = r$ and $v = r_{c/d}$. Then $v(n+1) = r_{c/d}(n+1)$ in (44), and (13) implies (45). $\square$

**Lemma 5.3** *Suppose $c$ and $d$ in $N$ are relatively prime and $c > d$. Then*

$$r^*_{c/d}(i) = i + 1 \textit{ for } i = 1, 2, \ldots, r_{c/d}(1) - 2,$$

$$r^*_{c/d}(r_{c/d}(m) - m + J) = r_{c/d}(m) + J + 1 \tag{46}$$

$$\textit{for } J = 0, 1, \ldots, \lfloor (m+1)c/d \rfloor - 1, \ m = 1, 2, 3, \ldots. \tag{47}$$

**Proof:** Consider the sequence of numbers to which $r^*_{c/d}$ is applied: initially,

$$1, 2, \ldots, r_{c/d}(1) - 2, \text{ followed by}$$

$$r_{c/d}(m) - m + J, \text{ for } m = 1 \text{ and } J = 0, 1, \ldots, \lfloor 2c/d \rfloor - 1,$$
$$r_{c/d}(m) - m + J, \text{ for } m = 2 \text{ and } J = 0, 1, \ldots, \lfloor 3c/d \rfloor - 1,$$

and so on. Thus, the sequence in question is the sequence of positive integers. Their images under the function $r^*_{c/d}$ are determined by arranging all the numbers $r^*_{c/d}(i)$ in increasing order. This listing is conveniently broken into segments, first from 1 to $r_{c/d}(1) - 1$, then from $r_{c/d}(1) + 1$ to $r_{c/d}(2) - 1$, and so on. Thus, counting the first segment as segment 1, the $(m+1)$st segment, for $m \geq 1$, is as given by (46) and (47). □

**Conjecture 5.1** *Suppose $c$ and $d$ in $N$ are relatively prime. Suppose that $r \in R$ and that there exists $n \geq 1$ such that $r(i) = r_{c/d}(i)$ for $i = 0, 1, \ldots, n$, and*

$$r(n+1) \geq r_{c/d}(n+1). \tag{48}$$

*Then*

$$tr(i) = tr_{c/d}(i) \quad \text{for } i = 0, 1, 2, \ldots, \lfloor (n+1)c/d \rfloor. \tag{49}$$

**Conjecture 5.2** *Continuing, suppose, instead of (48), that*

$$r(n+1) \leq r_{c/d}(n+1) - 2. \tag{50}$$

*Then*

$$tr(n+1) = tr_{c/d}(n+1) - 1. \tag{51}$$

**Conjecture 5.3** *Continuing, suppose, instead of (50), that*

$$r(n+1) = r_{c/d}(n+1) - 1. \tag{52}$$

*Then*

$$tr(n+1) = tr_{c/d}(n+1). \tag{53}$$

**Conjecture 5.4** *Continuing, suppose*

$$z(i) = tr_{c/d}(i) \text{ for } i = 0, 1, \ldots, \lfloor (n+1)c/d \rfloor$$

*and*

$$z(i+1) - z(i) \geq z(i) - z(i-1) \tag{54}$$

*for $i \geq \lfloor (n+1)c/d \rfloor$. Then*

$$tz(i) = r_{c/d}(i) \quad \text{for } i = 0, 1, 2, \ldots, n+1. \tag{55}$$

**Conjecture 5.5** *Suppose $r \in R$. Then there exists a positive real number $\theta$ such that $\lim_{n\to\infty} t^{(2n)}r$ is one of the sequences $r_\theta$ and $r_{(\theta-)}$.*

**Possible method of proof:** If $r$ is $r_\theta$ or $r_{(\theta-)}$ for some then $\theta$, then $ttr = r$ by Corollaries 3, 12, and 13. Suppose then that $r$ is not any such $r_\theta$ or $r_{(\theta-)}$. Let

$$(\gamma, \delta) = \begin{cases} (1,2) & \text{if } r(1) = 2; \\ (r(1)-2, 1) & \text{if } r(1) \geq 3. \end{cases}$$

Then $r(0) = r_{\gamma/\delta}(0) = 1$ and $r(1) = r_{\gamma/\delta}(1)$, so that the set of rational numbers $\gamma/\delta$ satisfying

$$r(i) = r_{\gamma/\delta}(i) \quad \text{for } i = 0, 1, \ldots, m, \tag{56}$$

for some $m \geq 1$, is not empty. The set of numbers $m$ for which (56) holds for some $\gamma/\delta$ is not unbounded, for if it were, there would be a sequence $\gamma_m/\delta_m$ having limit $\theta$ such that $r \in \{r_\theta, r_{\theta^-}\}$, a contradiction. Let $n$ be the greatest $m$ for which (56) holds, and let $c/d$, where $c$ and $d$ are relatively prime, be a rational number such that

$$r(i) = r_{c/d}(i) \quad \text{for } 0, 1, 2, \ldots, n.$$

As $r(n+1) \neq r_{c/d}(n+1)$, one of the inequalities (48), (50), and (52) holds, so that we have cases:

*Case 1:* $r(n+1) \geq r_{c/d}(n+1)$. If Conjecture 22 is valid, then (49) holds.

*Case 2:* $r(n+1) \leq r_{c/d}(n+1) - 2$. If Conjecture 23 is valid, then by (51), the sequences $tr_{c/d}$ and $tr$ satisfy the hypothesis of Conjecture 24 (with $tr_{c/d}$ and $tr$ substituted for $r_{c/d}$ and $r$, respectively).

*Case 3:* $r(n+1) = r_{c/d}(n+1) - 1$. If Conjecture 24 is valid, then by (53), the sequences $tr_{c/d}$ and $tr$ satisfy the hypothesis of Conjecture 22 (with $tr_{c/d}$ and $tr$ substituted for $r_{c/d}$ and $r$, respectively).

Let

$$r' = \begin{cases} tr & \text{if (48) holds;} \\ tttr & \text{if (50) holds;} \\ tr & \text{if (52) holds.} \end{cases} \tag{57}$$

The discussion of the three cases shows that $r'$ satisfies (48), hence (49), if Conjecture 22 is valid. Let $z = r'$. It is easy to check that (54) holds (for $z = t\rho$, for every $\rho$ in $R$). Thus, if Conjecture 25 is valid, then, with reference to (56), the sequence $ttttr$ satisfies

$$ttttr(i) = r_{c/d}(i) \quad \text{for } i = 0, 1, \ldots n, n+1$$

(whereas $r(n+1)$ may not be equal to $r_{c/d}(n+1)$), and a proof of Conjecture 26 follows by repeated applications of Conjectures 22-25. It is hoped that someone will prove those conjectures!

The interested reader may wish to use the following website:

**http://csserver.evansville.edu/~brownie/cgi-bin/transpose**.

There, the reader can submit the first ten to thirty terms of a sequence $r$. The dispersion of the complementary sequence will appear, of which the submitted sequence is the first column. The reader can then request iterations and see, in the successive first columns, initial terms of the sequences $tr, ttr, tttr, \ldots$

**References**


[1] John H. Conway and Richard K. Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996.

[2] Louis Comtet, *Advanced Combinatorics*, revised and enlarged edition, D. Reidel, Dordrecht, 1974.

[3] Clark Kimberling, Interspersions and dispersions, *Proc. Amer. Math. Soc.* **117** (1993), 313–321.

[4] Clark Kimberling, Fractal sequences and interspersions, *Ars Combinatoria* **45** (1997), 157–168.

[5] Stéphanie Petit, editor (2003) *Encyclopedia of Combinatorial Structures*, `http://pauillac.inria.fr/algo/encyclopedia/index.html` .

[6] N. J. A. Sloane, editor (2003), *The On-Line Encyclopedia of Integer Sequences,* `http://www.research.att.com/~njas/sequences/` .

[7] N. J. A. Sloane, (2003) Classic Sequences In *The On-Line Encyclopedia of Integer Sequences*: (Part 1:) The Wythoff Array and The Para-Fibonacci Sequence, `http://www.research.att.com/~njas/sequences/classic.html` .


---


2000 *Mathematics Subject Classification*: Primary 11B36; Secondary 11B25, 11B37.
*Keywords:* central polygonal numbers, dispersion, interspersion, Farey sequence, Farey tree, fractal sequence, partial complement, polygonal numbers, rank array, rank sequence.


---

(Concerned with sequences A000124, A000217, A000290, A000326, A001401, A001840, A001844, A001972, A002061, A002620, A005448, A008748, A077043, A082152, A082153, A082154, A082155, A082156, A086270, A086271, A086272, A086273, A087465, A087466, A087467, A087468, A087469, A087470.)

---


Received October 14 2003; revised version received January 19 2004. Published in *Journal of Integer Sequences*, February 18 2004.


---


Return to Journal of Integer Sequences home page.