



# On a Number Pyramid Related to the Binomial, Deleham, Eulerian, MacMahon and Stirling number triangles

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## Abstract

We study a particular number pyramid  $b_{n,k,l}$  that relates the binomial, Deleham, Eulerian, MacMahon-type and Stirling number triangles. The numbers  $b_{n,k,l}$  are generated by a function  $B^c(x, y, t)$ ,  $c \in \mathbb{C}$ , that appears in the calculation of derivatives of a class of functions whose derivatives can be expressed as polynomials in the function itself or a related function. Based on the properties of the numbers  $b_{n,k,l}$ , we derive several new relations related to these triangles. In particular, we show that the number triangle  $T_{n,k}$ , recently constructed by Deleham (Sloane's [A088874](#)), is generated by the Maclaurin series of  $\operatorname{sech}^c t$ ,  $c \in \mathbb{C}$ . We also give explicit expressions and various partial sums for the triangle  $T_{n,k}$ . Further, we find that  $e_{2p}^m$ , the numbers appearing in the Maclaurin series of  $\cosh^m t$ , for all  $m \in \mathbb{N}$ , equal the number of closed walks, based at a vertex, of length  $2p$  along the edges of an  $m$ -dimensional cube.

## 1 Introduction

In this work we study a function  $B^c(x, y, t)$ , the  $c$ -th power of  $B(x, y, t)$  defined in Eq. (3.1), that plays a central role in the calculation of derivatives, of a class of functions whose derivatives can be expressed as polynomials in the function itself or a related function. The construction of these polynomials, in terms of the function  $B^c(x, y, t)$ , is treated in a separate paper [3]. Here we focus on  $B^c(x, y, t)$  as a generating function in its own right, and derive from it some interesting number-theoretic results.

We show that the function  $B^c(x, y, t)$  generates a number pyramid  $b_{n,k,l}$ , of which various partial sums are closely related to some important number triangles, including the binomial coefficients  $\binom{n}{k}$ , a number triangle  $T_{n,k}$  recently constructed by Deleham [11, A088874], the Eulerian numbers  $A_{n,k}$  [2], a particular kind of MacMahon numbers  $B_{n,k}$  [5, p. 331], and Stirling numbers of the first kind  $s(n, k)$  [1, p. 824, 24.1.3].

We derive several new expressions related to these triangles. For the triangles  $A_{n,k}$  and  $B_{n,k}$ , we obtain new generating functions. We show in particular that the so far unstudied triangle  $T_{n,k}$  is generated by the Maclaurin series of  $\operatorname{sech}^c t$ , for all  $c \in \mathbb{C}$ . The numbers  $T_{n,k}$  are thus as fundamental for  $\operatorname{sech}^c t$  as the Euler numbers  $E_n$  are for  $\operatorname{sech} t$  [1, p. 804, 23.1.2]. We give explicit expressions and various partial sums for the numbers  $T_{n,k}$ .

Moreover, the special cases  $c = m \in \mathbb{Z}_+$  and  $c = -m \in \mathbb{Z}_-$  give rise to a particular generalization of the Euler numbers  $E_n$ , here denoted  $E_n^m$  and called “multinomial Euler numbers”, and a generalization of even parity numbers  $e_n$  (defined in Eq. (2.2)), here denoted  $e_n^m$  and called “even multinomial parity numbers”, respectively. The  $E_n^m$  are generated by the Maclaurin series of  $\operatorname{sech}^m t$  (so  $E_n^1 = E_n$ ) and the  $e_n^m$  by the Maclaurin series of  $\cosh^m t$  (so  $e_n^1 = e_n$ ). Obviously,  $E_{2p+1}^m = 0$  and  $e_{2p+1}^m = 0$ , for all  $p \in \mathbb{N}$ , because  $\operatorname{sech}^m t$  and  $\cosh^m t$  are even functions of  $t$ . We obtain explicit formulas for the numbers  $E_{2p}^m$  and  $e_{2p}^m$ , as well as relations between them. The numbers  $e_{2p}^m$  turn out to have as combinatorial interpretation, the number of closed walks, based at a vertex, of length  $2p$  along the edges of an  $m$ -dimensional cube.

## 2 Notation and definitions

1. Define the sets of positive odd and even integers  $\mathbb{Z}_{o,+}$  and  $\mathbb{Z}_{e,+}$ , the negative odd and even integers  $\mathbb{Z}_{o,-}$  and  $\mathbb{Z}_{e,-}$ , the odd integers  $\mathbb{Z}_o \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{o,+}$  and the even integers  $\mathbb{Z}_e \triangleq \mathbb{Z}_{e,-} \cup \{0\} \cup \mathbb{Z}_{e,+}$ , the positive integers  $\mathbb{Z}_+ \triangleq \mathbb{Z}_{o,+} \cup \mathbb{Z}_{e,+}$  and negative integers  $\mathbb{Z}_- \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{e,-}$ , the natural numbers  $\mathbb{N} \triangleq \{0\} \cup \mathbb{Z}_+$  and the integers  $\mathbb{Z} \triangleq \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+$ . Let  $\mathbb{Z}_{+,n} \triangleq \{1, 2, \dots, n\}$ ,  $\mathbb{Z}_{-,n} \triangleq \{-n, -(n-1), \dots, -1\}$ ,  $\mathbb{N}_n \triangleq \{0\} \cup \mathbb{Z}_{+,n}$  and denote by  $\mathbb{C}$  the complex numbers.

2. Define

$$\delta_{condition} \triangleq \begin{cases} 1, & \text{if } condition \text{ is true;} \\ 0, & \text{if } condition \text{ is false,} \end{cases} \quad (2.1)$$

and for all  $n \in \mathbb{Z}$  the even and odd parity numbers

$$e_n \triangleq \delta_{n \in \mathbb{Z}_e}, \quad (2.2)$$

$$o_n \triangleq \delta_{n \in \mathbb{Z}_o}. \quad (2.3)$$

3. Denote the  $n$ -th derivative with respect to  $x$  by  $D_x^n$ .
4. We define  $0^n \triangleq \delta_{n=0}$ , for all  $n \in \mathbb{N}$ , and  $z^0 \triangleq 1$ , for all  $z \in \mathbb{C}$ .

5. Let  $n \in \mathbb{N}$  and  $z \in \mathbb{C}$ . Denote by

$$z^{(n)} \triangleq \delta_{n=0} + \delta_{n>0} z(z+1)(z+2)\dots(z+(n-1)), \quad (2.4)$$

$$= \frac{\Gamma(z+n)}{\Gamma(z)}, \quad (2.5)$$

$$= \sum_{k=0}^n (-1)^{n-k} s(n, k) z^k, \quad (2.6)$$

the rising factorial polynomial (Pochhammer's symbol). In particular,  $0^{(n)} = \delta_{n=0}$  and  $m^{(n)} = (m-1+n)!/(m-1)!$  for  $m \in \mathbb{Z}_+$ .

Also, denote by

$$z_{(n)} \triangleq \delta_{n=0} + \delta_{n>0} z(z-1)(z-2)\dots(z-(n-1)), \quad (2.7)$$

$$= \frac{\Gamma(z+1)}{\Gamma(z+1-n)}, \quad (2.8)$$

$$= \sum_{k=0}^n s(n, k) z^k, \quad (2.9)$$

the falling factorial polynomial. In particular,  $0_{(n)} = \delta_{n=0}$  and  $m_{(n)} = (m!/(m-n)!) \delta_{n \leq m}$  for  $m \in \mathbb{Z}_+$ . In Eqs. (2.6) and (2.9),  $s(n, k)$  are Stirling numbers of the first kind. We have  $z_{(n)} = (-1)^n (-z)^{(n)}$ .

6. We will need

$$\frac{1}{(1-z)^c} = \sum_{n=0}^{+\infty} c^{(n)} \frac{z^n}{n!}, \quad (2.10)$$

$$(1+z)^c = \sum_{n=0}^{+\infty} c_{(n)} \frac{z^n}{n!}, \quad (2.11)$$

being absolutely and uniformly convergent series for all  $z \in \{z \in \mathbb{C} : |z| < 1\}$  and for all  $c \in \mathbb{C}$ . We have for all  $n \in \mathbb{N}$  and for all  $a, b \in \mathbb{C}$ ,

$$(a+b)^{(n)} = \sum_{k=0}^n \binom{n}{k} a^{(n-k)} b^{(k)}, \quad (2.12)$$

$$(a+b)_{(n)} = \sum_{k=0}^n \binom{n}{k} a_{(n-k)} b_{(k)}. \quad (2.13)$$

In particular, for  $a = c = -b$ , we get the orthogonality relations, for all  $n \in \mathbb{N}$  and for all  $c \in \mathbb{C}$ ,

$$\sum_{k=0}^n \binom{n}{k} c^{(n-k)} (-c)^{(k)} = \delta_{n=0}, \quad (2.14)$$

$$\sum_{k=0}^n \binom{n}{k} c_{(n-k)} (-c)_{(k)} = \delta_{n=0}. \quad (2.15)$$

7. With  $m, n \in \mathbb{N}$  and  $K \triangleq \{k_1, k_2, \dots, k_m \in \mathbb{N}\}$ , define  $|K| \triangleq k_1 + k_2 + \dots + k_m$ ,  $\#(K) \triangleq m$  and  $\binom{n}{K} \triangleq n! / (k_1! k_2! \dots k_m!)$ , expressions that are used in the last section.

### 3 The generating function $B^c(x, y, t)$

For all  $x, y, t \in \mathbb{C}$  define

$$B(x, y, t) \triangleq \begin{cases} \frac{x-y}{xe^{-\frac{x-y}{2}t} - ye^{\frac{x-y}{2}t}}, & \text{if } x \neq y; \\ \frac{1}{1-xt}, & \text{if } x = y. \end{cases} \quad (3.1)$$

**Proposition 3.1.** *The Maclaurin series of the  $c$ -th power of  $B(x, y, t)$ , for all  $c \in \mathbb{C}$ , is given by*

$$B^c(x, y, t) = \sum_{n=0}^{+\infty} 2^{-n} B_n(x, y; c) \frac{t^n}{n!}, \quad (3.2)$$

and converges absolutely and uniformly for  $|t| < \left| \frac{\ln x - \ln y}{x - y} \right|$ . For all  $n \in \mathbb{N}$ ,

$$B_n(x, y; c) = \sum_{k=0}^n B_{n,k}(c) x^{n-k} y^k, \quad (3.3)$$

with the coefficients  $B_{n,k}(c)$  satisfying, for all  $k \in \mathbb{N}_n$ ,

$$B_{n+1,k+1}(c) = (2(k+1) + c) B_{n,k+1}(c) + (2(n-k) + c) B_{n,k}(c), \quad (3.4)$$

with  $B_{0,0}(c) = 1$  and we define  $B_{n,k}(c) \triangleq 0$ , for all  $k \notin \mathbb{N}_n$ .

*Proof.* The point  $t = 0$  is an ordinary point of  $B^c(x, y, t)$ , so  $B^c(x, y, t)$  has a Maclaurin power series, converging absolutely and uniformly for  $|t| < \left| \frac{\ln x - \ln y}{x - y} \right|$ .

Define the partial differential operator

$$D(x, y, t; c) \triangleq \left(1 - \frac{x+y}{2}t\right) \frac{\partial}{\partial t} + \frac{x-y}{2} \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) - c \frac{x+y}{2}. \quad (3.5)$$

A direct calculation shows that

$$D(x, y, t; c) B^c(x, y, t) = 0. \quad (3.6)$$

Substituting in Eq. (3.6) for  $B^c(x, y, t)$  the uniformly convergent series (3.2) gives

$$\sum_{n=0}^{+\infty} 2^{-n} (D_n(x, y; c) B_n(x, y; c)) \frac{t^n}{n!} = 0,$$

wherein

$$D_n(x, y; c) \triangleq \frac{1}{2} T_1 + \frac{x-y}{2} \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) - (n+c) \frac{x+y}{2} \quad (3.7)$$

and  $T_p$  is the difference shift operator such that  $T_p B_n(x, y; c) = B_{n+p}(x, y; c)$ . This holds for any  $t$ , so we have

$$D_n(x, y; c)B_n(x, y; c) = 0. \quad (3.8)$$

Substituting in Eq. (3.8) for  $B_n(x, y; c)$  the bivariate homogeneous polynomial (3.3) gives

$$\sum_{k=0}^n (D_{n,k}(x, y; c)B_{n,k}(c)) x^{n-k} y^k = 0,$$

wherein

$$D_{n,k}(x, y; c) \triangleq \frac{1}{2}T_{1,0} - ((n - k + 1) + c/2)T_{0,-1} - (k + c/2) \quad (3.9)$$

and  $T_{p,q}$  is the bivariate difference shift operator such that  $T_{p,q}B_{n,k}(c) = B_{n+p,k+q}(c)$ . This holds for any  $x$  and  $y$ , so we have

$$D_{n,k}(x, y; c)B_{n,k}(c) = 0, \quad (3.10)$$

which is just Eq. (3.4).

From the fact that  $B^c(x, y, 0) = 1$ , we obtain  $B_0(x, y; c) = B_{0,0}(c) = 1$ .  $\square$

We have that  $B(x, y, t) = B(y, x, t)$  for all  $x, y, t \in \mathbb{C}$ , hence  $B_n(x, y; c) = B_n(y, x; c)$  for all  $n \in \mathbb{N}$ , and  $B_{n,k}(c) = B_{n,n-k}(c)$  for all  $c \in \mathbb{C}$ .

### 3.1 Special cases

(i) For  $x = 0$  or  $y = 0$ , we get

$$B^c(0, z, t) = B^c(z, 0, t) = e^{\frac{c}{2}zt}. \quad (3.11)$$

This implies that

$$B_n(0, z; c) = B_n(z, 0; c) = (cz)^n, \quad (3.12)$$

and this yields in turn that

$$B_{n,k}(c) = c^n \delta_{n=k}. \quad (3.13)$$

(ii) For  $y = \pm x$ , we get

$$B^c(x, x, t) = \frac{1}{(1 - xt)^c}, \quad (3.14)$$

$$B^c(x, -x, t) = \operatorname{sech}^c(xt). \quad (3.15)$$

This gives

$$B_n(x, x; c) = 2^n c^{(n)} x^n, \quad (3.16)$$

$$B_n(x, -x; c) = 2^n \left( \lim_{t \rightarrow 0} D_t^n \operatorname{sech}^c t \right) x^n, \quad (3.17)$$

and this yields in turn

$$\sum_{k=0}^n B_{n,k}(c) = 2^n c^{(n)}, \quad (3.18)$$

$$\sum_{k=0}^n (-1)^k B_{n,k}(c) = 2^n \left( \lim_{t \rightarrow 0} D_t^n \operatorname{sech}^c t \right). \quad (3.19)$$

n \ k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	6	1			
4	1	23	23	1		
5	1	76	230	76	1	
6	1	237	1682	1682	237	1

Table 1: The number triangle  $B_{n,k}$

### 3.2 The numbers $B_{n,k}(1)$ and $B_{n,k}(2)$

Putting  $c = 0$  in Eq. (3.2) shows that  $B_n(x, y; 0) = \delta_{n=0}$ , so  $B_{n,k}(0) = \delta_{n=0}$ .

(i) For  $c = 1$ , Eq. (3.4) becomes

$$B_{n+1,k+1}(1) = (2(k+1) + 1) B_{n,k+1}(1) + (2(n-k) + 1) B_{n,k}(1),$$

so

$$B_{n,k}(1) = B_{n+1,k+1}, \quad (3.20)$$

with  $B_{n,k}$  the numbers derived by MacMahon [5, p. 331], (Sloane's [A060187](#)), cf. Table 1.

In this case, Eqs. (3.18) and (3.19) become

$$\sum_{k=0}^n B_{n+1,k+1} = 2^n n!, \quad (3.21)$$

$$\sum_{k=0}^n (-1)^k B_{n+1,k+1} = 2^n E_n, \quad (3.22)$$

with  $E_n$  the Euler (or secant) numbers [1, p. 804, 23.1.2], ( $|E_{2n}|$  are Sloane's [A000364](#)). The numbers  $B_{n,k}$  are thus (also) generated by (for  $|t| < \frac{1}{2} \left| \frac{\ln y}{1-y} \right|$  and  $y \neq 1$ )

$$\frac{1-y}{e^{-(1-y)t} - ye^{+(1-y)t}} = \sum_{n=0}^{+\infty} \sum_{k=0}^n B_{n+1,k+1} y^k \frac{t^n}{n!}. \quad (3.23)$$

We can also obtain from Eq. (3.23) the following more standard generating function for the  $B_{n,k}$ , (i.e., on the same footing as Eq. (3.29) below), (for  $|t| < \left| \frac{\ln y}{1-y} \right|$  and  $y \neq 1$ )

$$\frac{1}{2} \ln \frac{e^{-\frac{1-y^2}{2}t} + ye^{+\frac{1-y^2}{2}t}}{1+y} \frac{1+y}{e^{-\frac{1-y^2}{2}t} - ye^{+\frac{1-y^2}{2}t}} = \sum_{n=1}^{+\infty} \sum_{k=1}^n B_{n,k} y^{2k-1} \frac{t^n}{n!}. \quad (3.24)$$

Eqs. (3.23) and (3.24) appear to be new.

(ii) For  $c = 2$ , Eq. (3.4) becomes

$$B_{n+1,k+1}(2) = (k+2) 2B_{n,k+1}(2) + (n-k+1) 2B_{n,k}(2),$$

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Table 2: The number triangle  $A_{n,k}$

so

$$B_{n,k}(2) = 2^n A_{n+1,k+1}, \quad (3.25)$$

with  $A_{n,k}$  the Eulerian numbers [2], (Sloane's [A008292](#)), cf. Table 2. Another notation for the Eulerian numbers is  $\langle n \rangle_k = A_{n,k+1}$ .

In this case, Eqs. (3.18) and (3.19) become

$$\sum_{k=0}^n A_{n+1,k+1} = (n+1)!, \quad (3.26)$$

$$\sum_{k=0}^n (-1)^k A_{n+1,k+1} = 2^{n+2} (2^{n+2} - 1) \frac{B_{n+2}}{n+2}, \quad (3.27)$$

with  $B_n$  the Bernoulli numbers [1, p. 804, 23.1.2], ( $|B_n|$  are Sloane's [A027641](#) and [A027642](#)). In Eq. (3.27) we used  $D_t^n \operatorname{sech}^2 t = D_t^{n+1} \tanh t$ . The Eulerian numbers  $A_{n,k}$  are thus (also) generated by

$$\frac{1}{\left(\frac{e^{-\frac{1-y}{2}t} - ye^{\frac{1-y}{2}t}}{1-y}\right)^2} = \sum_{n=0}^{+\infty} \sum_{k=0}^n A_{n+1,k+1} y^k \frac{t^n}{n!}. \quad (3.28)$$

The well-known standard generating function for the Eulerian numbers is

$$\frac{1-y}{1-ye^{(1-y)t}} = 1 + \sum_{n=1}^{+\infty} \sum_{k=1}^n A_{n,k} y^k \frac{t^n}{n!}. \quad (3.29)$$

For further convenience we define  $A_{n,k} \triangleq 0$  and  $B_{n,k} \triangleq 0$ , for all  $k \notin \mathbb{Z}_{+,n}$ .

### 3.3 Examples of some $B_n(x, y; c)$

The first six  $B_n(x, y; c)$  are:

$$\begin{aligned} B_0(x, y; c) &= 1, \\ B_1(x, y; c) &= cx + cy, \\ B_2(x, y; c) &= c^2 x^2 + 2c(2+c)xy + c^2 y^2, \\ B_3(x, y; c) &= c^3 x^3 + c(3c^2 + 12c + 8)x^2 y + c(3c^2 + 12c + 8)xy^2 + c^3 y^3, \end{aligned}$$

$$\begin{aligned}
B_4(x, y; c) &= \begin{aligned} & c^4 x^4 \\ & + 4c(c^3 + 6c^2 + 8c + 4)x^3 y \\ & + 2c(3c^3 + 24c^2 + 56c + 32)x^2 y^2, \\ & + 4c(c^3 + 6c^2 + 8c + 4)xy^3 \\ & + c^4 y^4 \end{aligned} \\
B_5(x, y; c) &= \begin{aligned} & c^5 x^5 \\ & + c(5c^4 + 40c^3 + 80c^2 + 80c + 32)x^4 y \\ & + c(10c^4 + 120c^3 + 480c^2 + 720c + 352)x^3 y^2 \\ & + c(10c^4 + 120c^3 + 480c^2 + 720c + 352)x^2 y^3 \cdot \\ & + c(5c^4 + 40c^3 + 80c^2 + 80c + 32)xy^4 \\ & + c^5 y^5 \end{aligned}
\end{aligned}$$

## 4 Properties of the $B_n(x, y; c)$

### 4.1 Additive property with respect to the parameter $c$

Obviously, for all  $a, b \in \mathbb{C}$ ,

$$B^{a+b}(x, y, t) = B^a(x, y, t)B^b(x, y, t), \quad (4.1)$$

and from this follows, for all  $n \in \mathbb{N}$ ,

$$B_n(x, y; a + b) = \sum_{k=0}^n \binom{n}{k} B_{n-k}(x, y; a) B_k(x, y; b). \quad (4.2)$$

Substituting Eq. (3.3) in Eq. (4.2) gives

$$B_{n,k}(a + b) = \sum_{p=0}^n \binom{n}{p} \sum_{q=0}^k B_{n-p,k-q}(a) B_{p,q}(b). \quad (4.3)$$

For instance, by letting  $a = b = 1$ , Eq. (4.3) yields the following quadratic expansion of Eulerian numbers  $A_{n,k}$  in the MacMahon numbers  $B_{n,k}$ ,

$$A_{n+1,k+1} = \frac{1}{2^n} \sum_{p=0}^n \binom{n}{p} \sum_{q=0}^k B_{n-p+1,k-q+1} B_{p+1,q+1}. \quad (4.4)$$

### 4.2 Infinite series

**Proposition 4.1.** *For all  $c \in \mathbb{C}$  and for all  $n \in \mathbb{N}$ ,*

$$B_n(x, y; c) = \begin{cases} \frac{(x-y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)} (2k+c)^n \frac{(y/x)^k}{k!} & \text{if } |y| < |x|; \\ \frac{(y-x)^{n+c}}{y^c} \sum_{k=0}^{+\infty} c^{(k)} (2k+c)^n \frac{(x/y)^k}{k!} & \text{if } |x| < |y|, \end{cases} \quad (4.5)$$

where  $x, y \in \mathbb{C}$  and the series converges absolutely.

*Proof.* (i) Applying Eq. (2.10) to  $B^c(x, y, t)$  gives, for all  $(x, y) \in D_{|y| < |x|} \triangleq \{(x, y) \in \mathbb{C}^2 : |y| < |x|\}$  and for all  $t \in \Lambda_t(x, y) \triangleq \{t \in \mathbb{C} : \operatorname{Re}((1 - y/x)t) < (\ln|x| - \ln|y|)\}$ , the absolutely convergent series

$$B^c(x, y, t) = \frac{(x - y)^c}{x^c} \sum_{k=0}^{+\infty} c^{(k)} \frac{(y/x)^k}{k!} e^{+(k+c/2)(x-y)t}.$$

Expanding herein  $e^{+(k+c/2)(x-y)t}$  in Maclaurin series gives

$$B^c(x, y, t) = \frac{(x - y)^c}{x^c} \sum_{k=0}^{+\infty} c^{(k)} \frac{(y/x)^k}{k!} \sum_{n=0}^{+\infty} (k + c/2)^n (x - y)^n \frac{t^n}{n!}.$$

Both series are absolutely convergent, so we may interchange the order of summation [10, p. 175, Theorem 8.3], yielding

$$B^c(x, y, t) = \sum_{n=0}^{+\infty} \left( \frac{(x - y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)} (k + c/2)^n \frac{(y/x)^k}{k!} \right) \frac{t^n}{n!}.$$

On the other hand holds by Proposition 3.1, for all  $t \in \Omega_t(x, y) \triangleq \left\{ t \in \mathbb{C} : |t| < \left| \frac{\ln x - \ln y}{x - y} \right| \right\}$ , that

$$B^c(x, y, t) = \sum_{n=0}^{+\infty} 2^{-n} B_n(x, y; c) \frac{t^n}{n!}.$$

For  $(x, y) \in D_{|y| < |x|}$ ,  $\Lambda_t(x, y) \cap \Omega_t(x, y) \neq \emptyset$ . Then for all  $t \in \Lambda_t(x, y) \cap \Omega_t(x, y)$  holds

$$\sum_{n=0}^{+\infty} 2^{-n} B_n(x, y; c) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \left( \frac{(x - y)^{n+c}}{x^c} \sum_{k=0}^{+\infty} c^{(k)} (k + c/2)^n \frac{(y/x)^k}{k!} \right) \frac{t^n}{n!},$$

and the first part of Eq. (4.5) follows.

(ii) Similar. □

In particular, Eq. (4.5) becomes, for  $c = m \in \mathbb{Z}_+$ ,

$$B_n(x, y; m) = (x - y)^{n+m} \sum_{k=0}^{+\infty} \binom{m-1+k}{k} (2k + m)^n x^{-(k+m)} y^k, \quad (4.6)$$

and for  $c = -m \in \mathbb{Z}_-$ ,

$$B_n(x, y; -m) = (x - y)^{n-m} \sum_{k=0}^m (-1)^k \binom{m}{k} (2k - m)^n x^{m-k} y^k. \quad (4.7)$$

Moreover, Eq. (4.5) reduces to the following special form, for all  $c \in \mathbb{C}$ ,

$$\sum_{k=0}^{+\infty} c^{(k)} (2k + c)^n \frac{z^k}{k!} = \frac{B_n(1, z; c)}{(1 - z)^{n+c}}, |z| < 1, \quad (4.8)$$

$$\sum_{k=0}^{+\infty} c^{(k)} (2k + c)^n \frac{z^{-k}}{k!} = z^c \frac{B_n(z, 1; c)}{(z - 1)^{n+c}}, |z| > 1. \quad (4.9)$$

In particular, Eqs. (4.8) and (4.9) become,  
(i) for  $c = 1$ , for all  $n \in \mathbb{N}$ ,

$$\sum_{k=0}^{+\infty} (2k+1)^n z^k = \frac{B_n(1, z)}{(1-z)^{n+1}}, |z| < 1, \quad (4.10)$$

$$\sum_{k=0}^{+\infty} (2k+1)^n z^{-k} = z \frac{B_n(z, 1)}{(z-1)^{n+1}}, |z| > 1. \quad (4.11)$$

Herein is  $B_n(x, y)$  the MacMahon homogeneous bivariate polynomial,

$$B_n(x, y; 1) \triangleq B_n(x, y) = \sum_{k=0}^n B_{n+1, k+1} x^{n-k} y^k. \quad (4.12)$$

(ii) for  $c = 2$ , for all  $n \in \mathbb{Z}_+$ ,

$$\sum_{l=1}^{+\infty} l^n z^l = \frac{z}{(1-z)^{n+1}} A_{n-1}(1, z), |z| < 1, \quad (4.13)$$

$$\sum_{l=1}^{+\infty} l^n z^{-l} = \frac{z}{(z-1)^{n+1}} A_{n-1}(z, 1), |z| > 1. \quad (4.14)$$

Herein is  $A_n(x, y)$  the Eulerian homogeneous bivariate polynomial,

$$2^{-n} B_n(x, y; 2) \triangleq A_n(x, y) = \sum_{k=0}^n A_{n+1, k+1} x^{n-k} y^k. \quad (4.15)$$

Notice that the left hand side of Eq. (4.13) is by definition the polylogarithm of negative integer order,  $\text{Li}_{-n}(z)$  [6]. Further, combining Eq. (4.13) with [13, Eq. (14)], we get the interesting identity, for all  $n \in \mathbb{N}$ ,

$$\sum_{p=1}^n (-1)^{n-p} p! S(n, p) z^{p-1} = A_{n-1}(z, z-1). \quad (4.16)$$

Taking in Eq. (4.7) the  $\lim_{y \rightarrow -x}$  and using Eq. (3.15) we obtain, for all  $m, n \in \mathbb{N}$ ,

$$\lim_{t \rightarrow 0} D_t^n \cosh^m t = \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} (2k-m)^n. \quad (4.17)$$

Taking in Eq. (4.7) the  $\lim_{y \rightarrow x}$  and using Eq. (3.16) yields, for all  $m \in \mathbb{N}$ ,

$$\frac{1}{2^m} \sum_{k=0}^m (-1)^k \binom{m}{k} (2k-m)^m = (-1)^m m!. \quad (4.18)$$

### 4.3 Generating expression

**Proposition 4.2.** For all  $n \in \mathbb{N}$  and for all  $c, z \in \mathbb{C}$ ,

$$B_n(1, z; c) = (1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c}. \quad (4.19)$$

*Proof.* It is easy to show from Eq. (2.11) that, for all  $n \in \mathbb{N}$ , for all  $b, x \in \mathbb{C}$  and for all  $z \in \{z \in \mathbb{C} : |z| < 1\}$ , the following identity holds

$$(x + 2zD_z)^n (1 + z)^b = \sum_{k=0}^{+\infty} (2k + x)^n b_{(k)} \frac{z^k}{k!}.$$

Then

$$(1 + z)^a (x + 2zD_z)^n (1 + z)^b = \sum_{k=0}^{+\infty} \left( \sum_{l=0}^k \binom{k}{l} a_{(k-l)} b_{(l)} (2l + x)^n \right) \frac{z^k}{k!}.$$

Putting  $x = c$ ,  $a = n + c$ ,  $b = -c$  and substituting  $z \rightarrow -z$ , we get

$$\begin{aligned} & (1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c} \\ &= \sum_{k=0}^{+\infty} \left( (-1)^k \sum_{l=0}^k \binom{k}{l} (n + c)_{(k-l)} (-c)_{(l)} (2l + c)^n \right) \frac{z^k}{k!}. \end{aligned}$$

Due to the fact that  $(1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c}$  is a polynomial of degree  $n$  in  $z$ , we must have that

$$(-1)^k \sum_{l=0}^k \binom{k}{l} (n + c)_{(k-l)} (-c)_{(l)} (2l + c)^n = 0,$$

for all  $k \notin \mathbb{Z}_{+,n}$ . Hence using Eq. (5.5) below, Eq. (3.3) and the fact that  $B_{n,k}(c) \triangleq 0$ , for all  $k \notin \mathbb{Z}_{+,n}$ , Eq. (4.19) follows.  $\square$

In particular, for  $c = 1$ , we obtain

$$(1 - z)^{n+1} (1 + 2zD_z)^n (1 - z)^{-1} = \sum_{k=0}^n B_{n+1,k+1} z^k, \quad (4.20)$$

and for  $c = 2$ , we obtain

$$(1 - z)^{n+2} (1 + zD_z)^n (1 - z)^{-2} = \sum_{k=0}^n A_{n+1,k+1} z^k. \quad (4.21)$$

These appear to be new generating expressions for the MacMahon and Eulerian numbers.

## 5 Properties of the $B_{n,k}(c)$

**Proposition 5.1.** For all  $m, n \in \mathbb{N}$  and for all  $c \in \mathbb{C}$ ,

$$\sum_{k=0}^m \binom{m}{k} (n+c)^{(m-k)} (k!B_{n,k}(c)) = c^{(m)} (2m+c)^n. \quad (5.1)$$

*Proof.* For all  $c \in \mathbb{C}$  and for all  $z \in \mathbb{C}$  such that  $|z| < 1$  we have the absolutely convergent series (4.8). As  $|z| < 1$ , we can apply Eq. (2.10) and get

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{k=0}^n k!B_{n,k}(c) \frac{z^k}{k!} \sum_{l=0}^{+\infty} (n+c)^{(l)} \frac{z^l}{l!}.$$

Interchanging the summation order gives

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{l=0}^{+\infty} \sum_{k=0}^n \binom{k+l}{k} (n+c)^{(l)} k!B_{n,k}(c) \frac{z^{k+l}}{(k+l)!}.$$

With the definition  $B_{n,k}(c) \triangleq 0$ , for all  $k \notin \mathbb{Z}_{+,n}$ , we can write this as

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{k+l}{k} (n+c)^{(l)} k!B_{n,k}(c) \frac{z^{k+l}}{(k+l)!}.$$

This is equivalent to

$$\sum_{m=0}^{+\infty} c^{(m)} (2m+c)^n \frac{z^m}{m!} = \sum_{m=0}^{+\infty} \sum_{k=0}^m \binom{m}{k} (n+c)^{(m-k)} k!B_{n,k}(c) \frac{z^m}{m!},$$

and since  $z$  is arbitrary, Eq. (5.1) follows.  $\square$

In particular, for  $c = 1$ , we obtain

$$\sum_{k=0}^m \binom{n+m-k}{n} B_{n+1,k+1} = (2m+1)^n, \quad (5.2)$$

and for  $c = 2$ , we obtain

$$\sum_{k=0}^m \binom{n+1+m-k}{n+1} A_{n+1,k+1} = (m+1)^{n+1}. \quad (5.3)$$

These are well-known partial sums of the MacMahon and Eulerian number triangles [5, p. 328 and p. 331].

## 5.1 Expressions

**Proposition 5.2.** For all  $n \in \mathbb{N}$ , for all  $k \in \mathbb{N}_n$  and for all  $c \in \mathbb{C}$ ,

$$k!B_{n,k}(c) = \sum_{l=0}^k \binom{k}{l} (-n+c)^{(k-l)} c^l (2l+c)^n, \quad (5.4)$$

$$= (-1)^k \sum_{l=0}^k \binom{k}{l} (n+c)_{(k-l)} (-c)_{(l)} (2l+c)^n. \quad (5.5)$$

*Proof.* (i) We will show that Eq. (5.4) is a solution of Eq. (5.1). Substitute Eq. (5.4) in Eq. (5.1) and get

$$\sum_{k=0}^m \binom{m}{k} (n+c)^{(m-k)} \sum_{l=0}^k \binom{k}{l} (-n+c)^{(k-l)} c^l (2l+c)^n = c^{(m)} (2m+c)^n.$$

Interchanging the summation order gives

$$\sum_{l=0}^m \binom{m}{l} \left( \sum_{q=0}^{m-l} \binom{m-l}{q} (n+c)^{(m-l-q)} (-n+c)^{(q)} \right) c^l (2l+c)^n = c^{(m)} (2m+c)^n.$$

Due to the orthogonality relation (2.14) this simplifies to

$$\sum_{l=0}^m \binom{m}{l} \delta_{l=m} c^l (2l+c)^n = c^{(m)} (2m+c)^n,$$

and this is an identity.

(ii) Use  $z_{(n)} = (-1)^n (-z)^{(n)}$ . □

In particular, for  $c = -m \in \mathbb{Z}_-$ ,

(i) for  $n \geq m$

$$B_{n,k}(-m) = (-1)^k \sum_{l=\max(0,k+m-n)}^{\min(k,m)} \binom{n-m}{k-l} \binom{m}{l} (2l-m)^n, \quad (5.6)$$

(ii) for  $n < m$

$$B_{n,k}(-m) = \sum_{l=0}^{\min(k,m)} (-1)^l \binom{m-n-1+(k-l)}{k-l} \binom{m}{l} (2l-m)^n. \quad (5.7)$$

An equivalent form of Eqs. (5.4) and (5.5) is

$$k!B_{n,k}(c) = \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \frac{\Gamma(n+c+1) \Gamma(c+l)}{\Gamma(n+c+1-(k-l)) \Gamma(c)} (2l+c)^n. \quad (5.8)$$

In particular, for  $c = 1$ , we obtain

$$B_{n+1,k+1} = \sum_{l=0}^k (-1)^{k-l} \binom{n+1}{k-l} (2l+1)^n. \quad (5.9)$$

Expression (5.9) coincides with that given by MacMahon [5, p. 331]. For  $c = 2$  and using Eq. (3.25), we get the familiar result

$$A_{n+1,k+1} = \sum_{l=0}^k (-1)^{k-l} \binom{n+2}{k-l} (l+1)^{n+1}, \quad (5.10)$$

or equivalently, for all  $n-1 \in \mathbb{Z}_+$  and for all  $k \in \mathbb{Z}_{+,n-1}$ ,

$$\langle \binom{n-1}{k-1} \rangle = A_{n-1,k} = \sum_{l=0}^k (-1)^l \binom{n}{l} (k-l)^{n-1}. \quad (5.11)$$

Let  $S(j, i)$  denote the Stirling numbers of the second kind (Sloane's [A008277](#)).

**Proposition 5.3.** *For all  $n \in \mathbb{N}$ , for all  $k \in \mathbb{N}_n$  and for all  $c \in \mathbb{C}$ ,*

$$B_{n,k}(c) = (-1)^k \sum_{j=0}^n \binom{n}{j} 2^j c^{n-j} \sum_{i=0}^{\min(k,j)} (-1)^i \binom{n-i}{k-i} S(j, i) c^{(i)}. \quad (5.12)$$

*Proof.* Using Eqs. (5.15) and (5.16) from Proposition 5.4 below, we have

$$\begin{aligned} B_{n,k}(c) &= \sum_{l=0}^n b_{n,k,l} c^l, \\ &= (-1)^k \sum_{l=0}^n (-1)^l \sum_{p=0}^l (-1)^p \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p, i) s(i, l-p) c^l. \end{aligned}$$

Interchanging the order of the first two summation gives

$$\begin{aligned} B_{n,k}(c) &= (-1)^k \sum_{p=0}^n \sum_{l=p}^n (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p, i) s(i, l-p) c^l, \\ &= (-1)^k \sum_{p=0}^n \sum_{l=p=0}^{n-p} (-1)^{l-p} \binom{n}{n-p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p, i) s(i, l-p) c^{l-p+p}, \\ &= (-1)^k \sum_{p=0}^n \sum_{j=0}^{n-p} (-1)^j \binom{n}{n-p} 2^{n-p} \sum_{i=j}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p, i) s(i, j) c^{j+p}, \\ &= (-1)^k \sum_{q=0}^n \binom{n}{q} 2^q c^{n-q} \sum_{j=0}^q (-c)^j \sum_{i=j}^{\min(k,q)} \binom{n-i}{k-i} S(q, i) s(i, j). \end{aligned}$$

Taking into account that  $S(n, k) = s(n, k) \triangleq 0$ , for all  $k \notin \mathbb{N}_n$ , we can write this as

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^n \binom{n}{q} 2^q c^{n-q} \sum_{j=0}^q (-c)^j \sum_{i=0}^k \binom{n-i}{k-i} S(q, i) s(i, j).$$

Interchanging the two last summations yields

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^n \binom{n}{q} 2^q c^{n-q} \sum_{i=0}^k \binom{n-i}{k-i} S(q, i) \sum_{j=0}^i s(i, j) (-c)^j.$$

Using the fundamental property of the Stirling numbers of the first kind [1, p. 824, 24.1.3, I, B, 1],

$$(-c)_{(i)} = \sum_{j=0}^i s(i, j) (-c)^j,$$

we obtain

$$B_{n,k}(c) = (-1)^k \sum_{q=0}^n \binom{n}{q} 2^q c^{n-q} \sum_{i=0}^k \binom{n-i}{k-i} S(q, i) (-c)_{(i)}.$$

By using the identity  $(-c)_{(i)} = (-1)^i c^{(i)}$ , writing  $j$  for  $q$  and replacing the upper limit in the second sum with  $\min(k, j)$ , Eq. (5.12) follows.  $\square$

In particular, for  $c = 1$ , we get

$$B_{n+1,k+1} = (-1)^k \sum_{j=0}^n \binom{n}{j} 2^j \sum_{i=0}^{\min(k,j)} (-1)^i \binom{n-i}{k-i} i! S(j, i), \quad (5.13)$$

and for  $c = 2$ , we get

$$A_{n+1,k+1} = (-1)^k \sum_{j=0}^n \binom{n}{j} \sum_{i=0}^{\min(k,j)} (-1)^i \binom{n-i}{k-i} (i+1)! S(j, i). \quad (5.14)$$

These appear to be new expressions for the MacMahon and Eulerian numbers, in terms of Stirling numbers of the second kind.

## 5.2 Polynomial expression

**Proposition 5.4.** *For all  $n \in \mathbb{N}$  and for all  $z \in \mathbb{C}$ ,*

$$B_{n,k}(c) = \sum_{l=0}^n b_{n,k,l} c^l, \quad (5.15)$$

where, for all  $k, l \in \mathbb{N}_n$ ,

$$b_{n,k,l} = (-1)^{k+l} \sum_{p=\max(0,l-k)}^l (-1)^p \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} s(i, l-p) S(n-p, i). \quad (5.16)$$

*Proof.* Using

$$(c + 2zD_z)^n = \sum_{l=0}^n \binom{n}{l} c^{n-l} 2^l (zD_z)^l,$$

we get

$$(1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c} = (1 - z)^{n+c} \sum_{l=0}^n \binom{n}{l} c^{n-l} 2^l (zD_z)^l (1 - z)^{-c}.$$

Using herein the formula [9, p. 144],

$$(zD_z)^l = \sum_{p=0}^l S(l, p) z^p D_z^p,$$

we obtain

$$\begin{aligned} & (1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c} \\ = & (1 - z)^{n+c} \sum_{l=0}^n \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^l S(l, p) z^p D_z^p (1 - z)^{-c}, \\ = & \sum_{l=0}^n \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^l S(l, p) z^p c^{(p)} (1 - z)^{n-p}, \\ = & \sum_{l=0}^n \binom{n}{l} c^{n-l} 2^l \sum_{p=0}^l S(l, p) (-c)_{(p)} (1 - z)^{n-p} (-z)^p, \\ = & \sum_{p=0}^n \sum_{l=p}^n \binom{n}{l} c^{n-l} 2^l S(l, p) (-c)_{(p)} (1 - z)^{n-p} (-z)^p. \end{aligned}$$

Substituting herein the binomial expansion for  $(1 - z)^{n-p}$  gives

$$\begin{aligned} & (1 - z)^{n+c} (c + 2zD_z)^n (1 - z)^{-c} \\ = & \sum_{p=0}^n \sum_{l=p}^n \binom{n}{l} c^{n-l} 2^l S(l, p) (-c)_{(p)} \sum_{q=0}^{n-p} \binom{n-p}{q} (-z)^q (-z)^p, \\ = & \sum_{p=0}^n \sum_{l=p}^n \sum_{m=p}^n \binom{n}{l} \binom{n-p}{m-p} c^{n-l} 2^l (-c)_{(p)} S(l, p) (-z)^m, \\ = & \sum_{p=0}^n \sum_{m=p}^n \sum_{l=p}^n \binom{n}{l} \binom{n-p}{m-p} c^{n-l} 2^l (-c)_{(p)} S(l, p) (-z)^m, \\ = & \sum_{m=0}^n \sum_{p=0}^m \binom{n-p}{m-p} (-c)_{(p)} \sum_{l=p}^n \binom{n}{l} c^{n-l} 2^l S(l, p) (-z)^m. \end{aligned}$$

Making use of the fundamental relation of the Stirling numbers of the first kind,

$$(-c)_{(p)} = \sum_{k=0}^p s(p, k) (-c)^k,$$

we get

$$\begin{aligned}
& (1-z)^{n+c} (c+2zD_z)^n (1-z)^{-c} \\
&= \sum_{m=0}^n \sum_{p=0}^m \binom{n-p}{m-p} \sum_{k=0}^p (-1)^k s(p,k) \sum_{q=p}^n \binom{n}{q} 2^q S(q,p) c^{n-q+k} (-z)^m, \\
&= \sum_{m=0}^n \sum_{p=0}^m \binom{n-p}{m-p} \sum_{k=0}^p (-1)^k s(p,k) \sum_{l=0}^{n-p} \binom{n}{n-l} 2^{n-l} S(n-l,p) c^{l+k} (-z)^m.
\end{aligned}$$

Summing over diagonals in the two last summations and putting  $S(n,k) = s(n,k) \triangleq 0$ , for all  $k \notin \mathbb{N}_n$ , this becomes

$$\begin{aligned}
& (1-z)^{n+c} (c+2zD_z)^n (1-z)^{-c} \\
&= \sum_{m=0}^n \sum_{p=0}^m \binom{n-p}{m-p} \sum_{q=0}^n \sum_{l=0}^{n-p} (-1)^{q-l} \binom{n}{l} 2^{n-l} S(n-l,p) s(p,q-l) c^q (-z)^m, \\
&= \sum_{m=0}^n \sum_{q=0}^n \sum_{p=0}^m \sum_{l=0}^{n-p} \binom{n-p}{m-p} (-1)^{q-l} \binom{n}{l} 2^{n-l} S(n-l,p) s(p,q-l) c^q (-z)^m, \\
&= \sum_{m=0}^n \sum_{q=0}^n \sum_{p=0}^m \sum_{l=0}^n \binom{n-p}{m-p} (-1)^{q-l} \binom{n}{l} 2^{n-l} S(n-l,p) s(p,q-l) c^q (-z)^m, \\
&= \sum_{m=0}^n \sum_{q=0}^n \sum_{l=0}^n \sum_{p=0}^m \binom{n-p}{m-p} (-1)^{q-l} \binom{n}{l} 2^{n-l} S(n-l,p) s(p,q-l) c^q (-z)^m, \\
&= \sum_{m=0}^n \sum_{q=0}^n \sum_{l=0}^n (-1)^{q-l} \binom{n}{l} 2^{n-l} \sum_{p=\max(0,q-l)}^{\min(m,n-l)} \binom{n-p}{m-p} S(n-l,p) s(p,q-l) c^q (-z)^m.
\end{aligned}$$

Renaming indexes gives

$$\begin{aligned}
& (1-z)^{n+c} (c+2zD_z)^n (1-z)^{-c} \\
&= \sum_{k=0}^n (-1)^k \sum_{l=0}^n (-1)^l \sum_{j=0}^n \binom{n}{j} (-1)^j \\
&\quad \left( 2^{n-j} \sum_{i=\max(0,l-n+n-j)}^{\min(k,n-j)} \binom{n-i}{k-i} S(n-j,i) s(i,l-n+n-j) \right) c^l z^k, \\
&= \sum_{k=0}^n (-1)^k \sum_{l=0}^n (-1)^l \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \\
&\quad \left( 2^j \sum_{i=\max(0,j+l-n)}^{\min(k,j)} \binom{n-i}{k-i} S(j,i) s(i,j+l-n) \right) c^l z^k.
\end{aligned}$$

Applying Proposition 4.2 and identification with

$$B_n(1, z; c) = \sum_{k=0}^n \left( \sum_{l=0}^n b_{n,k,l} c^l \right) z^k,$$

yields, for all  $k, l \in \mathbb{N}_n$ ,

$$b_{n,k,l} = (-1)^{n+k+l} \sum_{j=n-l}^n (-1)^j \binom{n}{j} \left( 2^j \sum_{i=j-(n-l)}^{\min(k,j)} \binom{n-i}{k-i} S(j, i) s(i, j - (n-l)) \right).$$

This can be rewritten as

$$\begin{aligned} b_{n,k,l} &= (-1)^{n+l+k} \sum_{j=n-l}^n (-1)^j \binom{n}{j} 2^j \sum_{i=j-(n-l)}^{\min(k,j)} \binom{n-i}{k-i} S(j, i) s(i, j - (n-l)), \\ &= (-1)^k \sum_{j-(n-l)=0}^l (-1)^{j-(n-l)} \binom{n}{j-(n-l)+(n-l)} \\ &\quad 2^{j-(n-l)+(n-l)} \sum_{i=j-(n-l)}^{\min(k, j-(n-l)+(n-l))} \binom{n-i}{k-i} S(j - (n-l) + (n-l), i) s(i, j - (n-l)), \\ &= (-1)^k \sum_{q=0}^l (-1)^q \binom{n}{q+(n-l)} \\ &\quad 2^{q+(n-l)} \sum_{i=q}^{\min(k, q+(n-l))} \binom{n-i}{k-i} S(q + (n-l), i) s(i, q), \\ &= (-1)^k \sum_{q=0}^l (-1)^q \binom{n}{l-q} 2^{n-(l-q)} \sum_{i=q}^{\min(k, n-(l-q))} \binom{n-i}{k-i} S(n - (l-q), i) s(i, q), \\ &= (-1)^k \sum_{p=\max(0, l-k)}^l (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{i=l-p}^{\min(k, n-p)} \binom{n-i}{k-i} S(n - p, i) s(i, l - p). \end{aligned}$$

□

Using basic properties of the Stirling numbers, it is easy to derive the following special values for the  $b_{n,k,l}$ ,

$$b_{n,n,l} = b_{n,0,l} = \delta_{n=l}, \quad (5.17)$$

$$b_{n,k,n} = \binom{n}{k}, \quad (5.18)$$

$$b_{n,k,0} = \delta_{k=0} \delta_{n=0}. \quad (5.19)$$

### 5.3 Symmetry

Recall Eq. (5.5) and a variant of it obtained by replacing  $k$  with  $n - k$ ,

$$\begin{aligned} k!B_{n,k}(c) &= (-1)^k \sum_{l=0}^k \binom{k}{l} (n+c)_{(k-l)} (-c)_{(l)} (2l+c)^n, \\ (n-k)!B_{n,n-k}(c) &= (-1)^{n-k} \sum_{l=0}^{n-k} \binom{n-k}{l} (n+c)_{(n-k-l)} (-c)_{(l)} (2l+c)^n. \end{aligned}$$

Due to the symmetry  $B_{n,k}(c) = B_{n,n-k}(c)$ , there must hold, for all  $n \in \mathbb{N}$ , for all  $k \in \mathbb{N}_n$  and for all  $c \in \mathbb{C}$ , that

$$\begin{aligned} &\frac{(-1)^k}{k!} \sum_{l=0}^k \binom{k}{l} (n+c)_{(k-l)} (-c)_{(l)} (2l+c)^n \\ &= \frac{(-1)^{n-k}}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (n+c)_{(n-k-l)} (-c)_{(l)} (2l+c)^n. \end{aligned} \quad (5.20)$$

In particular, for  $k = 0$ , Eq. (5.20) yields

$$\frac{(-1)^n}{n!} \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{c+n}{c+l} (c+2l)^n = \frac{c^n}{c^{(n)}}, \quad (5.21)$$

and for  $k = 1$ ,

$$\begin{aligned} &\frac{(-1)^n}{n!} \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{(c+n+1)(c+n)}{(c+l+1)(c+l)} (c+2l)^{n+1} \\ &= \frac{c(c+2)^{n+1} - (c+n+1)c^{n+1}}{c^{(n)}}. \end{aligned} \quad (5.22)$$

### 5.4 A result of Ruiz

Summing Eq. (5.8) over  $k$  from 0 to  $n$  and using result (3.18), we get, for all  $n \in \mathbb{N}$  and for all  $c \in \mathbb{C}$ ,

$$\begin{aligned} c^{(n)} &= 2^{-n} \sum_{k=0}^n B_{n,k}(c), \\ &= \sum_{k=0}^n \sum_{l=0}^k (-1)^{k-l} \binom{n+c}{k-l} \binom{c-1+l}{l} (l+c/2)^n, \end{aligned}$$

or

$$\sum_{l=0}^n \left( \sum_{q=0}^{n-l} (-1)^q \binom{n+c}{q} \right) \binom{c-1+l}{l} (l+c/2)^n = c^{(n)}.$$

Using the binomial identity

$$\sum_{q=0}^{n-l} (-1)^q \binom{n+c}{q} = (-1)^{n-l} \binom{n+c-1}{n-l},$$

we get

$$\sum_{l=0}^n (-1)^{n-l} \binom{n+c-1}{n-l} \binom{c-1+l}{l} (l+c/2)^n = c^{(n)},$$

or

$$\binom{n+c-1}{n} \sum_{l=0}^n (-1)^{n-l} \binom{n}{l} (l+c/2)^n = c^{(n)},$$

or

$$\sum_{l=0}^n (-1)^{n-l} \binom{n}{l} (l+c/2)^n = n!,$$

or finally

$$\frac{1}{n!} \sum_{l=0}^n (-1)^l \binom{n}{l} ((-c/2) - l)^n = 1. \quad (5.23)$$

Identity (5.23) is a result of Ruiz [8]. Written in the form

$$\frac{1}{n!} \sum_{l=0}^n (-1)^{n-l} \binom{n}{l} (c+2l)^n = 2^n, \quad (5.24)$$

it can be regarded as the first identity in a series of identities of which Eqs. (5.21) and (5.22) are the next two successors. Ruiz's result however is special because the sum in Eq. (5.24) is independent of  $c$ .

In addition, applying  $D_c^m$  to Eq. (5.24) we obtain the following derived identities, for all  $n \in \mathbb{N}$ , for all  $m \in \mathbb{N}_n$  and for all  $c \in \mathbb{C}$ ,

$$\sum_{l=0}^n (-1)^l \binom{n}{l} (c+2l)^{n-m} = (-1)^n 2^n n! \delta_{m=0}. \quad (5.25)$$

## 6 Properties of the $b_{n,k,l}$

Due to the symmetry relation  $B_{n,k}(c) = B_{n,n-k}(c)$ , we have that  $b_{n,n-k,l} = b_{n,k,l}$ , for all  $n \in \mathbb{N}$  and for all  $k, l \in \mathbb{N}_n$ .

### 6.1 Recursion relation for the $b_{n,k,l}$

**Proposition 6.1.** *For all  $n \in \mathbb{N}$  and for all  $k, l \in \mathbb{N}_n$ ,*

$$b_{n+1,k+1,l+1} = 2(k+1)b_{n,k+1,l+1} + b_{n,k+1,l} + 2(n-k)b_{n,k,l+1} + b_{n,k,l}, \quad (6.1)$$

with  $b_{0,0,0} = 1$  and  $b_{n,k,l} \triangleq 0$  if  $k, l \notin \mathbb{Z}_{+,n}$ .

*Proof.* From the fact that  $B_{n,k}(0) = \delta_{n=0}$  we find that  $b_{n,k,0} = \delta_{k=0}\delta_{n=0}$  and hence  $b_{0,0,0} = 1$ . Substituting Eq. (5.15) in Eq. (3.4), we get

$$\sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l = (2(k+1) + c) \sum_{l=0}^n b_{n,k+1,l}c^l + (2(n-k) + c) \sum_{l=0}^n b_{n,k,l}c^l.$$

This is equivalent to

$$\begin{aligned} \sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{l=0}^n b_{n,k+1,l}c^{l+1} + \sum_{l=0}^n b_{n,k,l}c^{l+1} \\ &\quad + 2(k+1) \sum_{l=0}^n b_{n,k+1,l}c^l + 2(n-k) \sum_{l=0}^n b_{n,k,l}c^l, \end{aligned}$$

or

$$\begin{aligned} \sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{q=1}^{n+1} b_{n,k+1,q-1}c^q + \sum_{q=1}^{n+1} b_{n,k,q-1}c^q \\ &\quad + 2(k+1) \sum_{l=0}^n b_{n,k+1,l}c^l + 2(n-k) \sum_{l=0}^n b_{n,k,l}c^l, \end{aligned}$$

or, because  $b_{n,k,l} \triangleq 0$  if  $k, l \notin \mathbb{Z}_{+,n}$ , we get

$$\begin{aligned} \sum_{l=0}^{n+1} b_{n+1,k+1,l}c^l &= \sum_{q=0}^{n+1} b_{n,k+1,q-1}c^q + \sum_{q=0}^{n+1} b_{n,k,q-1}c^q \\ &\quad + 2(k+1) \sum_{l=0}^{n+1} b_{n,k+1,l}c^l + 2(n-k) \sum_{l=0}^{n+1} b_{n,k,l}c^l. \end{aligned}$$

This holds for all  $c$ , so we obtain

$$b_{n+1,k+1,l} = 2(k+1)b_{n,k+1,l} + b_{n,k+1,l-1} + 2(n-k)b_{n,k,l} + b_{n,k,l-1}.$$

□

## 6.2 Partial sums of the $b_{n,k,l}$

The following result shows that various partial sums over the number pyramid  $b_{n,k,l}$  are related to several important number triangles.

**Proposition 6.2.** For all  $n \in \mathbb{N}$  and for all  $k, l \in \mathbb{N}_n$ ,

$$\sum_{l=0}^n b_{n,k,l} = B_{n+1,k+1}, \quad (6.2)$$

$$\sum_{l=0}^n (-1)^l b_{n,k,l} = (-1)^{n-k} \left( \binom{n}{k} - 2o_n \delta_{k>0} \binom{n-1}{k-1} \right), \quad (6.3)$$

$$\sum_{k=0}^n b_{n,k,l} = (-1)^{n-l} 2^n s(n, l), \quad (6.4)$$

$$\sum_{k=0}^n (-1)^k b_{n,k,l} = e_n (-1)^{n/2} 2^n \delta_{l \leq n/2} T_{n/2, l}. \quad (6.5)$$

*Proof.* (i) This immediately follows from Eq. (3.20).

(ii) We have, for all  $z, t \in \mathbb{C}$ ,

$$\begin{aligned} & B^{-1}(1, z, t) \\ &= \frac{1}{1-z} e^{-\frac{1-z}{2}t} - \frac{z}{1-z} e^{+\frac{1-z}{2}t}, \\ &= \sum_{n=0}^{+\infty} 2^{-n} (-1)^n (1-z)^{n-1} \frac{t^n}{n!} - z \sum_{n=0}^{+\infty} 2^{-n} (1-z)^{n-1} \frac{t^n}{n!}, \\ &= \sum_{n=0}^{+\infty} 2^{-n} ((-1)^n - z) (1-z)^{n-1} \frac{t^n}{n!}, \\ &= \sum_{n=0}^{+\infty} 2^{-n} \left( \delta_{n=0} + \delta_{n>0} ((-1)^n - z) \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k \right) \frac{t^n}{n!}, \\ &= \sum_{n=0}^{+\infty} 2^{-n} \left( \delta_{n=0} + \delta_{n>0} \left( (-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k + \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^{k+1} \right) \right) \frac{t^n}{n!}, \\ &= \sum_{n=0}^{+\infty} 2^{-n} \left( \delta_{n=0} + \delta_{n>0} \left( (-1)^n \sum_{k=0}^{n-1} \binom{n-1}{k} (-z)^k + \sum_{l=1}^n \binom{n-1}{l-1} (-z)^l \right) \right) \frac{t^n}{n!}, \end{aligned}$$

or

$$\begin{aligned} & B^{-1}(1, z, t) \\ &= \sum_{n=0}^{+\infty} 2^{-n} \left( \delta_{n=0} + \delta_{n>0} \left( \begin{array}{c} (-1)^n \\ + \sum_{k=1}^{n-1} \left( (-1)^n \binom{n-1}{k} + \binom{n-1}{k-1} \right) (-z)^k \\ + (-z)^n \end{array} \right) \right) \frac{t^n}{n!}. \end{aligned}$$

With the identity

$$(-1)^n \binom{n-1}{k} + \binom{n-1}{k-1} = \left( (-1)^n + (1 - (-1)^n) \frac{k}{n} \right) \binom{n}{k},$$

the expression inside the first pair of parentheses becomes

$$\begin{aligned}
& \delta_{n=0} + \delta_{n>0} \left( (-1)^n + \sum_{k=1}^{n-1} \left( (-1)^n + (1 - (-1)^n) \frac{k}{n} \right) \binom{n}{k} (-z)^k + (-z)^n \right) \\
&= \sum_{k=0}^n \left( \delta_{n=0} + \delta_{n>0} \left( (-1)^n + (1 - (-1)^n) \frac{k}{n} \right) \right) \binom{n}{k} (-z)^k, \\
&= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left( 1 - \delta_{n>0} (1 - (-1)^n) \frac{k}{n} \right) z^k,
\end{aligned}$$

and we get for the full expression

$$B^{-1}(1, z, t) = \sum_{n=0}^{+\infty} 2^{-n} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left( 1 - \delta_{n>0} (1 - (-1)^n) \frac{k}{n} \right) z^k \frac{t^n}{n!},$$

or

$$B^{-1}(1, z, t) = \sum_{n=0}^{+\infty} 2^{-n} \sum_{k=0}^n (-1)^{n-k} \left( \binom{n}{k} - 2o_n \delta_{k>0} \binom{n-1}{k-1} \right) z^k \frac{t^n}{n!}.$$

Identifying this expression with

$$B^{-1}(1, z, t) = \sum_{n=0}^{+\infty} \sum_{k=0}^n \left( 2^{-n} \sum_{l=0}^n b_{n,k,l} (-1)^l \right) z^k \frac{t^n}{n!}$$

yields

$$\sum_{l=0}^n (-1)^l b_{n,k,l} = (-1)^{n-k} \left( \binom{n}{k} - 2o_n \delta_{k>0} \binom{n-1}{k-1} \right).$$

(iii) Applying Eq. (3.18) to the following double sum we obtain, for all  $c \in \mathbb{C}$ ,

$$\sum_{k=0}^n \sum_{l=0}^n b_{n,k,l} c^l = \sum_{k=0}^n B_{n,k}(c) = 2^n c^{(n)},$$

or

$$\sum_{l=0}^n \left( (-1)^{n-l} 2^{-n} \sum_{k=0}^n b_{n,k,l} \right) c^l = c^{(n)}.$$

Identifying this with the definition of the Stirling numbers of the first kind gives

$$(-1)^{n-l} 2^{-n} \sum_{k=0}^n b_{n,k,l} = s(n, l).$$

(iv) This result is related to the Maclaurin series of  $\operatorname{sech}^c t$ , considered in the next section. There it is shown that (i)

$$\sum_{k=0}^{2m+1} (-1)^k b_{2m+1,k,l} = 0,$$

and (ii) (see Proposition 7.1 below)

$$\sum_{k=0}^n (-1)^k b_{n,k,l} = (-1)^{n/2} 2^n e_n \delta_{l \leq n/2} T_{n/2,l},$$

where the number triangle  $T_{m,l}$  is Sloane's sequence [A088874](#).  $\square$

We can add to the sums given by Proposition 6.2, Eq. (3.25), which expressed in terms of the  $b_{n,k,l}$  reads

$$\sum_{l=0}^n b_{n,k,l} 2^l = 2^n A_{n+1,k+1}. \quad (6.6)$$

### 6.3 The numbers $T_{n,k}$

We now give an explicit expression for the numbers [A088874](#).

**Proposition 6.3.** *For all  $n \in \mathbb{N}$  and for all  $l \in \mathbb{N}_n$ ,*

$$e_n \delta_{l \leq n/2} (-1)^{n/2-l} T_{n/2,l} = \sum_{p=0}^l (-1)^p \binom{n}{p} w_{n-p,l-p}, \quad (6.7)$$

wherein

$$w_{n,m} \triangleq 2^n \sum_{k=m}^n S(n,k) s(k,m) (1/2)^k. \quad (6.8)$$

*Proof.* Summing over  $k$  from 0 to  $n$  in the expression for  $b_{n,k,l}$  given by Proposition 5.4, we get

$$\sum_{k=0}^n (-1)^k b_{n,k,l} = \sum_{p=0}^l (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{k=0}^n \sum_{i=l-p}^{\min(k,n-p)} \binom{n-i}{k-i} S(n-p,i) s(i,l-p).$$

Using  $S(n,k) = s(n,k) \triangleq 0, \forall k \notin \mathbb{N}_n$ , we can write this as

$$\sum_{k=0}^n (-1)^k b_{n,k,l} = \sum_{p=0}^l (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{k=0}^n \sum_{i=0}^k \binom{n-i}{k-i} S(n-p,i) s(i,l-p).$$

Interchanging the last two summations gives

$$\begin{aligned} \sum_{k=0}^n (-1)^k b_{n,k,l} &= \sum_{p=0}^l (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{i=0}^n \left( \sum_{k=i}^n \binom{n-i}{k-i} \right) S(n-p,i) s(i,l-p), \\ &= \sum_{p=0}^l (-1)^{l-p} \binom{n}{p} 2^{n-p} \sum_{i=0}^n 2^{n-i} S(n-p,i) s(i,l-p), \end{aligned}$$

or

$$2^{-n} \sum_{k=0}^n (-1)^k b_{n,k,l} = \sum_{p=0}^l (-1)^{l-p} \binom{n}{p} \sum_{i=l-p}^{n-p} 2^{n-p-i} S(n-p,i) s(i,l-p).$$

Using herein definition (6.8) and the result (7.8) from Proposition 7.1 below, Eq. (6.7) follows.  $\square$

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	2	3						
3	0	16	30	15					
4	0	272	588	420	105				
5	0	7 936	18 960	16 380	6 300	945			
6	0	353 792	911 328	893 640	429 660	103 950	10 395		
7	0	22 368 256	61 152 000	65 825 760	36 636 600	11 351 340	1 891 890	135 135	
8	0	1 903 757 312	5 464 904 448	6 327 135 360	3 918 554 640	1 427 025 600	310 269 960	37 837 800	2 027 025

Table 3: The number triangle  $T_{n,k}$

We give the onset of the number triangle  $T_{n,k}$  in Table 3. In particular,  $T_{n,0} = \delta_{n=0}$ ,  $T_{n,1}$  are the tangent numbers (Sloane's [A000182](#)), and  $T_{n,n} = 1.3.5\dots(2n-1)$  are the double factorial numbers (Sloane's [A001147](#)).

## 7 The Maclaurin series of $\operatorname{sech}^c t$

Substituting  $x = -y = 1$  in Eqs. (3.2) and (3.3), and using Eq. (3.15), shows that the Maclaurin series of the  $c$ -th power of  $\operatorname{sech} t$  is given by, for all  $c \in \mathbb{C}$ ,

$$\operatorname{sech}^c t = \sum_{n=0}^{+\infty} p_n(c) \frac{t^n}{n!}, \quad (7.1)$$

where

$$p_n(c) = 2^{-n} \sum_{k=0}^n (-1)^k B_{n,k}(c), \quad (7.2)$$

$$= \sum_{l=0}^n \left( 2^{-n} \sum_{k=0}^n (-1)^k b_{n,k,l} \right) c^l. \quad (7.3)$$

The polynomials  $p_n(c)$  have the following properties.

- (i)  $p_{2k+1}(c) = 0$ , for all  $k \in \mathbb{N}$  and for all  $c \in \mathbb{C}$ , because  $\operatorname{sech}^c t$  is even in  $t$ .
- (ii)  $p_n(c)$  has degree  $n/2$ . This can be seen as follows. We have, for all  $m \in \mathbb{N}$ ,  $\lim_{c \rightarrow 0} D_c^m \operatorname{sech}^c t = (\ln \operatorname{sech} t)^m$  and  $(\ln \operatorname{sech} t)^m = O(t^{2m})$ . Hence  $p_n(c)$  must have degree  $n/2$ .
- (iii)  $p_n(0) = \delta_{n=0}$ .

Then due to (i) we have, for all  $m \in \mathbb{N}$ ,

$$\sum_{k=0}^{2m+1} (-1)^k b_{2m+1,k,l} = 0, \quad (7.4)$$

and we define, in accordance with (ii),

$$\sum_{k=0}^{2m} (-1)^k b_{2m,k,l} \triangleq (-1)^m 2^{2m} \delta_{l \leq m} T_{m,l}. \quad (7.5)$$

Hence

$$p_{2m}(c) = \lim_{t \rightarrow 0} D_t^{2m} \operatorname{sech}^c t = (-1)^m \sum_{l=0}^m T_{m,l} c^l, \quad (7.6)$$

so

$$\operatorname{sech}^c t = \sum_{m=0}^{+\infty} \left( (-1)^m \sum_{l=0}^m T_{m,l} c^l \right) \frac{t^{2m}}{(2m)!}. \quad (7.7)$$

Due to (iii),  $T_{n,0} = \delta_{n=0}$ , for all  $n \in \mathbb{N}$ .

We now clarify the nature of the numbers  $T_{m,l}$  defined in Eq. (7.5).

**Proposition 7.1.** *The number triangle  $T_{n,l}$ , for all  $n \in \mathbb{N}$  and for all  $l \in \mathbb{N}_n$ , satisfies the recursion relation*

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left( \binom{p+l+1}{p+1} + \delta_{l>0} \binom{p+l}{p+1} \right) T_{n,l+p}, \quad (7.8)$$

with  $T_{n,0} = \delta_{n=0}$ .

*Proof.* Starting from the identity

$$D_t^2 \operatorname{sech}^c t = c^2 \operatorname{sech}^c t - c(c+1) \operatorname{sech}^{c+2} t,$$

and substituting herein the series expansion for  $\operatorname{sech}^c t$ , Eq. (7.7), we get

$$\begin{aligned} & \sum_{n=1}^{+\infty} (-1)^n \sum_{l=0}^n T_{n,l} c^l \frac{t^{2(n-1)}}{(2(n-1))!} \\ = & c^2 \sum_{n=0}^{+\infty} (-1)^n \sum_{l=0}^n T_{n,l} c^l \frac{t^{2n}}{(2n)!} - c(c+1) \sum_{n=0}^{+\infty} (-1)^n \sum_{l=0}^n T_{n,l} (c+2)^l \frac{t^{2n}}{(2n)!}. \end{aligned}$$

As this holds for all  $t$ , we must have that

$$- \sum_{l=0}^{n+1} T_{n+1,l} c^l = c^2 \sum_{l=0}^n T_{n,l} c^l - c(c+1) \sum_{l=0}^n T_{n,l} (c+2)^l.$$

This can be rearranged in the form

$$\sum_{l=0}^n T_{n+1,l+1} c^l = (c+1) \sum_{l=0}^n T_{n,l} (c+2)^l - \sum_{l=0}^n T_{n,l} c^{l+1}.$$

By the binomial theorem,

$$\begin{aligned} \sum_{l=0}^n T_{n+1,l+1} c^l &= \sum_{l=0}^n T_{n,l} \sum_{k=0}^l \binom{l}{k} 2^{l-k} c^{k+1} \\ &\quad + \sum_{l=0}^n T_{n,l} \sum_{k=0}^l \binom{l}{k} 2^{l-k} c^k - \sum_{l=0}^n T_{n,l} c^{l+1}. \end{aligned}$$

Interchanging the order of summation in the double sum terms gives

$$\begin{aligned} \sum_{l=0}^n T_{n+1,l+1} c^l &= \sum_{k=0}^n \sum_{q=0}^{n-k} 2^q \binom{q+k}{k} T_{n,q+k} c^{k+1} \\ &\quad + \sum_{k=0}^n \sum_{q=0}^{n-k} 2^q \binom{q+k}{k} T_{n,q+k} c^k - \sum_{k=1}^{n+1} T_{n,k-1} c^k. \end{aligned}$$

We can rearrange this further into

$$\begin{aligned} \sum_{l=0}^n T_{n+1,l+1} c^l &= \sum_{l=1}^{n+1} \sum_{q=0}^{n-(l-1)} 2^q \binom{q+l-1}{l-1} T_{n,q+l-1} c^l \\ &\quad + \sum_{l=0}^n \sum_{q=0}^{n-l} 2^q \binom{q+l}{l} T_{n,q+l} c^l - \sum_{l=1}^{n+1} T_{n,l-1} c^l. \end{aligned}$$

As this holds for all  $c$ , we must have that

$$\begin{aligned} T_{n+1,1} &= \sum_{q=0}^n 2^q T_{n,q}; \\ \sum_{l=1}^n T_{n+1,l+1} c^l &= \sum_{l=1}^n \left( \begin{aligned} &\sum_{q=0}^{n-(l-1)} 2^q \binom{q+l-1}{l-1} T_{n,q+l-1} \\ &+ \sum_{q=0}^{n-l} 2^q \binom{q+l}{l} T_{n,q+l} - T_{n,l-1} \end{aligned} \right) c^l \\ &\quad + T_{n,n} c^{n+1} - T_{n,n} c^{n+1}, \end{aligned}$$

or

$$\begin{aligned} T_{n+1,1} &= \sum_{q=0}^n 2^q T_{n,q}; \\ T_{n+1,l+1} &= \sum_{q=0}^{n-(l-1)} 2^q \binom{q+l-1}{l-1} T_{n,q+l-1} + \sum_{q=0}^{n-l} 2^q \binom{q+l}{l} T_{n,q+l} - T_{n,l-1}, l \in \mathbb{Z}_{+,n}, \end{aligned}$$

or

$$\begin{aligned} T_{n+1,1} &= \sum_{q=0}^n 2^q T_{n,q}; \\ T_{n+1,l+1} &= \sum_{q=1}^{n-l+1} 2^q \binom{q+l-1}{l-1} T_{n,q+l-1} + \sum_{q=0}^{n-l} 2^q \binom{q+l}{l} T_{n,q+l}, l \in \mathbb{Z}_{+,n}, \end{aligned}$$

or

$$T_{n+1,1} = \sum_{q=0}^n 2^q T_{n,q};$$

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^{p+1} \binom{p+l}{l-1} T_{n,p+l} + \sum_{p=0}^{n-l} 2^p \binom{p+l}{l} T_{n,p+l}, l \in \mathbb{Z}_{+,n},$$

or

$$T_{n+1,1} = \sum_{p=0}^n 2^p T_{n,p}, l = 0;$$

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left( 2 \binom{p+l}{l-1} + \binom{p+l}{l} \right) T_{n,p+l}, l \in \mathbb{Z}_{+,n},$$

or

$$T_{n+1,1} = \sum_{p=0}^n 2^p T_{n,p}, l = 0;$$

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left( 2 \binom{p+l}{p+1} + \binom{p+l}{p} \right) T_{n,p+l}, l \in \mathbb{Z}_{+,n}.$$

With the basic additive (Pascal's first) binomial identity

$$\binom{p+l}{p+1} + \binom{p+l}{p} = \binom{p+l+1}{p+1},$$

we can write this also as

$$T_{n+1,1} = \sum_{p=0}^n 2^p T_{n,p};$$

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left( \binom{p+l+1}{p+1} + \binom{p+l}{p+1} \right) T_{n,l+p}, l \in \mathbb{Z}_{+,n}.$$

Both these equations can be combined into the following single equation, for all  $l \in \mathbb{N}_n$ ,

$$T_{n+1,l+1} = \sum_{p=0}^{n-l} 2^p \left( \binom{p+l+1}{p+1} + \delta_{l>0} \binom{p+l}{p+1} \right) T_{n,l+p}.$$

Finally, from the fact that  $\operatorname{sech}^0 t = 1$  we conclude that  $T_{n,0} = \delta_{n=0}$ . □

Identifying our results with those given in Sloane [11, [A085734](#) and [A088874](#)] we see that the  $T_{n,k}$ , satisfying recursion relation (7.8) and boundary condition  $T_{n,0} = \delta_{n=0}$ , are indeed Sloane's sequence [A088874](#) (there constructed by Deleham using a delta operator). Thus

Eq. (7.7) shows that the number triangle  $T_{n,k}$  is as fundamental for the Maclaurin series of  $\operatorname{sech}^c t$  as the Euler numbers  $E_n$  are for the Maclaurin series of  $\operatorname{sech} t$ .

Eq. (7.7) also immediately leads to the orthogonality relation, for all  $n, k \in \mathbb{N}$ ,

$$\sum_{p=0}^n \binom{2n}{2p} \sum_{l=\max(0, k+p-n)}^{\min(k,p)} (-1)^l T_{n-p, k-l} T_{p,l} = \delta_{n=0} \delta_{k=0}. \quad (7.9)$$

The series (7.7) shows that the numbers  $E_{2m}^{(c)} \triangleq (-1)^m \sum_{l=0}^m T_{m,l} c^l$ , for all  $m \in \mathbb{N}$  and for all  $c \in \mathbb{C}$ , are a set of generalized Euler numbers (although they are polynomials in  $c$ ) in the sense of Luo, et al. [7]. It seems more natural however, to regard the integer numbers  $T_{m,l}$  as a more fundamental set, because they are independent of  $c$ .

## 7.1 Multinomial Euler numbers $E_n^m$

For the important special case that  $c = m \in \mathbb{N}$ , it might be convenient to introduce a generalization of the Euler numbers that are also integers.

By identifying Eq. (7.7) with the Maclaurin series of  $\operatorname{sech}^m t$ , written in the form

$$\operatorname{sech}^m t = \sum_{n=0}^{+\infty} E_n^m \frac{t^n}{n!}, \quad (7.10)$$

we get, for all  $p \in \mathbb{N}$ ,  $E_{2p+1}^m = 0$  and

$$E_{2p}^m = (-1)^p \sum_{l=0}^p T_{p,l} m^l. \quad (7.11)$$

On the other hand, we also have that  $\operatorname{sech}^m t = \left( \sum_{n=0}^{+\infty} E_n \frac{t^n}{n!} \right)^m$ , so we obtain,

$$E_n^m = \delta_{m=0} \delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} E_{k_i}. \quad (7.12)$$

Expression (7.12) suggests that the  $E_n^m$  be called *multinomial Euler numbers*. Equating the right hand sides of Eq. (7.11) and Eq. (7.12) gives

$$(-1)^n \sum_{l=0}^n T_{n,l} m^l = \delta_{m=0} \delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} E_{k_i}. \quad (7.13)$$

This reduces, for  $m = 1$ , to

$$(-1)^n \sum_{l=0}^n T_{n,l} = E_{2n}. \quad (7.14)$$

Some particular multinomial Euler numbers are mentioned in Sloane [11]. Eq. (7.11) reduces to the following particular cases:  $E_0^m = 1$ ,  $E_2^m = -m$ ,  $E_4^m = m(3m+2)$  (rhombic matchstick numbers, Sloane's [A045944](#)) and  $E_6^m = -m(15m^2 + 30m + 16)$  (not in Sloane). Also,  $E_n^0 = \delta_{n=0}$ ,  $E_n^1 = E_n$  (Euler or sech numbers,  $|E_n|$  is Sloane's [A000364](#)),  $E_{n-1}^2$  are the tanh numbers (due to  $D_t^n \operatorname{sech}^2 t = D_t^{n+1} \tanh t$ , with  $|E_{n-1}^2|$  being Sloane's [A000182](#)).

We give the onset of the number square  $E_n^m$ , for even  $n$ , in Table 4.

n \ m	1	2	3	4	5	6
0	1	1	1	1	1	1
2	-1	-2	-3	-4	-5	-6
4	5	16	33	56	85	120
6	-61	-272	-723	-1 504	-2 705	-4 416
8	1 385	7 936	25 953	64 256	134 185	249 600
10	-50 521	-353 792	-1 376 643	-3 963 904	-9 451 805	-19 781 376

Table 4: The number square  $E_n^m$

## 7.2 Even multinomial parity numbers $e_n^m$

By identifying Eq. (7.7) with the Maclaurin series of  $\cosh^m t$ , written in the form

$$\cosh^m t = \sum_{n=0}^{+\infty} e_n^m \frac{t^n}{n!}, \quad (7.15)$$

we get, for all  $p \in \mathbb{N}$ ,  $e_{2p+1}^m = 0$  and

$$e_{2p}^m = (-1)^p \sum_{l=0}^p T_{p,l}(-m)^l. \quad (7.16)$$

On the other hand, we also have that  $\cosh^m t = \left(\sum_{n=0}^{+\infty} e_n \frac{t^n}{n!}\right)^m$ , so we obtain,

$$e_n^m = \delta_{m=0} \delta_{n=0} + \delta_{m>0} \sum_{K:|K|=n} \binom{n}{K} \prod_{i=1}^{\#(K)} e_{k_i}. \quad (7.17)$$

Expression (7.17) suggests that the  $e_n^m$  be called *even multinomial parity numbers*. Eq. (4.17) gives us the following explicit expression for the  $e_n^m$  numbers,

$$e_n^m = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (m-2j)^n. \quad (7.18)$$

Equating the right hand sides of Eq. (7.16) and Eq. (7.18) gives

$$(-1)^n \sum_{l=0}^n T_{n,l}(-m)^l = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (m-2j)^{2n}. \quad (7.19)$$

In particular, for  $m = 1$ , Eq. (7.19) reduces to

$$(-1)^n \sum_{l=0}^n (-1)^l T_{n,l} = 1, \quad (7.20)$$

and, for  $m = 2$ , to

$$(-1)^n \sum_{l=0}^n T_{n,l}(-2)^l = 2^{2n-1}. \quad (7.21)$$

$n \setminus m$	1	2	3	4	5	6
0	1	1	1	1	1	1
2	1	2	3	4	5	6
4	1	8	21	40	65	96
6	1	32	183	544	1 205	2 256
8	1	128	1 641	8 320	26 465	64 896
10	1	512	14 763	131 584	628 805	2 086 656

Table 5: The number square  $e_n^m$

Various particular even multinomial parity numbers are mentioned in Sloane [11]. For instance,  $e_0^m = 1$ ,  $e_2^m = m$ ,  $e_4^m = m(3m - 2)$  (octagonal numbers, Sloane's [A000567](#)) and  $e_6^m = m(15m^2 - 30m + 16)$  (not in Sloane). Also,  $e_n^0 = \delta_{n=0}$ ,  $e_n^1 = e_n$ ,  $e_n^2 = \delta_{n=0} + \delta_{n>0} e_n 2^{n-1}$  ( $e_{2p}^2$  is Sloane's [A009117](#)),  $e_n^3 = e_n(3^n + 3)/4$  ( $e_{2p}^3$  is Sloane's [A054879](#)),  $e_n^4 = e_n(4^n + 4 \cdot 2^n)/8$  ( $e_{2p}^4$  is Sloane's [A092812](#)) and  $e_n^5 = e_n(10 + 5 \cdot 3^n + 5^n)/16$  (not in Sloane). In general, expression (7.18) shows that  $e_{2p}^m$  equals the number of closed walks, based at a vertex, of length  $2p$  along the edges of an  $m$ -dimensional cube [12].

We give the onset of the number square  $e_n^m$ , for even  $n$ , in Table 5.

### 7.3 Relations between the $E_n^m$ and the $e_n^m$ numbers

(i) Evidently, due to the fact that  $\cosh^m t \operatorname{sech}^m t = 1$ , for all  $m \in \mathbb{N}$ , holds the following orthogonality relation, for all  $n, m \in \mathbb{N}$ ,

$$\sum_{i=0}^n \binom{n}{i} e_{n-i}^m E_i^m = \delta_{n=0}. \quad (7.22)$$

Combining Eqs. (7.22) and (7.18), and using the fact that  $e_0^m = 1$ , for all  $m \in \mathbb{N}$ , we obtain the following recursion relation for the  $E_n^m$ ,

$$E_n^m = \delta_{n=0} - \delta_{n>0} \sum_{i=0}^{n-1} \binom{n}{i} \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (m-2j)^{n-i} E_i^m. \quad (7.23)$$

In particular, for  $m = 1$ , Eq. (7.23) yields

$$E_{2p} = \delta_{p=0} - \delta_{p>0} \sum_{j=0}^{p-1} \binom{2p}{2j} E_{2j}. \quad (7.24)$$

Due to the symmetry of the binomial expression in Eq. (7.22) and because  $E_0^m = 1$ , for all  $m \in \mathbb{N}$ , we get equivalently a recursion relation for the  $e_n^m$  in terms of the  $E_n^m$  numbers,

$$e_n^m = \delta_{n=0} - \delta_{n>0} \sum_{i=0}^{n-1} \binom{n}{i} E_{n-i}^m e_i^m. \quad (7.25)$$

In particular, for  $m = 1$ , by using  $e_n^1 = e_n$  and  $e_0 = 1 = E_0$ , we get

$$E_0 = \delta_{p=0} - \delta_{p>0} \sum_{j=0}^{p-1} \binom{2p}{2j} E_{2(p-j)}. \quad (7.26)$$

(ii) Comparing Eq. (7.11) with Eq. (7.16) we see that the numbers  $E_{2n}^m$  are the extension to negative integers  $m$  of the numbers  $e_{2n}^m$ , as expected from their generating functions given in Eqs. (7.10) and (7.15).

(iii) Recall Faa di Bruno's formula for the  $n$ -th derivative of a composition of two functions [1, p. 823, 24.1.2, II, C.], for all  $n \in \mathbb{Z}_+$ ,

$$D_t^n f(g(t)) = \sum_{k=1}^n (D_g^k f(g)) (t) \sum_{P(n):|K|=k} n! \prod_{i=1}^n \frac{(D_t^i g(t))^{k_i}}{(i!)^{k_i} k_i!},$$

where  $P(n) \triangleq \{K \triangleq \{k_1, k_2, \dots, k_n \in \mathbb{N}\} : 1k_1 + 2k_2 + \dots + nk_n = n\}$ . An element  $K \in P(n)$  represents a partition of a set of cardinality  $n$  into  $k_1$  classes of cardinality 1,  $k_2$  classes of cardinality 2, up to  $k_n$  classes of cardinality  $n$ .

Applied to  $f \circ g$ , with  $g(t) = \cosh^m t$  and  $f(g) = 1/g$ , we get

$$\begin{aligned} E_n^m &= \lim_{t \rightarrow 0} D_t^n f(g(t)), \\ &= \sum_{k=1}^n (-1)^k k! \sum_{P(n):|K|=k} n! \prod_{i=1}^n \frac{(e_i^m)^{k_i}}{(i!)^{k_i} k_i!}. \end{aligned}$$

Define, for all  $n \in \mathbb{N}$ ,  $S_e^m(n, 0) \triangleq \delta_{n=0}$  and if  $n > 0$ , for all  $k \in \mathbb{Z}_{+,n}$ ,

$$S_e^m(n, k) \triangleq \sum_{P(n):|K|=k} n! \prod_{i=1}^n \frac{(e_i^m)^{k_i}}{(i!)^{k_i} k_i!}. \quad (7.27)$$

Then

$$E_n^m = \sum_{k=1}^n (-1)^k k! S_e^m(n, k). \quad (7.28)$$

Eq. (7.28) expresses the multinomial Euler numbers in terms of the even multinomial parity numbers through the intermediate numbers  $S_e^m(n, k)$ .

The even parity symbol  $e_i$  in the product in Eq. (7.27) makes that all  $k_i$  with odd index  $i$  must be taken zero, so  $S_e^m(n, k)$  and hence  $E_n^m$  are both zero for odd  $n$ . With  $n = 2p$ , we get, for all  $p \in \mathbb{Z}_+$ ,

$$S_e^m(2p, k) = \sum_{P_e(2p):|K_e|=k} (2p)! \prod_{j=1}^p \frac{(e_{2j}^m)^{k_{2j}}}{((2j)!)^{k_{2j}} k_{2j}!},$$

where  $P_e(2p) \triangleq \{K_e \triangleq \{k_2, k_4, \dots, k_n \in \mathbb{N}\} : k_2 + 2k_4 + \dots + pk_{2p} = p\}$ . An element  $K_e \in P_e(2p)$  represents a partition of a set of  $2p$  elements into 0 classes of cardinality 1,  $k_2$  classes

of cardinality 2, 0 classes of cardinality 3,  $k_4$  classes of cardinality 4, up to  $k_{2p}$  classes of cardinality  $2p$ .

In particular, for  $m = 1$ , we get from Eq. (7.28), for all  $p \in \mathbb{Z}_+$ ,

$$S_e(2p, k) \triangleq S_e^1(2p, k) = \sum_{P_e(2p):|K_e|=k} (2p)! \prod_{j=1}^p \frac{1}{((2j)!)^{k_{2j}} k_{2j}!},$$

i.e., the number of ways of partitioning a set of  $2p$  elements into  $k$  non-empty subsets, each of even cardinality, and

$$E_{2p} = \sum_{k=1}^{2p} (-1)^k k! S_e(2p, k). \quad (7.29)$$

This seems to be a new expression for the (even) Euler numbers. Here the sum involves partitions into subsets of even cardinality. A similar sum, involving partitions into subsets of any cardinality, is the well-known result for the Stirling numbers of the second kind,

$$1 = \sum_{k=0}^{2p} (-1)^k k! S(2p, k).$$

It thus turns out that the numbers  $S_e^m(n, k)$ , (which by comparing Eq. (7.27) with Eq. (7.30) might be called “even multinomial Stirling numbers of the second kind”), are more natural to the  $E_n^m$  than the  $S(n, k)$ . This can be seen by applying Faa di Bruno’s formula to  $f \circ g$ , with  $g(t) = e^t$  and  $f(g) = (\frac{1}{2}(g + 1/g))^{-m}$ , and using [1, p. 823, 24.1.2, II B],

$$S(n, k) = \sum_{P(n):|K|=k} \frac{n!}{\prod_{i=1}^n (i!)^{k_i} k_i!}. \quad (7.30)$$

We get

$$E_n^m = \sum_{k=1}^n \left( \lim_{t \rightarrow 0} D_t^k \operatorname{sech}^m(\ln(1+t)) \right) S(n, k), \quad (7.31)$$

an expression more complicated than  $E_n^m = \lim_{t \rightarrow 0} D_t^n \operatorname{sech}^m(t)$ . For  $m = 1$ , the numbers defined by the expression inside the parentheses in (7.31) are Sloane’s [A009014](#).

We can derive another expression for the  $E_n^m$  in terms of the  $S(n, k)$ , directly from the generating function  $\operatorname{sech}^m(x)$ , as was done in Luo, et al. [7], but it turns out to involve a double sum. In the particular case  $m = 1$  however, we can obtain this other expression from our results by combining Eqs. (3.22) and (5.13), and then it reads

$$E_n = \sum_{k=0}^n \binom{n}{k} \sum_{l=0}^k (-1)^l l! 2^{k-l} S(k, l). \quad (7.32)$$

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