Journal of Lie Theory
Volume **11** (2001) 505–543

Cartan-Decomposition Subgroups of SU(2,n)

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Communicated by E. B. Vinberg

Abstract. We give explicit, practical conditions that determine whether or not a closed, connected subgroup H of $G = \mathrm{SU}(2,n)$ has the property that there exists a compact subset C of G with $CHC = G$. To do this, we fix a Cartan decomposition $G = KA^+K$ of G, and then carry out an approximate calculation of $(KHK) \cap A^+$ for each closed, connected subgroup H of G. This generalizes the work of H. Oh and D. Witte for $G = \mathrm{SO}(2,n)$.

1 Introduction

Definition 1.1. [14, Defn. 1.2] Let H be a closed subgroup of a connected, simple, linear, real Lie group G. We say that H is a *Cartan-decomposition subgroup* of G if

- H is connected, and
- there is a compact subset C of G, such that $CHC = G$.

(Note that C is only assumed to be a sub*set* of G; it need not be a sub*group*.)

Example 1.2. The Cartan decomposition $G = KAK$ shows that the maximal split torus A is a Cartan-decomposition subgroup of G.

It is known that $G = KNK$ [9, Thm. 5.1], so the maximal unipotent subgroup N is also a Cartan-decomposition subgroup.

If $\mathbb{R}\text{-rank}\, G = 0$ (that is, if G is compact), then every (closed, connected) subgroup of G is a Cartan-decomposition subgroup.

If $\mathbb{R}\text{-rank}\, G = 1$, then it not difficult to see that every (closed, connected) noncompact subgroup of G is a Cartan-decomposition subgroup (cf. [5, Lem. 3.2]).

It is more difficult to characterize the Cartan-decomposition subgroups when $\mathbb{R}\text{-rank}\, G = 2$, but H. Oh and D. Witte [14] studied two examples in detail. Namely, they described all the Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$ and of $\mathrm{SO}(2,n)$, and they also explicitly described the closed, connected subgroups that are *not* Cartan-decomposition subgroups. Here, we obtain similar results for $\mathrm{SU}(2,n)$. Unfortunately, the results are rather complicated to state.

ISSN 0949–5932 / \$2.50

Notation 1.3. Let $G = \mathrm{SU}(2,n)$ and fix an Iwasawa decomposition $G = KAN$ and a corresponding Cartan decomposition $G = KA^+K$, where A^+ is the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive. Thus, K is a maximal compact subgroup, A is the identity component of a maximal split torus, and N is a maximal unipotent subgroup.

To simplify, let us restrict our attention here to subgroups of N.

Theorem 1.4. (cf. 3.4) *Let $G = \mathrm{SU}(2,n)$ and let H be a closed, connected subgroup of N. Then H is a Cartan-decomposition subgroup of G if and only if*

1. *H satisfies at least one of the eight conditions in Proposition* 4.1*; and*
2. *H satisfies at least one of the five conditions in Proposition* 5.1.

Theorem 1.5. *Let $G = \mathrm{SU}(2,n)$ and let H be a closed, connected, nontrivial subgroup of N. Then H is **not** a Cartan-decomposition subgroup of G if and only if H belongs to one of the eleven types of subgroups explicitly described in Theorem* 6.1.

For subgroups H that are not contained in N, there is no loss of generality in assuming that $H \subset AN$ (see 7.1), and that H satisfies the additional technical condition of being compatible with A (see 7.3). Under these assumptions, Theorem 7.4, Proposition 7.6, and Lemma 7.8, taken together, list the possibilities for H and, in each case, determine whether H is a Cartan-decomposition subgroup or not.

Our results require an effective method to determine whether a subgroup is a Cartan-decomposition subgroup or not. This is provided by the Cartan projection.

Definition 1.6. (Cartan projection) For each element g of G, the Cartan decomposition $G = KA^+K$ implies that there is an element a of A^+ with $g \in KaK$. In fact, the element a is unique, so there is a well-defined function

$$\mu\colon G \to A^+ \text{ given by } g \in K\,\mu(g)\,K.$$

The function μ is continuous and proper (that is, the inverse image of any compact set is compact). Some properties of the Cartan projection are discussed in [1] and [7].

We have $\mu(H) = A^+$ if and only if $KHK = G$. This immediately implies that if $\mu(H) = A^+$, then H is a Cartan-decomposition subgroup. Y. Benoist and T. Kobayashi proved the deeper statement that, in the general case, H is a Cartan-decomposition subgroup if and only if $\mu(H)$ comes within a bounded distance of every point in A^+.

Notation 1.7. For subsets U and V of A^+, we write $U \approx V$ if there is a compact subset C of A, such that $U \subset VC$ and $V \subset UC$. This is an equivalence relation.

Theorem 1.8. (Benoist [1, Prop. 5.1], Kobayashi [8, Thm. 1.1]) *A closed, connected subgroup H of G is a Cartan-decomposition subgroup if and only if $\mu(H) \approx A^+$.*

Remark 1.9. We may consider $SO(2,n)$ to be the subgroup of $SU(2,n)$ consisting of the real matrices. Then, because $A \subset SO(2,n)^\circ$, we see that $SO(2,n)^\circ$ is a Cartan-decomposition subgroup of $SU(2,n)$. More generally, for any subgroup H of $SO(2,n)^\circ$, we see that H is a Cartan-decomposition subgroup of $SO(2,n)^\circ$ if and only if H is a Cartan-decomposition subgroup of $SU(2,n)$. (For example, this follows from the fact that the Cartan projection for $SO(2,n)^\circ$ is the restriction of the Cartan projection for $SU(2,n)$.) Thus, our results generalize those theorems of H. Oh and D. Witte [14] that are directed toward $SO(2,n)$.

Remark 1.10. One may define a preorder $\prec$ on the set of closed, connected subgroups of G by

$$H_1 \prec H_2 \text{ if there is a compact subset } C \text{ of } G, \text{ such that } H_1 \subset C H_2 C.$$

(So H is a Cartan-decomposition subgroup of G if and only if $G \prec H$.) We see from [1, Prop. 5.1] that $H_1 \prec H_2$ if and only if there is a compact subset C of A, such that $\mu(H_1) \subset \mu(H_2)C$. Thus, it is of interest to calculate $\mu(H)$, for each subgroup H of G. Our results solve this problem: for each (closed, connected) subgroup H, we give an explicit subset U of A^+, such that $\mu(H) \approx U$. For the cases where $\mu(H) \not\approx A^+$, these results are summarized in Tables 1, 2, and 3 of Section 8, and the subset U is given in a standard form that makes it easy to determine whether $H_1 \prec H_2$. Thus, we determine the order structure of the relation $\prec$, and also determine precisely where each subgroup lies in this preorder.

The interest in Cartan-decomposition subgroups is largely due to the following basic observation that, to construct nicely behaved actions on homogeneous spaces, one must find subgroups that are *not* Cartan-decomposition subgroups. (See [7, §3] for some historical background on this result.)

Proposition 1.11. (Calabi-Markus phenomenon, cf. [10, pf. of Thm. A.1.2]) *If H is a Cartan-decomposition subgroup of G, then no closed, noncompact subgroup of G acts properly on G/H.*

H. Oh and D. Witte [15, 16] used this proposition as a starting point to study the existence of tessellations. (A homogeneous space G/H is said to have a tessellation if there is a discrete subgroup Γ of G, such that Γ acts properly on G/H, and $\Gamma\backslash G/H$ is compact.) In particular, when n is even, they determined exactly which homogeneous spaces $SO(2,n)/H$ have a tessellation (under the assumption that H is connected). These results depend not only on the characterization of Cartan-decomposition subgroups, but also on the calculation of $\mu(H)$ for each subgroup H, and on the maximum possible dimension of subgroups with a given image under the Cartan projection. In [4] we use some of the results of the current paper to study tessellations of homogeneous spaces of $SU(2,n)$.

Here is an outline of the paper. Section 2 describes the notation we use to specify elements of $SU(2,n)$. Section 3 recalls some general results on Cartan-decomposition subgroups, and defines a representation ρ. Section 4 determines whether H contains large elements with $\|\rho(h)\|$ approximately equal to $\|h\|^2$. Similarly, Section 5 determines whether H contains large elements with $\|\rho(h)\|$ approximately equal to $\|h\|$. By combining the calculations of the preceding two sections, Section 6 determines which subgroups of N are Cartan-decomposition subgroups. Then Section 7 determines which other subgroups of G are Cartan-decomposition subgroups. Section 8 determines the maximum possible dimension of a subgroup of H with any given image under the Cartan projection.

Acknowledgments 1.12. This research was partially supported by a grant from the National Science Foundation (DMS-9801136). Much of the work was carried out during productive visits to the University of Bielefeld (Germany) and the Isaac Newton Institute for Mathematical Sciences (Cambridge, U.K.). We would like to thank the German-Israeli Foundation for Research and Development for financial support that made the visit to Bielefeld possible, and an anonymous referee for helpful comments. D.W. would also like to thank the mathematics department of the University of Maryland for its hospitality during the visit that initiated this project.

2 Explicit coordinates in $\mathrm{SU}(2,n)$

Notation 2.1. We realize $\mathrm{SU}(2,n)$ as isometries of the indefinite Hermitian form

$$\langle v \mid w \rangle = v_1\overline{w_{n+2}} + v_2\overline{w_{n+1}} + \sum_{i=3}^{n} v_i\overline{w_i} + v_{n+1}\overline{w_2} + v_{n+2}\overline{w_1}$$

on $\mathbb{C}^{n+2}$. The virtue of this particular realization is that we may choose A to consist of the diagonal matrices in $\mathrm{SU}(2,n)$ that have nonnegative real entries, and N to consist of the upper-triangular matrices in $\mathrm{SU}(2,n)$ with only 1's on the diagonal. Thus, the Lie algebra of AN is

$$\mathfrak{a}+\mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & x & \eta & i\mathsf{x} \\ 0 & t_2 & y & i\mathsf{y} & -\overline{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & -t_2 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & -t_1 \end{pmatrix} \;\middle|\; \begin{array}{c} t_1, t_2 \in \mathbb{R}, \\ \phi, \eta \in \mathbb{C}, \\ x, y \in \mathbb{C}^{n-2}, \\ \mathsf{x}, \mathsf{y} \in \mathbb{R} \end{array} \right\}, \tag{2.1}$$

where $\overline{\phi}$ or $\overline{\eta}$ denotes the conjugate of a complex number ϕ or η, and $x^\dagger$ or $y^\dagger$ denotes the conjugate-transpose of a row vector x or y. Note that the first two rows of any element of $\mathfrak{a}+\mathfrak{n}$ are sufficient to determine the entire matrix.

Notation 2.2. Because the exponential map is a diffeomorphism from $\mathfrak{n}$ to N, each element of N has a unique representation in the form $\exp u$ with $u \in \mathfrak{n}$. Thus, each element h of N determines corresponding values of ϕ, x, y, η, x and y (with $t_1 = t_2 = 0$). We write

$$\phi_h, x_h, y_h, \eta_h, \mathsf{x}_h, \mathsf{y}_h$$

for these values.

Notation 2.3. We let α and β be the simple real roots of $\mathrm{SU}(2,n)$, defined by $\alpha(a) = a_1/a_2$ and $\beta(a) = a_2$, for an element a of A of the form

$$a = \mathrm{diag}(a_1, a_2, 1, 1, \dots, 1, 1, a_2^{-1}, a_1^{-1}).$$

Thus,

- the root space $\mathfrak{u}_\alpha$ is the ϕ-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_\beta$ is the y-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+\beta}$ is the x-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+2\beta}$ is the η-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{2\beta}$ is the y-subspace in $\mathfrak{n}$, and
- the root space $\mathfrak{u}_{2\alpha+2\beta}$ is the x-subspace in $\mathfrak{n}$.

Notation 2.4. For a given Lie algebra $\mathfrak{h} \subset \mathfrak{n}$, we use $\mathfrak{z}$ to denote $\mathfrak{h} \cap (\mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta} + \mathfrak{u}_{2\beta})$. In other words,

$$\mathfrak{z} = \{\, u \in \mathfrak{h} \mid \phi_u = 0 \text{ and } x_u = y_u = 0 \,\}.$$

(We remark that if $\phi_u = 0$ for every $u \in \mathfrak{h}$, then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{z}$ and $\mathfrak{z}$ is contained in the center of $\mathfrak{h}$.)

Notation 2.5. For $h \in \mathrm{SU}(2, n)$, define

$$\Delta(h) = \det \begin{pmatrix} h_{1,n+1} & h_{1,n+2} \\ h_{2,n+1} & h_{2,n+2} \end{pmatrix}.$$

The following results collect some straightforward calculations that will be used repeatedly throughout the paper.

Remark 2.6. For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & i\mathsf{x} \\ 0 & 0 & y & i\mathsf{y} & -\overline{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 0 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{n} \qquad \text{and} \qquad h = \exp u \in N,$$

we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x + \frac{1}{2}\phi y & \begin{matrix} \eta - \frac{1}{2}xy^\dagger \\ + \frac{1}{2}i\phi\mathsf{y} - \frac{1}{6}\phi|y|^2 \end{matrix} & \begin{matrix} -\frac{1}{2}|x|^2 - \operatorname{Re}(\phi\overline{\eta}) + \frac{1}{24}|\phi|^2|y|^2 \\ + i\left(\mathsf{x} - \frac{1}{6}|\phi|^2\mathsf{y} + \frac{1}{3}\operatorname{Im}(\overline{\phi}xy^\dagger)\right) \end{matrix} \\ 0 & 1 & y & i\mathsf{y} - \frac{1}{2}|y|^2 & -\overline{\eta} - \frac{1}{2}yx^\dagger - \frac{1}{2}i\overline{\phi}\mathsf{y} + \frac{1}{6}\overline{\phi}|y|^2 \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger + \frac{1}{2}\overline{\phi}y^\dagger \\ 0 & 0 & 0 & 1 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Delta(h) = \begin{matrix} -|\eta|^2 + \mathsf{x}\mathsf{y} - \frac{1}{4}|x|^2|y|^2 + \frac{1}{4}|xy^\dagger|^2 - \frac{1}{6}|y|^2\operatorname{Re}(\eta\overline{\phi}) \\ - \frac{1}{6}\mathsf{y}\operatorname{Im}(xy^\dagger\overline{\phi}) + \frac{1}{12}\mathsf{y}^2|\phi|^2 - \frac{1}{144}|y|^4|\phi|^2 \\ + i\left(\frac{1}{24}\mathsf{y}|\phi|^2|y|^2 + \operatorname{Im}(xy^\dagger\overline{\eta}) + \frac{1}{2}\mathsf{x}|y|^2 + \frac{1}{2}\mathsf{y}|x|^2\right). \end{matrix}$$

When $\phi = 0$, these simplify to:

$$\exp(u) = \begin{pmatrix} 1 & 0 & x & \eta - \frac{1}{2}xy^\dagger & i\mathsf{x} - \frac{1}{2}|x|^2 \\ 0 & 1 & y & i\mathsf{y} - \frac{1}{2}|y|^2 & -\overline{\eta} - \frac{1}{2}yx^\dagger \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Delta(h) = \begin{array}{c} -|\eta|^2 + \mathsf{xy} - \frac{1}{4}|x|^2|y|^2 + \frac{1}{4}|xy^\dagger|^2 \\ + i\left(\mathrm{Im}(xy^\dagger\overline{\eta}) + \frac{1}{2}\mathsf{x}|y|^2 + \frac{1}{2}\mathsf{y}|x|^2\right). \end{array}$$

Similarly, when $y = 0$, we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x & \eta + \frac{1}{2}i\phi\mathsf{y} & \begin{array}{c} -\frac{1}{2}|x|^2 - \mathrm{Re}(\phi\overline{\eta}) \\ + i\left(\mathsf{x} - \frac{1}{6}|\phi|^2\mathsf{y}\right) \end{array} \\ 0 & 1 & 0 & i\mathsf{y} & -\overline{\eta} - \frac{1}{2}i\overline{\phi}\mathsf{y} \\ 0 & 0 & \mathrm{Id} & 0 & -x^\dagger \\ 0 & 0 & 0 & 1 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\Delta(h) = \left(\mathsf{xy} + \tfrac{1}{12}|\phi|^2\mathsf{y}^2 - |\eta|^2\right) + i\left(\tfrac{1}{2}|x|^2\mathsf{y}\right). \tag{2.2}$$

Remark 2.7. For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & i\mathsf{x} \\ & 0 & y & i\mathsf{y} & -\overline{\eta} \\ & & \cdots & & \end{pmatrix} \quad \text{and} \quad \tilde{u} = \begin{pmatrix} 0 & \tilde{\phi} & \tilde{x} & \tilde{\eta} & i\tilde{\mathsf{x}} \\ & 0 & \tilde{y} & i\tilde{\mathsf{y}} & -\overline{\tilde{\eta}} \\ & & \cdots & & \end{pmatrix}, \tag{2.3}$$

we have

$$[u, \tilde{u}] = \begin{pmatrix} 0 & 0 & \phi\tilde{y} - \tilde{\phi}y & -x\tilde{y}^\dagger + \tilde{x}y^\dagger + i\phi\tilde{\mathsf{y}} - i\tilde{\phi}\mathsf{y} & -2i\,\mathrm{Im}(x\tilde{x}^\dagger + \phi\overline{\tilde{\eta}} - \tilde{\phi}\overline{\eta}) \\ & 0 & 0 & -2i\,\mathrm{Im}(y\tilde{y}^\dagger) & \tilde{y}x^\dagger - y\tilde{x}^\dagger + i\overline{\phi}\tilde{\mathsf{y}} - i\overline{\tilde{\phi}}\mathsf{y} \\ & & & \cdots & \end{pmatrix},$$

and

$$\big[[u, \tilde{u}], \hat{u}\big] = \begin{pmatrix} 0 & 0 & 0 & -(\phi\tilde{y} - \tilde{\phi}y)\hat{y}^\dagger + 2i\hat{\phi}\,\mathrm{Im}(y\tilde{y}^\dagger) & * \\ & 0 & 0 & 0 & * \\ & & & \cdots & \end{pmatrix}. \tag{2.4}$$

3 Preliminaries on Cartan-decomposition subgroups

Notation 3.1. We employ the usual Big Oh and little oh notation: for functions f_1, f_2 on H, and a subset Z of H, we say $f_1 = O(f_2)$ *for* $z \in Z$ if there is a constant C, such that, for all $z \in Z$ with $\|z\| > C$, we have $\|f_1(z)\| \le C\|f_2(z)\|$. (The values of each f_i are assumed to belong to some finite-dimensional normed vector space, typically either $\mathbb{C}$ or a space of complex matrices. Which particular norm is used does not matter, because all norms are equivalent up to a bounded factor.) We say $f_1 = o(f_2)$ *for* $z \in Z$ if $\|f_1(z)\|/\|f_2(z)\| \to 0$ as $\|z\| \to \infty$. (For a sequence $\{h_m\}$ in H, we may write $h_m \to \infty$ if $\|h_m\| \to \infty$.) Also, we write $f_1 \asymp f_2$ if $f_1 = O(f_2)$ and $f_2 = O(f_1)$.

Definition 3.2. Define $\rho\colon \mathrm{SU}(2,n) \to \mathrm{GL}(\mathbb{C}^{n+2} \wedge \mathbb{C}^{n+2})$ by $\rho(h) = h \wedge h$, so ρ is the second exterior power of the standard representation of $\mathrm{SU}(2,n)$. Thus, we may define $\|\rho(h)\|$ to be the maximum absolute value among the determinants of all the 2×2 submatrices of the matrix h.

We now introduce convenient notation for describing the image of a subgroup under the Cartan projection μ.

Notation 3.3. For functions $f_1, f_2\colon \mathbb{R}^+ \to \mathbb{R}^+$, and a subgroup H of $\mathrm{SU}(2,n)$, we write $\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|)\big]$ if, for every sufficiently large $C > 1$, we have

$$\mu(H) \approx \left\{\, a \in A^+ \;\middle|\; C^{-1} f_1\big(\|a\|\big) \le \|\rho(a)\| \le C f_2\big(\|a\|\big) \,\right\}.$$

(If f_1 and f_2 are monomials, or other very tame functions, then it does not matter which particular norm is used.)

We have $A^+ = \{\, a \in A \mid a_{1,1} \ge a_{2,2} \ge 1 \,\}$, so, for $a \in A^+$, we have

$$\|a\| = a_{1,1} \le a_{1,1}\, a_{2,2} = \|\rho(a)\| \le a_{1,1}^2 = \|a\|^2.$$

Thus $A^+ \approx \big[\|h\|, \|h\|^2\big]$, so, from Theorem 1.8, we see that H is a Cartan-decomposition subgroup of G if and only if $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$. This observation, which is essentially due to Y. Benoist (in a much more general context, cf. [1, Lem. 2.4]), leads to the following result.

Proposition 3.4. (cf. [14, Prop. 3.14]) *A closed, connected subgroup H of $\mathrm{SU}(2,n)$ is a Cartan-decomposition subgroup if and only if*

1. *there is a sequence $\{h_m\}$ in H, such that $h_m \to \infty$ as $n \to \infty$, and $\rho(h_m) \asymp \|h_m\|^2$; and*
2. *there is a sequence $\{h_m\}$ in H, such that $h_m \to \infty$ as $n \to \infty$, and $\rho(h_m) \asymp h_m$.*

Proof. ($\Rightarrow$) Fix $k \in \{1,2\}$. There is a sequence $a_m \to \infty$ in A^+, such that $\|\rho(a_m)\| \asymp \|a_m\|^k$. Also, because $\mu(H) \approx A^+$ (see 1.8), there is a compact subset C of A, such that $A^+ \subset \mu(H)C$. Therefore, there is a sequence $h_m \to \infty$ in H, such that $a_m \in \mu(h_m)C$. So

$$\|\rho(h_m)\| \asymp \|\rho\big(\mu(h_m)\big)\| \asymp \|\rho(a_m)\| \asymp \|a_m\|^k \asymp \|\mu(h_m)\|^k \asymp \|h_m\|^k.$$

($\Leftarrow$) For simplicity, let us assume that, instead of only sequences, there are continuous curves $h_1'(t)$ and $h_2'(t)$, $t \in [0,\infty)$, in H, such that $h_k'(t) \to \infty$ as $t \to \infty$, and, for each $k = 1,2$, we have $\rho_2\big(h_k'(t)\big) \asymp \|h_k'(t)\|^k$, for $t \in [0,\infty)$. (In our applications of the proposition, this stronger assumption will be known to be true. Also, the proof of [14, Prop. 3.14] shows how to prove the existence of continuous curves from the existence of sequences, by using the theory of o-minimal structures [18, 19, 20].)

There is no loss of generality in assuming that $H \subset AN$ (see 7.1). This implies that H is homeomorphic to some Euclidean space $\mathbb{R}^m$ (with $m \geq 2$), so it is easy to find a continuous and proper map $\Phi\colon [1,2] \times \mathbb{R}^+ \to H$, such that $\Phi(k,t) = h_i'(t)$ for $k = 1,2$ and for all $t \in \mathbb{R}^+$. Then an elementary homotopy argument shows that there is some $C > 1$, such that

$$\big\{\, a \in A^+ \;\big|\; C\|a\| \leq \|\rho(a)\| \leq \|a\|^2/C \,\big\} \subset \mu\big[\Phi\big([1,2] \times \mathbb{R}^+\big)\big] \subset \mu(H),$$

so $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$. Therefore H is a Cartan-decomposition subgroup of G. ∎

The following result allows us to replace H by a conjugate subgroup whenever it is convenient.

Lemma 3.5. (cf. [1, Prop. 1.5], [8, Cor. 3.5]) *Let H be any closed, connected subgroup of $\mathrm{SU}(2,n)$. For every $g \in G$, we have $\mu(g^{-1}Hg) \approx \mu(H)$.*

In particular, H is a Cartan-decomposition subgroup if and only if $g^{-1}Hg$ is a Cartan-decomposition subgroup.

4 When is the size of $\rho(h)$ quadratic?

In this section, Proposition 4.1 is a list of types of subgroups of N that contain a sequence $\{h_m\}$ with $\rho(h_m) \asymp \|h_m\|^2$, and Proposition 4.3 is a list of types of subgroups of N that do not contain such a sequence. Then Proposition 4.4 shows that both lists are complete.

Proposition 4.1. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected subgroup of N. There is a sequence $h_m \to \infty$ in H with $\rho(h_m) \asymp \|h_m\|^2$ if either*

1. *there is an element u of $\mathfrak{h}$ with $\phi_u = 0$, such that the vectors x_u and y_u are linearly independent over $\mathbb{C}$; or*

2. *there is an element z of $\mathfrak{z}$, such that $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$; or*

3. *there are elements u of $\mathfrak{h}$ and z of $\mathfrak{z}$, such that $\phi_u = 0$, and $\mathsf{x}_z|y_u|^2 + \mathsf{y}_z|x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \overline{\eta_z}) \neq 0$; or*

4. *there is an element u of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u = 0$, and $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) = 0$; or*

5. *$\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{h}$ and there is an element u of $\mathfrak{h}$, such that $\phi_u \neq 0$, $\mathsf{y}_u \neq 0$, and $y_u = 0$; or*

6. *there are elements u and v of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $x_v \neq 0$, $\mathsf{y}_v = 0$, and $x_v y_u^\dagger = 0$; or*

7. *$\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{z}$, and there are nonzero elements u and v of $\mathfrak{h}$, satisfying $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $\mathsf{y}_v \neq 0$, and $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v$; or*

8. *$\dim \mathfrak{h} = 3$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, there exist $u, v \in \mathfrak{h} \setminus \mathfrak{z}$, such that $y_u \neq 0$, $y_v = 0$, $\mathsf{y}_v = 0$, $|x_v|^2 + 2\operatorname{Re}(\phi_v \overline{\eta_v}) > 0$, and we have $\phi_h \neq 0$ for every $h \in \mathfrak{h} \setminus \mathfrak{z}$.*

Remark 4.2. In Conclusions (6) and (7), the restriction on $x_v y_u^\dagger$ is not necessary; it was included to avoid overlap with Conclusion (2). Namely, if $x_v y_u^\dagger \neq -i\phi_u \mathsf{y}_v$, then $[u,v]$ satisfies $\mathsf{y} = 0$ and $\eta \neq 0$, so Conclusion (2) holds. Also, it is not necessary to assume $\mathsf{y}_v \neq 0$ in Conclusion (7), because Conclusion (6) holds if $\mathsf{y}_v = 0$ (and $x_v \neq 0$). Thus, (6) and (7) may be replaced with the following:

(6*) there are elements u and v of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $x_v \neq 0$, and $\mathsf{y}_v = 0$; or

(7*) $\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{z}$, and there are nonzero elements u and v of $\mathfrak{h}$, satisfying $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$.

Proof. We separately consider each of the eight cases in the statement of the proposition.

(1) Let $h^t = \exp(tu)$. Replacing H by a conjugate under U_α, we may assume that x_u is orthogonal to y_u; that is, $x_u y_u^\dagger = 0$. Then it is clear that $\rho(h^t) \asymp \Delta(h^t) \asymp t^4 \asymp \|h^t\|^2$.

(2) Let $h^t = \exp(tz)$. We have $h^t \asymp t$ and

$$\Delta(h^t) = \mathsf{x}_{tz}\mathsf{y}_{tz} - |\eta_{tz}|^2 = t^2(\mathsf{x}_z \mathsf{y}_z - |\eta_z|^2) \asymp t^2.$$

Therefore $\rho(h^t) \asymp \Delta(h^t) \asymp t^2 \asymp \|h^t\|^2$.

(3) For any large t, let $h = \exp(tu + t^2 z)$. Clearly, we have $|x_h| + |y_h| = O(t)$ and $|\mathsf{x}_h| + |\mathsf{y}_h| + |\eta_h| = O(t^2)$, so $h = O(t^2)$.

We have

$$\operatorname{Im}\Delta(h^t) = t^4\left[\frac{1}{2}\left(2\operatorname{Im}(x_u y_u^\dagger \overline{\eta_z} + \mathsf{x}_z|y_u|^2 + \mathsf{y}_z|x_u|^2\right)\right] + O(t^3) \asymp t^4.$$

Therefore, $\rho(h^t) \asymp t^4 \asymp \|h^t\|^2$.

(4) For any large t, let $h = \exp(tu)$. Then $h_{1,n+2} = it\mathsf{x}_u$, so it is easy to see that $h \asymp t$. We have $\rho(h) \asymp t^2 \asymp \|h\|^2$.

(5) Replacing H by a conjugate (under a diagonal matrix), we may assume that $\phi_u = \mathsf{y}_u$. Then, by renormalizing, we may assume that $\phi_u = \mathsf{y}_u = 1$. Let z be the element of $\mathfrak{u}_{2\alpha+2\beta}$ with $\mathsf{x}_z = 1$. By subtracting a multiple of z from u, we may assume $\mathsf{x}_u = 0$. For any large t, let $h = \exp(6tu + 36t^3 z)$, so $h_{1,n+2}$ is real. We have

$$\operatorname{Re}\Delta(h) = (36t^3)(6t) + \frac{1}{12}(6t)^2(6t)^2 + O(t^2) \asymp t^4,$$

so $\rho(h) \asymp t^4 \asymp \|h\|^2$.

(6) For each large t, let h be an element of $\exp(tu + \mathbb{R}v)$, such that $h_{1,n+2}$ is pure imaginary. (This exists because the sign of $-\frac{1}{2}|x|^2$ is opposite that of

$\frac{1}{24}|\phi|^2|y|^2$.) We note that $x_h \asymp t^2$ and $|\eta_h| + |\mathsf{x}_h| = O(t^2)$, but $\phi_h \asymp y_h \asymp t$ and $|\mathsf{y}_h| + |x_h y_h^\dagger| = O(t)$. Thus $h = O(t^3)$ and

$$\rho(h) \asymp \operatorname{Re}\Delta(h) = -\frac{1}{4}|x_h|^2|y_h|^2 - \frac{1}{144}|y_h|^4|\phi_h|^2 + O(t^5) \asymp t^6 \asymp \|h\|^2.$$

(7) Because $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v$, we have $x_v \neq 0$, so, for any large t, we may choose $h \in \exp(tu + \mathbb{R}v + \mathfrak{u}_{2\alpha+2\beta})$, such that $h_{1,n+2} = 0$. Thus $\phi_h \asymp y_h \asymp t$, but $x_h \asymp \mathsf{y}_h \asymp t^2$ and $|\eta_h| + |\mathsf{x}_h| = O(t^2)$. Then (because $h_{1,n+2} = 0$) it is easy to verify that $h = O(t^3)$. However

$$\operatorname{Im}\Delta(h) = \frac{1}{24}\mathsf{y}_h|\phi_h|^2|y_h|^2 + \frac{1}{2}\mathsf{y}_h|x_h|^2 + O(t^5) \asymp t^6.$$

So $\rho(h) \asymp \|h\|^2$.

(8) For any large t, choose $s = O(1)$, such that $\operatorname{Re}\big(\exp(su+tv)_{1,n+2}\big) = 0$. (This is possible, because $-\frac{1}{2}|x_v|^2 - \operatorname{Re}(\phi_v\overline{\eta_v}) < 0$.) Then we may choose $h \in \exp(su + tv + \mathfrak{z})$, such that $h_{1,n+2} = 0$. Then $\phi_h \asymp t$, $|x_h| + |\eta_h| = O(t)$, and $|y_h| + |\mathsf{y}_h| = O(1)$, so we have $\rho(h) \asymp t^2 \asymp \|h\|^2$. ∎

Proposition 4.3. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected, nontrivial subgroup of N.*

1. *If $\dim\mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h\mathsf{y}_h$ for every $h \in H$, then $\rho(h) \asymp h$ for every $h \in H$.*
2. *If $\phi_h = 0$ and $y_h = 0$ for every $h \in \mathfrak{h}$, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, and there is some $u \in \mathfrak{h}$, such that $\mathsf{y}_u \neq 0$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 1$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
3. *Suppose $\phi_h = 0$ for every $h \in \mathfrak{h}$, and there is some $\lambda \in \mathbb{C}$, such that $x_h = \lambda y_h$ for every $h \in H$, and we have $\eta_z = i\lambda\mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2\mathsf{y}_z$ for every $z \in \mathfrak{z}$.*
   (a) *If there is some $u \in \mathfrak{h}$, such that $\mathsf{x}_u + |\lambda|^2\mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \neq 0$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 1$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
   (b) *Otherwise, $\rho(h) \asymp h$ for every $h \in H$.*
4. *If $y_h = 0$, $\mathsf{y}_h = 0$, and $|x_h|^2 + 2\operatorname{Re}(\phi_h\overline{\eta_h}) \neq 0$ for every $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$ (so $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$), then $\rho(h) \asymp h$ for every $h \in H$.*
5. *If $\mathfrak{z} = 0$, there is some $u \in \mathfrak{h}$ and some nonzero $\phi_0 \in \mathbb{C}$, such that $\phi_u \neq 0$, and we have $\phi_h = \phi_0\mathsf{y}_h$ and $y_h = 0$, for every $h \in \mathfrak{h}$, then $\mu(H) \approx \big[\|h\|, \|h\|^{4/3}\big]$, unless $\dim H = 1$, in which case, $\rho(h) \asymp \|h\|^{4/3}$ for every $h \in H$.*
6. *If $\dim\mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, we have $\phi_v \asymp y_v$ and $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ for every $v \in \mathfrak{h}$, and there exists $u \in \mathfrak{h}$, such that $\phi_u \neq 0$, then $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
7. *If $\dim\mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, and $\phi_h \neq 0$ and $y_h \neq 0$ for every $h \in \mathfrak{h} \setminus \mathfrak{z}$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.*

Proof. We separately consider each of the seven cases in the statement of the proposition.

(1) Because $\Delta(h) = 0$ for every $h \in H$, it is clear that $\rho(h) \asymp h$ for every $h \in H$.

(2) We have $|\eta_h| + |\mathsf{y}_h| = O(x_h)$, so $h_{1,n+2} \asymp |x_h|^2 + |\mathsf{x}_h|$ and $h_{i,j} = O(x_h) = O\big(|h_{1,n+2}|^{1/2}\big)$ whenever $(i,j) \neq (1, n+2)$. Thus, $\rho(h) = O\big(\|h\|^{3/2}\big)$.

We have $\rho\big(\exp(tu)\big) \asymp \operatorname{Im}\Delta\big(\exp(tu)\big) \asymp t^3 \asymp \|\exp(tu)\|^{3/2}$. If $\dim H > 1$, then there is some nonzero $v \in \mathfrak{h}$, such that $\mathsf{y}_v = 0$. Then, for $h \in \exp(\mathbb{R}v)$, we have $\rho(h) \asymp |x_h|^2 + |\mathsf{x}_h| \asymp h$.

(3) Replacing H by a conjugate under U_α, we may assume that $\lambda = 0$, so $x_h = 0$ for every $h \in H$, and $\eta_z = \mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$ (which means $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$). Therefore, the Weyl reflection corresponding to the root α conjugates $\mathfrak{h}$ to a subalgebra either of type (2) or of type (4), depending on whether or not there is some $u \in \mathfrak{h}$, such that $\mathsf{x}_u + |\lambda|^2 \mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \neq 0$.

(4) By assumption, the quadratic form $|x|^2 + 2\operatorname{Re}(\phi\overline{\eta})$ is definite on $\mathfrak{h}/\mathfrak{z}$, so $|x|^2 + |\phi|^2 + |\eta|^2 = O\big(|x|^2 + 2\operatorname{Re}(\phi\overline{\eta})\big)$. Therefore, $h_{i,j} = O(|h_{1,n+2}|^{1/2})$ whenever $(i,j) \neq (1, n+2)$. Furthermore, $h_{i,j} = O(1)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $\rho(h) \asymp h$.

(5) For any sequence $\{h_m\} \to \infty$ in H, we write $\phi_m, x_m, y_m, \mathsf{y}_m, \eta_m, \mathsf{x}_m$ for ϕ_{h_m}, etc.

We have $\phi_m \asymp \mathsf{y}_m$. If $x_m = O(|\mathsf{y}_m|^{3/2})$, then $\rho(h_m) \asymp \operatorname{Re}\Delta(h_m) \asymp \mathsf{y}_m^4 \asymp \|h_m\|^{4/3}$. (This completes the proof if $\dim H = 1$.) If $|\mathsf{y}_m|^{3/2} = o(x_m)$, then $h_m \asymp h_{1,n+2} \asymp |x_m|^2$, but $h_{i,j} = O\big(|x_m| + \mathsf{y}_m^2\big) = O\big(|x_m|^{4/3}\big)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\mathsf{y}_m) = O\big(|x_m|^{2/3}\big)$ whenever $i \neq 1$ and $j \neq n+2$. Therefore

$$\rho(h_m) = O\big(|x_m|^2 |x_m|^{2/3} + |x_m|^{4/3}|x_m|^{4/3}\big) = O\big(|x_m|^{8/3}\big) = O\big(\|h_m\|^{4/3}\big).$$

If $\dim H > 1$, then there is some (large) $h \in H$ with $\mathsf{y}_h = 0$ (and hence $\phi_h = 0$). Thus $\rho(h) \asymp |x_h|^2 \asymp h$.

(6) For any sequence $\{h_m\} \to \infty$ in H, we show that $\rho(h_m) \asymp \Delta(h_m) \asymp \|h_m\|^{3/2}$. We write $\phi_m, x_m, y_m, \mathsf{y}_m, \eta_m, \mathsf{x}_m$ for ϕ_{h_m}, etc.

If $\mathsf{y}_m = o(\phi_m^2)$, then $h_{1,n+2} \asymp \phi_m^4$, but $h_{i,j} = O(\phi_m^3)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\phi_m^2)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $\rho(h_m) \asymp \operatorname{Re}\Delta(h_m) \asymp \phi_m^6 \asymp \|h_m\|^{3/2}$.

We may now assume that $\phi_m^2 = O(\mathsf{y}_m)$. Thus, there is some $v \in \mathfrak{h}$, such that $\phi_v = 0$ and $\mathsf{y}_v = 1$. (Note that, because $y_v \asymp \phi_v$, we have $y_v = 0$.) Because $[u,v] \in \mathfrak{z} = 0$, we must have $\eta_{[u,v]} = 0$, so $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v \neq 0$. In particular, $x_v \neq 0$, so $x_m \asymp \mathsf{y}_m$.

We have $h_{1,n+2} = O\big(|x_m|^2\big) = O(\mathsf{y}_m^2)$, but $h_{i,j} = O\big(|\phi_m \mathsf{y}_m| + |\mathsf{y}_m|\big) = O\big(|\mathsf{y}_m|^{3/2}\big)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\mathsf{y}_m)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $h_m = O(\mathsf{y}_m^2)$ and $\rho(h_m) = O(\mathsf{y}_m^3)$.

Furthermore, we have

$$\operatorname{Im}\Delta(h_m) = \frac{1}{24}\mathsf{y}_m |\phi_m|^2 |y_m|^2 + \frac{1}{2}\mathsf{y}_m |x_m|^2 + O(\mathsf{y}_m^2 \phi_m) \asymp \mathsf{y}_m^3,$$

because $\mathsf{y}_m |x_m|^2 \asymp \mathsf{y}_m^3$, and the terms $\frac{1}{24}\mathsf{y}_m |\phi_m|^2 |y_m|^2$ and $\frac{1}{2}\mathsf{y}_m |x_m|^2$ cannot cancel (since they both have the same sign as y_m). We conclude that $\rho(h_m) \asymp \Delta(h_m) \asymp \mathsf{y}_m^3$.

All that remains is to show $\mathsf{y}_m^2 = O(h_m)$. If $\phi_m^2 = o(\mathsf{y}_m)$, then

$$\operatorname{Re} h_{1,n+2} = -\frac{1}{2}|x|^2 + O(\phi_m^2 \mathsf{y}) \asymp \mathsf{y}_m^2,$$

as desired. If $\mathsf{y}_m = o(\phi_m^2)$, then

$$\operatorname{Re} h_{1,n+2} \asymp o(\phi_m^4) + o(\phi_m^3) + |\phi_m^4| \asymp \phi_m^4,$$

so $\mathsf{y}_m = o(\phi_m^2) = o(\phi_m^4) = o(h_m)$, as desired. Thus, we may assume that $\mathsf{y}_m \asymp \phi_m^2$. Because $x_m = \mathsf{y}_m x_v + O(\phi_m)$ and $x_v y_m^\dagger = -i\phi_m \mathsf{y}_v = -i\phi_m$, we have

$$\begin{aligned} \operatorname{Im}(h_{1,n+2}) &= O(\mathsf{y}_m) - \frac{1}{6}|\phi_m|^2 \mathsf{y}_m + \left[\frac{1}{3}\operatorname{Im}\big(\overline{\phi_m}(\mathsf{y}_m x_v) y_m^\dagger\big) + O(\phi_m^3)\right] \\ &= -\frac{1}{6}|\phi_m|^2 \mathsf{y}_m - \frac{1}{3}|\phi_m|^2 \mathsf{y}_m + O(\phi_m^3) \asymp \mathsf{y}_m^2, \end{aligned}$$

as desired.

(7) For $z \in \mathfrak{z}$, we have $\rho(z) \asymp z$. For $u \in \mathfrak{h} \setminus \mathfrak{z}$ with $y_u \neq 0$, we have $\rho\big(\exp(tu)\big) \asymp t^6 \asymp \|\exp(tu)\|^{3/2}$. All that remains is to show $\rho(h) = O\big(\|h\|^{3/2}\big)$ for every $h \in H$.

Note that $\phi_h \asymp y_h$, and $|x_h| + |\eta_h| + |\mathsf{y}_h| = O(\phi_h)$. If $\phi_h = O(1)$, then it is obvious that $\rho(h) \asymp h$. Thus, we may assume $|\phi_h| \to \infty$. Then, because $\operatorname{Re} h_{1,n+2} \asymp |\phi_h|^2 |y_h|^2 \asymp \phi_h^4$, but $h_{i,j} = O\big(\phi_h |y_h|^2\big) = O(\phi_h^3)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\phi_h^2)$ whenever $i \neq 1$ and $j \neq n+2$, we have

$$\rho(h) = O\big[|\phi_h|^4 |\phi_h|^2 + \big(|\phi_h|^3\big)^2\big] = O\big(|\phi_h|^6\big) = O\big(|h_{1,n+2}|^{3/2}\big) = O\big(\|h\|^{3/2}\big). \quad \blacksquare$$

Proposition 4.4. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected, nontrivial subgroup of N.*

1. *There is a sequence $h_m \to \infty$ in H with $\rho(h_m) \asymp \|h_m\|^2$ if and only if H is one of the subgroups described in Proposition* 4.1.
2. *There is* ***not*** *a sequence $h_m \to \infty$ in H with $\rho(h_m) \asymp \|h_m\|^2$ if and only if H is one of the subgroups described in Proposition* 4.3.

Proof. It suffices to show that H is described in either Proposition 4.1 or Proposition 4.3.

We may assume

$$|\eta_z|^2 = \mathsf{x}_z \mathsf{y}_z \text{ for every } z \in \mathfrak{z} \tag{4.1}$$

(otherwise, 4.1(2) holds). Because $|\eta|^2 - \mathsf{xy}$ is a quadratic form of signature $(3,1)$ on $\mathfrak{u}_{2\beta} + \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$, then we must have $\dim \mathfrak{z} \leq 1$. Thus, we may assume $\mathfrak{h} \neq \mathfrak{z}$ (otherwise 4.3(1) holds).

Case 1. Assume $\phi_h = 0$ and $y_h = 0$ for every $h \in H$ (and $\mathfrak{h} \neq \mathfrak{z}$). We may assume $\mathsf{y}_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(3) holds. Then, from Eq. (4.1), we have $\eta_z = 0$ for every $z \in \mathfrak{z}$. Thus, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. We may assume $\mathsf{y}_h = 0$ for every $h \in H$, for otherwise Conclusion 4.3(2) holds. We conclude that 4.3(4) holds.

Case 2. Assume $\phi_h = 0$ for every $h \in H$, and there is some $u \in \mathfrak{h}$ with $y_u \neq 0$. We may assume that x_h and y_h are linearly dependent over $\mathbb{C}$ for every $h \in H$ (otherwise 4.1(1) holds). In particular, there exists $\lambda \in \mathbb{C}$, such that $x_u = \lambda y_u$.

Subcase 2.1. Assume $\mathfrak{z} = 0$.

Subsubcase 2.1.1. Assume there exists $v \in \mathfrak{h}$, such that either $x_v \notin \mathbb{C}y_u$ or $y_v \notin \mathbb{C}y_u$. We may assume there exists $w \in \mathfrak{h}$, such that $x_w \neq \lambda y_w$ (otherwise 4.3(3) holds). Furthermore, by adding a small linear combination of u and v to w, we may assume that $y_w \neq 0$ and that either $x_w \notin \mathbb{C}y_u$ or $y_w \notin \mathbb{C}y_u$. Because x_w and y_w are linearly dependent, there exists λ_1 $(\neq \lambda)$ such that $x_w = \lambda_1 y_w$. (Then note that we must have $y_w \notin \mathbb{C}y_u$.) Then

$$x_{u+w} = x_u + x_w = \lambda y_u + \lambda_1 y_w \notin \mathbb{C}(y_u + y_w) = \mathbb{C}y_{u+w}$$

(because $\lambda \neq \lambda_1$ and $\{y_u, y_w\}$ is linearly independent over $\mathbb{C}$). This contradicts the fact that x_{u+w} and y_{u+w} are linearly dependent over $\mathbb{C}$.

Subsubcase 2.1.2. Assume $x_h, y_h \in \mathbb{C}y_u$, for every $h \in \mathfrak{h}$. For each $h \in \mathfrak{h}$, there exist $\lambda_x, \lambda_y \in \mathbb{C}$, such that $x_h = \lambda_x y_u$ and $y_h = \lambda_y y_u$. Because $\mathfrak{z} = 0$, we must have $\mathsf{y}_{[h,u]} = 0$, so $\operatorname{Im}(y_h y_u^\dagger) = 0$, which means that λ_y is real. We must also have $\eta_{[h,u]} = 0$, so

$$0 = -x_h y_u^\dagger + x_u y_h^\dagger = (-\lambda_x + \lambda\overline{\lambda_y})|y_u|^2 = (-\lambda_x + \lambda\lambda_y)|y_u|^2.$$

Thus $\lambda_x = \lambda\lambda_y$, so

$$x_h = \lambda_x y_u = \lambda\lambda_y y_u = \lambda y_h.$$

Therefore 4.3(3) holds.

Subcase 2.2. Assume $\mathfrak{z} \neq 0$. We show that either 4.1(2), 4.1(3) or 4.3(3) holds. Straightforward calculations show that conditions 4.1(2), 4.1(3) and 4.3(3) are invariant under conjugation by U_α, so we may assume that $\lambda = 0$; that is, $x_u = 0$. Thus, we may assume $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(3) holds. Then we may assume $\eta_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(2) holds; therefore $\mathfrak{z} = U_{2\beta}$. We may now assume $x_h = 0$ for every $h \in \mathfrak{h}$, for, otherwise, 4.1(3) holds. Thus, 4.3(3) holds (with $\lambda = 0$).

Case 3. Assume there exists $u \in \mathfrak{h}$ with $\phi_u \neq 0$. We claim that $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. If not, then there is some $z \in \mathfrak{z}$, such that either $\eta_z \neq 0$ or $\mathsf{y}_z \neq 0$. If $\mathsf{y}_z = 0$, then $|\eta_z|^2 \neq 0 = \mathsf{x}_z\mathsf{y}_z$, so 4.1(2) holds. On the other hand, if $\mathsf{y}_z \neq 0$, then, letting $z' = [u, z]$, we have $\mathsf{y}_{z'} = 0$ and $\eta_{z'} \neq 0$, so 4.1(2) holds once again.

Subcase 3.1. Assume $y_h = 0$ for every $h \in \mathfrak{h}$. We may assume that there is some $v \in \mathfrak{h}$, such that $\mathsf{y}_v \neq 0$ (otherwise, either 4.1(4) or 4.3(4) holds). Then we may assume $\mathfrak{z} = 0$ (otherwise, 4.1(5) holds).

We claim that 4.3(5) holds. If not, then there is some $w \in \mathfrak{h}$, such that $\phi_w \neq 0$ and $\mathsf{y}_w = 0$. Then $\eta_{[v,w]} \neq 0$, which contradicts the assumption that $\mathfrak{z} = 0$.

Subcase 3.2. Assume there is some $v \in \mathfrak{h}$, such that $y_v \neq 0$.

Subsubcase 3.2.1. Assume $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$. Suppose, for the moment, that there exists $w \in \mathfrak{h} \setminus \mathfrak{z}$ with $\phi_w = 0$. We may assume that $y_w = 0$ (otherwise, 4.1(3) holds). Therefore $x_w \neq 0$, so 4.1(7*) holds.

We may now assume that $\phi_w \neq 0$ for every $w \in \mathfrak{h} \setminus \mathfrak{z}$. This implies that x, y, η, and y are functions of ϕ; in particular, $\dim \mathfrak{h} \leq 3$. Also, because $\mathfrak{z} \neq 0$ and $u, v \notin \mathfrak{z}$, we must have $\dim \mathfrak{h} \geq 2$.

We claim $\dim \mathfrak{h} = 2$ (so 4.3(7) holds). If not, then $\dim \mathfrak{h} = 3$, so there exist $u, w \in \mathfrak{h}$, such that $\phi_u = 1$ and $\phi_w = i$. Because $\phi_{[u,w]} = 0$, we must have $[u, w] \in \mathfrak{u}_{2\alpha+2\beta}$. Therefore $0 = x_{[u,w]} = y_w - iy_u$, so $y_w = iy_u$. Furthermore,

$$0 = \mathsf{y}_{[u,w]} = -2i \operatorname{Im}(y_u y_w^\dagger) = -2i \operatorname{Im}\bigl(-i|y_u|^2\bigr) = -2i|y_u|^2,$$

so $y_u = 0$. Then $y_w = iy_u$ is also 0. This implies $y_h = 0$ for every $h \in H$. This contradicts the fact that $y_v \neq 0$.

Subsubcase 3.2.2. Assume $\mathfrak{z} = 0$. Lemma 4.5 below implies that either 4.3(6) or 4.1(6*) holds. ∎

Lemma 4.5. *Let H be a closed, connected subgroup of N, such that $\mathfrak{z} = 0$, and assume there exist $u, v \in \mathfrak{h}$, such that $\phi_u \neq 0$ and $y_v \neq 0$. Then either H is described in* 4.3(6) (*and in* 5.2(4), *which is the same*), *or H is a Cartan-decomposition subgroup* (*and is described in* 4.1(6*) *and* 5.1(2)).

Proof. Let us begin by establishing that $\phi_h \asymp y_h$ for $h \in \mathfrak{h}$. If not, then we may assume either that $y_u = 0$ or that $\phi_v = 0$. Then, because $\bigl[[u,v], v\bigr] \in \mathfrak{z} = 0$, we see from Eq. (2.4) that

$$0 = -(\phi_u y_v - \phi_v y_u) y_v^\dagger + 2i\phi_v \operatorname{Im}(y_u y_v^\dagger) = -\phi_u |y_v|^2 - 0 + 0 \neq 0.$$

This contradiction establishes the claim.

Case 1. Assume there is a nonzero $w \in \mathfrak{h}$, such that $\phi_w = 0$ and $\mathsf{y}_w = 0$. Note, from the preceding paragraph, that $y_w = 0$. Then, because $\mathfrak{z} = 0$, we must have $x_w \neq 0$. Therefore, 4.1(6*) and 5.1(2) hold, so $\mu(H) \approx \bigl[\|h\|, \|h\|^2\bigr]$, so H is a Cartan-decomposition subgroup.

Case 2. Assume there does not exist such an element $w \in \mathfrak{h}$. Then H is described in 4.3(6) and in 5.2(4). ∎

5 When is the size of $\rho(h)$ linear?

In this section, Proposition 5.1 is a list of types of subgroups of N that contain a sequence $\{h_m\}$ with $\rho(h_m) \asymp h_m$, and Proposition 5.2 is a list of types of subgroups of N that do not contain such a sequence. Then Proposition 5.3 shows that both lists are complete.

Proposition 5.1. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected subgroup of N. There is a sequence $h_m \to \infty$ in H with $\rho(h_m) \asymp h_m$ if either*

1. *there is a nonzero element z of $\mathfrak{z}$ with $|\eta_z|^2 = \mathsf{x}_z \mathsf{y}_z$; or*
2. *there is an element u of $\mathfrak{h}$, such that $\phi_u = 0$, $\dim_{\mathbb{C}}\langle x, y\rangle = 1$, and*

$$\mathsf{x}_u |y_u|^2 + \mathsf{y}_u |x_u|^2 + 2\operatorname{Im}(x_u y_u^\dagger \overline{\eta_u}) = 0;$$

or

3. *there is an element h of H with $y_h = 0$, $\mathsf{y}_h = 0$ and $|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) \neq 0$; or*

4. *there are elements u of $\mathfrak{h}$ and z of $\mathfrak{z}$, such that $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u \neq 0$, $\eta_z \neq 0$, and $\mathsf{y}_z = 0$; or*

5. *there are nonzero elements u of $\mathfrak{h}$ and z of $\mathfrak{z}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u \overline{\eta_z}$ is real, and*

$$\mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\operatorname{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) = 0.$$

Proof. We separately consider each of the five cases in the statement of the proposition.

(1) From 4.3(1), we have $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}z)$.

(2) Replacing H by a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume that $y_u = 0$ (and $x_u \neq 0$). Then, from the assumption of this case, we know that y_u is also 0. Therefore, 4.3(4) implies that $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}u)$.

(3) From 4.3(4), we have $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}u)$.

(4). For any large t, choose $h \in \exp(tu + \mathfrak{z})$, such that $\mathsf{x}_h \mathsf{y}_h + \frac{1}{12}|\phi_h|^2 \mathsf{y}_h^2 - |\eta_h|^2 = 0$. Note that $\eta_h \asymp |\phi_h y_h| \asymp t^2$, so $h \asymp \operatorname{Re} h_{1,n+2} \asymp t^3$, but $h_{i,j} = O(t^2)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(t)$ whenever $i \notin \{1,2\}$ or $j \notin \{n+1, n+2\}$. From the choice of h, we have

$$\Delta(h) = 0 + i\left(\tfrac{1}{2}|x_h|^2 \mathsf{y}_h\right) = O(t^3) = O(h),$$

so it is not difficult to see that $\rho(h) \asymp h$.

(5) Replacing $\mathfrak{h}$ by a conjugate, we may assume $u \in \mathfrak{u}_\alpha + \mathfrak{u}_\beta$. (First, conjugate by an element of U_β to make $\mathsf{y}_u = 0$. Then conjugate by an element of U_α to make x_u orthogonal to y_u. Then conjugate by an element of U_β that centralizes y_u, to make $x_u = 0$. Then conjugate by an element of $U_{\alpha+\beta}$ to make $\eta_u = 0$. Then conjugate by an element of $U_{\alpha+2\beta}$ to make $\mathsf{x}_u = 0$.) Then, by assumption, we must have $\mathsf{x}_z = 0$, because $\mathsf{y}_u = 0$ and $x_u = 0$.

Furthermore, replacing $\mathfrak{h}$ by a conjugate under a diagonal matrix (that belongs to G), we may assume that ϕ_u and y_u are real. Then η_z must also be real (because $\phi_u \overline{\eta_z}$ is real). Thus, we see that $u, z \in \mathfrak{so}(2,n)$. So [14, Thm. 5.3(1)] implies that H is a Cartan-decomposition subgroup. ∎

Proposition 5.2. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected, nontrivial subgroup of N such that*

$$|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z, \text{ for every nonzero } z \in \mathfrak{z}. \tag{5.1}$$

1. *If $\mathfrak{h} = \mathfrak{z}$ (so $\dim H \leq 3$), then $\rho(h) \asymp \|h\|^2$ for every $h \in H$.*

2. *If $\phi_h = 0$ and $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ for every $h \in \mathfrak{h}$, then $\rho(h) \asymp \|h\|^2$ for every $h \in H$.*

3. *If $\phi_h = 0$ for every $h \in \mathfrak{h}$, there exist nonzero u and v in $\mathfrak{h}$, such that $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ and $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, and $\mathsf{x}_v |y_v|^2 + \mathsf{y}_v |x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ for every such $v \in \mathfrak{h}$, then $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$.*

4. *If* $\dim \mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, *we have* $\phi_v \asymp y_v$ *and* $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ *for every* $v \in \mathfrak{h}$, *and there exists* $u \in \mathfrak{h}$, *such that* $\phi_u \neq 0$, *then* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

5. *If* $\dim \mathfrak{h} \leq 2$ *and* $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\,\mathrm{Re}(\phi_h \overline{\eta_h}) = 0$ *for every nonzero* $h \in \mathfrak{h}$, *then* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

6. *If* $\dim \mathfrak{h} = 2$ *and there exist nonzero* $u \in \mathfrak{h}$ *and* $z \in \mathfrak{z}$, *such that* $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u \overline{\eta_z}$ *is real, and* $\mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0$, *then* $\mu(H) \approx \big[\|h\|^{5/4}, \|h\|^2\big]$.

7. *If* $\dim \mathfrak{h} = 1$, *and we have* $\phi_h = 0$, $\dim_{\mathbb{C}}\langle x_h, y_h\rangle = 1$, *and*

$$\mathsf{x}_h |y_h|^2 + \mathsf{y}_h |x_h|^2 + 2\,\mathrm{Im}(x_h y_h^\dagger \overline{\eta_h}) \neq 0$$

for every nonzero $h \in \mathfrak{h}$, *then* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

8. *If* $\dim \mathfrak{h} = 1$, *and* $\phi_h \neq 0$, $y_h = 0$, *and* $\mathsf{y}_h \neq 0$, *for every nonzero* $h \in \mathfrak{h}$, *then* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$.

Proof. We separately consider each of the eight cases in the statement of the proposition.

(1) From Eq. (5.1), we know that the quadratic form $|\eta|^2 - \mathsf{x}\mathsf{y}$ is anisotropic on $\mathfrak{z} = \mathfrak{h}$, so

$$\Delta(h) = |\eta_h|^2 - \mathsf{x}_h \mathsf{y}_h \asymp |\eta_h|^2 + \mathsf{x}_h^2 + \mathsf{y}_h^2 \asymp \|h\|^2.$$

(2) Because $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$, we have

$$|x_h|^2 |y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4,$$

so Lemma 5.4 implies $\rho(h) \asymp \|h\|^2$.

(7) From either Proposition 4.3(2) or 4.3(3a) (depending on whether y_h is 0 or not), we have $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.

(3) From Lemma 5.4, we have $\|h\|^{3/2} = O\big(\rho(h)\big)$.

From (2), we see that $\rho(h) \asymp \|h\|^2$ for $h \in \exp(\mathbb{R}u)$.

From (7), we see that $\rho(h) \asymp \|h\|^{3/2}$ for $h \in \exp(\mathbb{R}v)$.

(4) See Proposition 4.3(6).

(5) Because $\mathrm{Re}\, h_{1,n+2} = 0$, it is easy to see that $\rho(h) \asymp \phi_h^2 \asymp \|h\|^2$.

(6) Replacing H by a conjugate, we may assume $x_u = 0$ and $\mathsf{y}_u = 0$. Therefore, $x_h = 0$ and $\mathsf{y}_h = 0$ for every $h \in H$. Thus

$$x_z |y_u|^2 = \mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0,$$

so $\mathsf{x}_z \neq 0$. From Eq. (5.1), we know $\eta_z \neq 0$.

We have $\rho(tz) \asymp \|tz\|^2$ (see 5.2(1)).

Because ϕ_u is a real multiple of η_z, we may let h be a large element of H, such that $\eta_h = -|y_h|^2 \phi_h / 12 + O(\phi_h)$. (So $y_h \asymp \phi_h$ and $\mathsf{x}_h \asymp \eta_h \asymp \phi_h^3$.) Then

$$\begin{aligned}
\Delta(h) &= \left(-|\eta_h|^2 - \frac{1}{6}|y_h|^2 \eta_h \overline{\phi_h} - \frac{1}{144}|y_h|^4 |\phi_h|^2\right) + i\left(\frac{1}{2}\mathsf{x}_h |y_h|^2\right) \\
&= O(\phi_h^4) + i\left(\frac{1}{2}\mathsf{x}_h |y_h|^2\right) \asymp \phi_h^5.
\end{aligned}$$

It is clear that all other matrix entries of $\rho(h)$ are $O(\phi_h^5)$. Thus, we have $\rho(h) \asymp \phi_h^5 \asymp \|h\|^{5/4}$.

Now suppose there is a sequence $h_m \to \infty$ in H with $\rho(h_m) = o\big(\|h_m\|^{5/4}\big)$.

Case 1. Assume $\eta_m = o(\phi_m^3)$. We have $h_m \asymp \phi_m^4$, so

$$\phi_m^6 \asymp \operatorname{Re}\Delta(h_m) = O\big(\rho(h_m)\big) = o(\|h_m\|^{5/4}) = o(\phi_m^5).$$

This is a contradiction.

Case 2. Assume $\phi_m^3 = o(\eta_m)$. We have $h_m \asymp \operatorname{Re} h_{1,n+2} \asymp \phi_m \eta_m$, so

$$\eta_m^2 \asymp \operatorname{Re}\Delta(h_m) = O\big(\rho(h_m)\big) = o(\|h_m\|^{5/4}) = o(\|h_m\|^{3/2}) = o(|\phi_m \eta_m|^{3/2}) = o(\eta_m^2).$$

This is a contradiction.

Case 3. Assume $\eta_m \asymp \phi_m^3$. We have $h_m = O(\phi_m^4)$, so

$$\phi_m^5 \asymp \mathsf{x}_m |y_m|^2 \asymp \operatorname{Im}\Delta(h_m) = O\big(\rho(h_m)\big) = o\big(\|h_m\|^{5/4}\big) = o(\phi_m^5).$$

This is a contradiction.

(8) See Proposition 4.3(5). ∎

Proposition 5.3. *Assume that* $G = \mathrm{SU}(2,n)$. *Let* H *be a closed, connected, nontrivial subgroup of* N.

1. *There is a sequence* $h_m \to \infty$ *in* H *with* $\rho(h_m) \asymp h_m$ *if and only if* H *is one of the subgroups described in Proposition* 5.1.
2. *There is **not** a sequence* $h_m \to \infty$ *in* H *with* $\rho(h_m) \asymp \|h_m\|^2$ *if and only if* H *is one of the subgroups described in Proposition* 5.2.

Proof. It suffices to show that H is described in either Proposition 5.1 or Proposition 5.2.

We may assume (5.1) holds (otherwise, Conclusion 5.1(1) holds).

Case 1. Assume $\phi_h = 0$ *for every* $h \in H$. We may assume there exists $v \in \mathfrak{h}$, such that $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$ (otherwise 5.2(2) holds). Furthermore, we may assume $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ for every such v (otherwise 5.1(2) holds). Then we may assume $\dim_{\mathbb{C}}\langle x_u, y_u\rangle = 1$ for every nonzero $u \in \mathfrak{h}$ (otherwise 5.2(3) holds).

The argument in Subsubcase 2.1.1 of the proof of Proposition 4.3 implies there exists $\lambda \in \mathbb{C}$, such that, for every $h \in H$, we have $x_h = \lambda y_h$ (or vice-versa: for every h, we have $y_h = \lambda x_h$). Thus, replacing H by a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume $x_h = 0$ for every $h \in H$.

If $\dim H > 1$, then there is some nonzero $u \in \mathfrak{h}$, such that $\mathsf{x}_h = 0$. This contradicts the fact that $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$. Thus, we conclude that $\dim H = 1$, so 5.2(7) holds.

Case 2. Assume the projection of $\mathfrak{h}$ *to* $\mathfrak{u}_\alpha$ *is one-dimensional.* Replacing H by a conjugate under A, we may assume ϕ_h is real for every $h \in H$. Fix some $u \in \mathfrak{h}$, such that $\phi_u \neq 0$.

We may assume that $\mathfrak{u}_{2\alpha+2\beta} \not\subset \mathfrak{h}$ (otherwise Conclusion 5.1(1) holds). Therefore $[\mathfrak{h}, u]$ must be zero, so $\mathsf{y}_z = 0$ and η_z is a nonzero real, for every nonzero $z \in \mathfrak{z}$. (This implies $\dim \mathfrak{z} \leq 1$.)

Subcase 2.1. Assume $y_h = 0$ for every $h \in H$. We may assume Conclusion 5.1(2) does not hold.

We claim that $\mathfrak{h} = \mathbb{R}u + \mathfrak{z}$. Suppose not. Then there is some $v \in \mathfrak{h}$, such that $\phi_v = 0$ and $x_v \neq 0$. Because Conclusion 5.1(2) does not hold, we must have $\mathsf{y}_v \neq 0$. Then $\big[[v,u],u\big]$ is a nonzero element of $\mathfrak{u}_{2\alpha+2\beta}$. (This can be seen easily by replacing H with a conjugate, so that $u \in \mathfrak{u}_\alpha$.) This contradicts our assumption that $\mathfrak{u}_{2\alpha+2\beta} \not\subset \mathfrak{h}$.

If $\mathsf{y}_u \neq 0$, then either Conclusion 5.2(8) or 5.1(4) holds (depending on whether $\mathfrak{z}$ is 0 or not). If $\mathsf{y}_u \neq 0$, then 5.1(3) or 5.2(6) holds.

Subcase 2.2. Assume the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta$ is nontrivial. Then we may assume $y_u \neq 0$.

Subsubcase 2.2.1. Assume there are nonzero $v \in \mathfrak{h}$ and $z \in \mathfrak{z}$, such that $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$. We may assume that Conclusion 5.1(5) does not hold. Therefore, for every real t, we must have

$$\begin{aligned} 0 &\neq \mathsf{x}_z |y_u|^2 - \phi_u(\mathsf{y}_u + t\mathsf{y}_v)\overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z}(x_u + t x_v) y_u^\dagger\big) \\ &= t\big[-\phi_u \mathsf{y}_v \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_v y_u^\dagger\big)\big] + \text{constant}. \end{aligned}$$

Thus, the coefficient of t must vanish, which (using the fact that η_z is real and nonzero) means

$$0 = -\phi_u \mathsf{y}_v + 2\,\mathrm{Im}\big(x_v y_u^\dagger\big). \tag{5.2}$$

We have $[u,v] \in \mathfrak{z}$, so $\eta_{[u,v]}$ is real. Thus,

$$0 = \mathrm{Im}\,\eta_{[u,v]} = \mathrm{Im}\big(x_v y_u^\dagger + i\phi_u \mathsf{y}_v\big) = \mathrm{Im}\big(x_v y_u^\dagger\big) + \phi_u \mathsf{y}_v.$$

Comparing this with Eq. (5.2), we conclude that $\phi_u \mathsf{y}_v = 0$. Therefore $\mathsf{y}_v = 0$, so Conclusion 5.1(2) holds (for the element v).

Subsubcase 2.2.2. Assume there do not exist nonzero $v \in \mathfrak{h}$ and $z \in \mathfrak{z}$, such that $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$. We must have

$$y_w = 0 \text{ for every } w \in \mathfrak{h}, \text{ such that } \phi_w = 0. \tag{5.3}$$

(Otherwise, we obtain a contradiction by setting $v = [u,w]$ and $z = \big[[u,w],w\big]$.) We may assume

$$\mathsf{y}_v \neq 0 \text{ for every } v \in \mathfrak{h} \text{ such that } \phi_v = 0,\ y_v = 0, \text{ and } x_v \neq 0. \tag{5.4}$$

(Otherwise, Conclusion 5.1(2) holds.)

We claim $\dim \mathfrak{h} \leq 2$. If not, then there exist linearly independent $v, w \in \mathfrak{h}$, such that $\phi_v = \phi_w = 0$. From (5.3), we know that $y_v = y_w = 0$. By replacing with a linear combination, we may assume $\mathsf{y}_w = 0$. Then, from (5.4), we know that $x_w = 0$, so $w \in \mathfrak{z}$. Because $\mathfrak{z}$ is (at most) one-dimensional, but v and w

are linearly independent, we know that $v \notin \mathfrak{z}$, so $x_v \neq 0$. This contradicts the assumption of this subsubcase.

We may now assume $\dim \mathfrak{h} = 2$ (otherwise Conclusion 5.2(5) holds). Choose a nonzero $v \in \mathfrak{h}$, such that $\phi_v = 0$. If $x_v \neq 0$, then Conclusion 5.2(5) holds. If $x_v = 0$, then $v \in \mathfrak{z}$, so either Conclusion 5.1(5) or 5.2(6) holds.

Case 3. Assume the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha$ is two-dimensional. We may assume $\mathfrak{z} = 0$ (otherwise, $\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{h}$, so Conclusion 5.1(1) holds). We may assume $y_h = 0$ for every $h \in H$ (otherwise Lemma 4.5 implies that either 5.2(5) or 5.1(2) applies. Therefore $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{z} = 0$, so $\mathfrak{h}$ is abelian.

Let $u, v \in \mathfrak{h}$ with $\phi_u = 1$ and $\phi_v = i$. Then

$$0 = \eta_{[u,v]} = i\mathsf{y}_v + \mathsf{y}_u,$$

so $\mathsf{y}_u = \mathsf{y}_v = 0$. Then, for every $w \in \mathfrak{h}$, we have $0 = \eta_{[u,w]} = i\mathsf{y}_w$, so $\mathsf{y}_w = 0$. We may assume

$$|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) = 0 \tag{5.5}$$

for every $h \in \mathfrak{h}$ (otherwise Conclusion 5.1(3) holds). This implies $\dim \mathfrak{h} = 2$ (otherwise, there is some $w \in \mathfrak{h}$ such that $\phi_w = 0$ and $x_w \neq 0$, and then Eq. (5.5) does not hold for $h = u + tw$ when t is sufficiently large). Thus, Conclusion 5.2(5) holds. ∎

Lemma 5.4. *Let H be a closed, connected, nontrivial subgroup of N. Assume $\phi_h = 0$ for every $h \in \mathfrak{h}$, that (5.1) holds, and that $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ for every $v \in \mathfrak{h}$ such that $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$. Then $\|h\|^{3/2} = O\big(\Delta(h)\big)$ for every $h \in H$.*

Furthermore, $\Delta(h) \asymp \|h\|^2$ whenever $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4$.

Proof. We have $h \asymp |x_h|^2 + |y_h|^2 + |\mathsf{x}_h| + |\mathsf{y}_h| + |\eta_h|$. Also, from Eq. (5.1), we have $|\eta_z|^2 - \mathsf{x}_z\mathsf{y}_z \asymp \big(|\mathsf{x}_z| + |\mathsf{y}_z| + |\eta_z|\big)^2$ for every $z \in \mathfrak{z}$. Also, $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \asymp |v|^3$ whenever $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$.

Case 1. Assume $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 = o\big(|x_h|^4 + |y_h|^4\big)$. Then there is some $v \in \mathfrak{h}$ such that $v - \log h = o\big(|x_h| + |y_h|\big)$ and $|x_v|^2|y_v|^2 - |x_v y_v^\dagger|^2 = 0$. We have $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$. Therefore

$$\begin{aligned}
\operatorname{Im}\Delta(h) &\asymp \mathsf{x}_h|y_h|^2 + \mathsf{y}_h|x_h|^2 + 2\operatorname{Im}(x_h y_h^\dagger \overline{\eta_h}) \\
&= \mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \\
&\qquad + o\big(|\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_h|^3 + |y_h|^3\big) \\
&\asymp |v|^3 + o\big(|\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_h|^3 + |y_h|^3\big) \\
&\asymp |\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_v|^3 + |y_v|^3 \\
&\neq o\big(\|h\|^{3/2}\big).
\end{aligned}$$

Thus, $\|h\|^{3/2} = O\big(\rho(h)\big)$.

Case 2. Assume $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4$. We may assume $\operatorname{Re}\Delta(h) = o\big(|x_h|^4 + |y_h|^4\big)$ for otherwise it is clear that $\operatorname{Re}\Delta(h) \asymp \|h\|^2$. (So we have

$\|h\| \asymp |\eta_h| + |\mathsf{x}_h| + |\mathsf{y}_h| \asymp |x_h|^2 + |y_h|^2$.) Thus, there is some $z \in \mathfrak{z}$, such that $z - \log h = o(\log h)$ and

$$|\eta_z|^2 - \mathsf{x}_z \mathsf{y}_z = -\frac{1}{4}\big(|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2\big) + o\big(|x_h|^4 + |y_h|^4\big) < 0.$$

(This implies that x_z and y_z must have the same sign.) From (5.1), we conclude that $|\eta_z|^2 - \mathsf{x}_z \mathsf{y}_z < 0$ for every $z \in \mathfrak{z}$. Thus, there is a constant $\epsilon < 1$, such that $|\eta_z| \le \epsilon\sqrt{\mathsf{x}_z \mathsf{y}_z}$ for every $z \in \mathfrak{z}$. Then

$$|\operatorname{Im}(x_h y_h^\dagger \overline{\eta_z})| \le |\eta_z||x_h||y_h| \le \frac{\epsilon}{2}\big|\mathsf{x}_z|y_h|^2 + \mathsf{y}_z|x_h|^2\big|,$$

so

$$\operatorname{Im}(x_h y_h^\dagger \overline{\eta_z}) + \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 \asymp \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2.$$

Therefore

$$\begin{aligned}
\operatorname{Im}\Delta(h) &= \operatorname{Im}(x_h y_h^\dagger \overline{\eta_h}) + \frac{1}{2}\mathsf{x}_h|y_h|^2 + \frac{1}{2}\mathsf{y}_h|x_h|^2 \\
&= \operatorname{Im}(x_h y_h^\dagger \overline{\eta_z}) + \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 + o\big((|x_h|^2 + |y_h|^2)\log h\big) \\
&\asymp \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 \\
&\asymp |x_h|^4 + |y_h|^4 \\
&\asymp \|h\|^2.
\end{aligned}$$

∎

6 Non-Cartan-decomposition subgroups contained in N

Theorem 6.1. *Assume that $G = \mathrm{SU}(2,n)$. Here is a complete list of the closed, connected, nontrivial subgroups H of N, such that H is* **not** *a Cartan-decomposition subgroup.*

1. *If $\dim \mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h \mathsf{y}_h$ for every $h \in H$, then $\rho(h) \asymp h$ for every $h \in H$.*

2. *If $\phi_h = 0$ and $y_h = 0$ for every $h \in \mathfrak{h}$, there is some $u \in \mathfrak{h}$, such that $\mathsf{y}_u \ne 0$, and $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 1$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*

3. *Suppose $\phi_h = 0$ for every $h \in \mathfrak{h}$, and there is some $\lambda \in \mathbb{C}$, such that $x_h = \lambda y_h$ for every $h \in H$, and we have $\eta_z = i\lambda \mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ for every $z \in \mathfrak{z}$.*

   (a) *If there is some $u \in \mathfrak{h}$, such that $\mathsf{x}_u + |\lambda|^2\mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \ne 0$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 1$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*

   (b) *Otherwise, $\rho(h) \asymp h$ for every $h \in H$.*

4. *If* $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$ *(so* $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$*), then* $\rho(h) \asymp h$ *for every* $h \in H$.

5. *If* $\mathfrak{z} = 0$, *there is some* $u \in \mathfrak{h}$ *and some nonzero* $\phi_0 \in \mathbb{C}$, *such that* $\phi_u \neq 0$, *and we have* $\phi_h = \phi_0 \mathsf{y}_h$ *and* $y_h = 0$, *for every* $h \in \mathfrak{h}$, *then* $\mu(H) \approx \big[\|h\|, \|h\|^{4/3}\big]$, *unless* $\dim H = 1$, *in which case,* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$.

6. *If* $\phi_h = 0$ *and* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *for every* $h \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$, *then* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

7. *If* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *there exist nonzero* u *and* v *in* $\mathfrak{h}$, *such that* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *and* $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, *and we have* $\mathsf{x}_v |y_v|^2 + \mathsf{y}_v |x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ *for every such* $v \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$ *then* $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$.

8. *If* $\dim \mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, *we have* $\phi_v \asymp y_v$ *and* $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ *for every* $v \in \mathfrak{h}$, *and there exists* $u \in \mathfrak{h}$, *such that* $\phi_u \neq 0$, *then* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

9. *If* $\dim \mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, $\phi_h \neq 0$ *and* $y_h \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{z}$, *then* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.

10. *If* $\dim \mathfrak{h} \leq 2$ *and* $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) = 0$ *for every nonzero* $h \in \mathfrak{h}$, *then* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

11. *If* $\dim \mathfrak{h} = 2$ *and there exist nonzero* $u \in \mathfrak{h}$ *and* $z \in \mathfrak{z}$, *such that* $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u \overline{\eta_z} \neq 0$ *is real, and* $\mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\operatorname{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0$, *then* $\mu(H) \approx \big[\|h\|^{5/4}, \|h\|^2\big]$.

Proof. The theorem is obtained by merging the statement of Proposition 4.3 with the statement of Proposition 5.2, and eliminating some redundancy (see 3.4). Specifically:

- 4.3(1) appears here as 6.1(1).
- 4.3(2) appears here as 6.1(2).
- 4.3(3) appears here as 6.1(3).
- 4.3(4) appears here as 6.1(4).
- 4.3(5) appears here as 6.1(5).
- 4.3(6) appears here as 6.1(8).
- 4.3(7) appears here as 6.1(9).
- 5.2(1) is a special case of 6.1(6).
- 5.2(2) appears here as 6.1(6).
- 5.2(3) appears here as 6.1(7).

- 5.2(4) appears here as 6.1(8).
- 5.2(5) appears here as 6.1(10).
- 5.2(6) appears here as 6.1(11).
- 5.2(7) is a special case of 6.1(3a) (with $\dim H = 1$).
- 5.2(8) is a special case of 6.1(5) (with $\dim H = 1$).

■

Corollary 6.2. *Assume that $G = \mathrm{SU}(2,n)$. Here is a complete list of the closed, connected, nontrivial subgroups H of N, such that H is* **not** *a Cartan-decomposition subgroup,* **and** *$N_A(H)$ is nontrivial.*

1. *Suppose $\dim \mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h \mathsf{y}_h$ for every $h \in H$.*
   - *(a) If $\mathfrak{h} = \mathfrak{u}_{2\beta}$ or $\mathfrak{h} = \mathfrak{u}_{2\alpha+2\beta}$, then $N_A(H) = A$.*
   - *(b) Otherwise, $N_A(H) = \ker(\alpha)$.*
2. *Suppose $\phi_h = 0$ and $y_h = 0$ for every $h \in \mathfrak{h}$, there is some $u \in \mathfrak{h}$, such that $\mathsf{y}_u \neq 0$, and $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, then $N_A(H) = \ker(\alpha - \beta)$.*
3. *Suppose $\phi_h = 0$ for every $h \in \mathfrak{h}$, and there is some nonzero $\lambda \in \mathbb{C}$, such that $x_h = \lambda y_h$ for every $h \in H$, and we have $\eta_z = i\lambda \mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ for every $z \in \mathfrak{z}$. If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z} \neq \mathfrak{z}$, then $N_A(H) = \ker(\alpha)$.*
4. *Suppose $\phi_h = 0$ and $x_h = 0$ for every $h \in \mathfrak{h}$, we have $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, and $\mathfrak{h} \neq \mathfrak{z}$.*
   - *(a) If $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_{\beta}) + \mathfrak{z}$, then $N_A(H) = A$.*
   - *(b) Otherwise:*
     - *i. If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, then $N_A(H) = \ker(\alpha + \beta)$.*
     - *ii. If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{2\alpha+2\beta})\big) + \mathfrak{z}$, then $N_A(H) = \ker(2\alpha + \beta)$.*
     - *iii. If $\mathfrak{z} = 0$ and $\mathfrak{h} \subset \mathfrak{u}_{\beta} + \mathfrak{u}_{2\beta}$, then $N_A(H) = \ker(\beta)$.*
5. *Suppose $y_h = 0$, $\mathsf{y}_h = 0$, and $|x_h|^2 + 2\,\mathrm{Re}(\phi_h \overline{\eta_h}) \neq 0$ for every $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$.*
   - *(a) If $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, then $N_A(H) = A$.*
   - *(b) If $\mathfrak{h} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, but $\mathfrak{h} \neq (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, then $N_A(H) = \ker(\alpha+\beta)$.*
   - *(c) If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha} + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, but $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, then $N_A(H) = \ker(\beta)$.*
6. *Suppose $\mathfrak{z} = 0$, there is some nonzero $\phi_0 \in \mathbb{C}$, such that $\phi_h = \phi_0 \mathsf{y}_h$ and $y_h = 0$, for every $h \in \mathfrak{h}$, and there is some $u \in \mathfrak{h}$, such that $\phi_u \neq 0$. If $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta})$, then $N_A(H) = \ker(\alpha - 2\beta)$.*
7. *Suppose $\phi_h = 0$ and $\dim_{\mathbb{C}}\langle x_u, y_u \rangle \neq 1$ for every $h \in \mathfrak{h}$, and $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, for every nonzero $z \in \mathfrak{z}$.*

(a) *If* $\mathfrak{h} \subset \mathfrak{u}_{\alpha+2\beta}$, *then* $N_A(H) = A$.

(b) *If* $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+2\beta}$, *and* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha)$.

8. *Suppose* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *there exist nonzero* $u, v \in \mathfrak{h}$, *such that* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *and* $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ *for every such* $v \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$.

   (a) *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $N_A(H) = \ker(\alpha - \beta)$ *(and* $\dim H \leq 3$*).*

   (b) *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $N_A(H) = \ker(2\alpha + \beta)$ *(and* $\dim H \leq 3$*).*

9. *Suppose* $\dim \mathfrak{h} \leq 3$, $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big)$, $\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta) \neq 0$, *and we have* $\phi_h \asymp y_h$ *and* $x_h \asymp \mathsf{y}_h$ *for* $h \in \mathfrak{h}$, *then* $N_A(H) = \ker(\alpha - \beta)$.

10. *Suppose* $\dim \mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, $\phi_h \neq 0$ *and* $y_h \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{z}$. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha - \beta)$.

11. *Suppose* $\dim \mathfrak{h} \leq 2$ *and* $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) = 0$ *for every nonzero* $h \in \mathfrak{h}$.

   (a) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha$, *then* $N_A(H) = A$.

   (b) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, *but* $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$, *then* $N_A(H) = \ker(\beta)$.

   (c) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$, *then* $N_A(H) = \ker(\alpha + 2\beta)$.

Proof. It is clear that each of the given subgroups is normalized by the indicated torus. We now show that the list is complete, and that no larger subtorus of A normalizes H.

Assume $N_A(H)$ is nontrivial. We proceed in cases, determined by Theorem 6.1.

Case 1. Assume 6.1(1). We may assume $\mathfrak{h}$ is neither $\mathfrak{u}_{2\beta}$ nor $\mathfrak{u}_{2\alpha+2\beta}$ (otherwise (1a) applies). Then, because $|\eta_u|^2 = \mathsf{x}_u\mathsf{y}_u$ for every $u \in \mathfrak{h}$, we see that $\eta_u \neq 0$ for every nonzero $u \in \mathfrak{h}$. Thus, the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+2\beta}$ is nontrivial. However, because $|\eta_u|^2 = \mathsf{x}_u\mathsf{y}_u$, we have $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$. We know that $\mathfrak{h} \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\beta} + \mathfrak{u}_{2\alpha+2\beta}$ (because $\mathfrak{h} = \mathfrak{z}$), so, because each of 2β and $2\alpha + 2\beta$ differs from $\alpha + 2\beta$ by α, we conclude that $N_A(H) = \ker(\alpha)$, so (1b) applies.

Case 2. Assume 6.1(2). Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta}$. Because $\mathsf{y}_u \neq 0$, we know that V projects nontrivially to $\mathfrak{u}_{2\beta}$. However, because $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, we also know that $V \cap \mathfrak{u}_{2\beta} = 0$. Therefore $N_A(H) = \ker(\alpha - \beta)$. Then, because neither $\alpha + 2\beta$ nor $2\alpha + 2\beta$ differs from $\alpha + \beta$ by a multiple of $\alpha - \beta$, we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, so (2) applies.

Case 3. Assume 6.1(3). We may assume $\mathfrak{h} \neq \mathfrak{z}$ (otherwise Case 1 applies).

Subcase 3.1. Assume $\lambda \neq 0$. Because $\mathfrak{h} \neq \mathfrak{z}$, the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$ is nontrivial. However, because $\lambda \neq 0$, this projection intersects neither $\mathfrak{u}_\beta$ nor $\mathfrak{u}_{\alpha+\beta}$. Therefore $N_A(H) \subset \ker(\alpha)$. Then, because neither 2β, $\alpha + 2\beta$, nor $2\alpha + 2\beta$ differs from β by a multiple of α, we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, so (3) applies.

Subcase 3.2. Assume $\lambda = 0$. This means $x_u = 0$ for every $u \in \mathfrak{h}$, and $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$.

Because $\mathfrak{h} \neq \mathfrak{z}$, we know that $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_{\beta}$. Because $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, we know that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = \mathfrak{h} \cap \mathfrak{u}_{2\alpha+2\beta} = 0$. Thus, it is easy to see that if $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_{\alpha+2\beta}$ or $\mathfrak{u}_{2\alpha+2\beta}$ then either (4(b)i) or (4(b)ii) applies.

Thus, we may assume $\mathfrak{h} \subset \mathfrak{u}_{\beta} + \mathfrak{u}_{2\beta}$. If $\mathfrak{z} \neq 0$, then $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_{\beta}) + \mathfrak{u}_{2\beta}$, so (4a) applies. Otherwise, (4(b)iii) applies.

Case 4. Assume 6.1(4).

Subcase 4.1. Assume the projection of $\mathfrak{h}$ *to* $\mathfrak{u}_{\alpha}$ *is trivial.* Because

$$|x_u|^2 = |x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$$

for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, we know that $x_u \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$. Thus, if the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+2\beta}$ is nontrivial, then $N_A(H) = \ker(\beta)$, and we see that (5c) applies. If not, then $\mathfrak{h} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, so either (5a) or (5b) applies.

Subcase 4.2. Assume the projection of $\mathfrak{h}$ *to* $\mathfrak{u}_{\alpha}$ *is nontrivial.* Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha} + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. Because $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, we know that $V \cap \mathfrak{u}_{\alpha} = 0$. Then, because α, $\alpha+\beta$, and $\alpha+2\beta$ all differ by multiples of β, we conclude that $N_A(H) = \ker(\beta)$. Therefore (5c) applies.

Case 5. Assume 6.1(5). Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha} + \mathfrak{u}_{2\beta}$. Because $\phi_h = \phi_0 \mathsf{y}_h$, we see that $V \cap \mathfrak{u}_{\alpha} = 0$ and $V \cap \mathfrak{u}_{2\beta} = 0$. Therefore $N_A(H) = \ker(\alpha - 2\beta)$.

Because no other roots differ by a multiple of $\alpha - 2\beta$ (and $\mathfrak{z} = 0$), we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta})$. Thus, (6) applies.

Case 6. Assume 6.1(6).

Subcase 6.1. Assume $\mathfrak{h} \neq \mathfrak{z}$. Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+\beta}$. From the assumption of this subcase, we know $V \neq 0$. However, because $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ for every $u \in \mathfrak{h}$, we know that $V \cap \mathfrak{u}_{\beta} = 0$ and $\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta} = 0$. Therefore $N_A(H) = \ker(\alpha)$, so (7b) applies.

Subcase 6.2. Assume $\mathfrak{h} = \mathfrak{z}$. We may assume $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+2\beta}$ (otherwise (7a) applies). Therefore, $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_{2\beta} + \mathfrak{u}_{2\alpha+2\beta}$. However, because $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, for every nonzero $z \in \mathfrak{z}$, we know that $V \cap \mathfrak{u}_{2\beta} = 0$ and $V \cap \mathfrak{u}_{2\alpha+2\beta} = 0$. Because 2β, $\alpha+2\beta$, and $2\alpha+2\beta$ all differ by multiples of α, we conclude that $N_A(H) = \ker(\alpha)$, so (7b) applies.

Case 7. Assume 6.1(7).

Subcase 7.1. Assume $N_A(H) = \ker(\alpha)$. Because $\alpha+\beta$ is the only root that differs from β by a multiple of α, we must have $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$. Thus, there is some $w \in \mathfrak{h}$, such that $x_w = x_v$ and $y_w = y_v$, but the projection of w to $\mathfrak{u}_{2\beta} + \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$ is zero. This contradicts the fact that $\mathsf{x}_w|y_w|^2 + \mathsf{y}_w|x_w|^2 + 2\,\mathrm{Im}(x_w y_w^{\dagger} \overline{\eta_w}) \neq 0$.

Subcase 7.2. Assume $N_A(H) \neq \ker(\alpha)$. Because 2β, $\alpha+2\beta$, and $2\alpha+2\beta$ all differ by multiples of α, we must have $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{u}_{2\beta}) + (\mathfrak{z} \cap \mathfrak{u}_{\alpha+2\beta}) + (\mathfrak{z} \cap \mathfrak{u}_{2\alpha+2\beta})$. Then, because $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$ for every nonzero $z \in \mathfrak{z}$, we conclude that $\mathfrak{z} \subset \mathfrak{u}_{\alpha+2\beta}$.

Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. Because β and $\alpha+\beta$ differ by α, we know that $V = (V \cap \mathfrak{u}_\beta) + (V \cap \mathfrak{u}_{\alpha+\beta})$.

Subsubcase 7.2.1. Assume $x_v \neq 0$. Because $V = (V \cap \mathfrak{u}_\beta) + (V \cap \mathfrak{u}_{\alpha+\beta})$, there is some $w \in V$, such that $x_w \neq 0$ and $y_w = 0$. For every such w, because $\mathsf{x}_w |y_w|^2 + \mathsf{y}_w |x_w|^2 + 2\,\mathrm{Im}(x_w y_w^\dagger \overline{\eta_w}) \neq 0$, we know that $\mathsf{y}_w \neq 0$. Thus, we see that $N_A(H) = \ker\big((\alpha+\beta) - 2\beta\big) = \ker(\alpha - \beta)$.

We know that $\mathfrak{h} \cap \mathfrak{u}_\beta = 0$, that $\mathfrak{h}$ projects trivially to $\mathfrak{u}_\alpha$, and that α is the only root that differs from β by a multiple of $\alpha - \beta$, so we conclude that $y_h = 0$ for every $h \in H$.

We now see that (8a) applies.

Subsubcase 7.2.2. Assume $y_v \neq 0$. This is similar to the preceding subsubcase (indeed, they are conjugate under the Weyl reflection corresponding to the root α); we see that (8b) applies.

Case 8. Assume 6.1(8). By considering the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_\beta$, and noting that $\phi_h \asymp y_h$ for every $h \in H$, we see that $N_A(H) = \ker(\alpha - \beta)$. The only other pair of roots that differ by a multiple of $\alpha - \beta$ is $\{\alpha+\beta, 2\beta\}$. Thus, we see that (9) applies.

Case 9. Assume 6.1(9). By considering the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_\beta$, we see that $N_A(H) = \ker(\alpha - \beta)$. Because $\phi_u \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, but β is the only root that differs from α by a multiple of $\alpha - \beta$, we conclude that $\mathfrak{h}$ projects trivially into every root space except $\mathfrak{u}_\alpha$, $\mathfrak{u}_\beta$, and $\mathfrak{u}_{2\alpha+2\beta}$. Thus (10) applies.

Case 10. Assume 6.1(10). We may assume $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$ (otherwise (11a) applies). Thus, there is some root $\sigma \neq \alpha$, such that the projection of $\mathfrak{h}$ to $\mathfrak{u}_\sigma$ is nontrivial. However, because $\phi_h \neq 0$ for every nonzero $h \in \mathfrak{h}$, we know that $\mathfrak{h} \cap \mathfrak{u}_\sigma = 0$. Thus, $N_A(H) = \ker(\alpha - \sigma)$.

Because $y_h = 0$ and $\mathsf{y}_h = 0$ for every nonzero $h \in \mathfrak{h}$, we know that $\sigma \neq \beta$ and $\sigma \neq 2\beta$. If $\sigma = \alpha + \beta$ or $\sigma = \alpha + 2\beta$, we obtain (11b). If $\sigma = 2\alpha + 2\beta$, we obtain (11c).

Case 11. Assume 6.1(11). Because $\phi_u \neq 0$ and $y_u \neq 0$, we must have $N_A(H) = \ker(\alpha - \beta)$. Then, because $\alpha + \beta$ does not differ from α by a multiple of $\alpha - \beta$, we conclude that $x_u = 0$.

Because $\eta_z \neq 0$, but no root differs from $\alpha + 2\beta$ by a multiple of $\alpha - \beta$, we conclude that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} \neq 0$. Because $\mathfrak{z}$ is one-dimensional, this implies $z \in \mathfrak{u}_{\alpha+2\beta}$, so $\mathsf{x}_z = 0$.

Since $\mathsf{x}_z = 0$ and $x_u = 0$, we conclude, from the inequality $\mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0$, that $\mathsf{y}_u \neq 0$. This is a contradiction, because 2β does not differ from α by a multiple of $\alpha - \beta$, and $\mathfrak{h} \cap \mathfrak{u}_{2\beta} = 0$ (because, as shown above, $\mathfrak{z} \subset \mathfrak{u}_{\alpha+2\beta}$). ■

7 Subgroups that are not contained in N

Let H be a closed, connected subgroup of G that is not contained in N. In this section, we determine whether H is a Cartan-decomposition subgroup or not (and, if not, we calculate $\mu(H)$).

Lemma 7.1 shows that we may assume $H \subset AN$, and then Lemma 7.3 shows that we may assume H satisfies the technical condition of being compatible with A. (Both of these lemmas are well known.) Furthermore, we may assume that $H \cap N$ is **not** a Cartan-decomposition subgroup, and that $A \not\subset H$ (otherwise, it is obvious that H is a Cartan-decomposition subgroup).

Theorem 7.4 describes $\mu(H)$ for every such subgroup that is a semidirect product $(H \cap A) \ltimes (H \cap N)$; and Proposition 7.6 describes $\mu(H)$ for the other subgroups (except that the one-dimensional case appears in Lemma 7.8).

Lemma 7.1. [14, Lem. 2.9] *Let H be a closed, connected subgroup of a connected, almost simple, linear, real Lie group G. There is a closed, connected subgroup H' of G and a compact subgroup C of G, such that $CH = CH'$, and H' is conjugate to a subgroup of AN.*

Definition 7.2. Let us say that a subgroup H of AN is *compatible* with A if $H \subset TUC_N(T)$, where $T = A \cap (HN)$, $U = H \cap N$, and $C_N(T)$ denotes the centralizer of T in N.

Lemma 7.3. [14, Lem. 2.3] *If H is a closed, connected subgroup of AN, then H is conjugate, via an element of N, to a subgroup that is compatible with A.*

Theorem 7.4. *Assume that $G = \mathrm{SU}(2,n)$. Here is a list of every closed, connected, nontrivial subgroup H of AN, such that H is of the form $H = T \ltimes U$, where T is a one-dimensional subgroup of A, and U is a nontrivial subgroup of N that is **not** a Cartan-decomposition subgroup.*

1. *Suppose $\dim \mathfrak{u} = 1$, $\mathfrak{u} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h \mathsf{y}_h$ for every $h \in U$.*
   (a) *If $\mathfrak{u} = \mathfrak{u}_{2\beta}$ or $\mathfrak{u} = \mathfrak{u}_{2\alpha+2\beta}$, then $\mu(H)$ is described in [14, Prop. 3.17 or Cor. 3.18].*
   (b) *Otherwise, $T = \ker(\alpha)$, and H is a Cartan-decomposition subgroup.*
2. *Suppose $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, there is some $v \in \mathfrak{u}$, such that $\mathsf{y}_v \neq 0$, and $T = \ker(\alpha - \beta)$. Then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 2$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
3. *Suppose $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, $T = \ker(\alpha)$, and there is some nonzero $\lambda \in \mathbb{C}$, such that we have $x_u = \lambda y_u$ for every $u \in U$, and we have $\eta_z = i\lambda \mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ for every $z \in \mathfrak{z}$. Then H is a Cartan-decomposition subgroup.*
4. *Suppose $\phi_u = 0$ and $x_u = 0$ for every $u \in \mathfrak{u}$, we have $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, and $\mathfrak{u} \neq \mathfrak{z}$.*
   (a) *If $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{u}_{\beta}) + \mathfrak{z}$, then $\mu(H)$ is described in [14, Prop. 3.17 or Cor. 3.18].*
   (b) *Otherwise:*
      i. *If $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, then $T = \ker(\alpha + \beta)$, and $\rho(h) \asymp h$ for every $h \in H$.*

ii. *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + \mathfrak{z}$*, then* $T = \ker(2\alpha+\beta)$*, and* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$*, unless* $\dim H = 2$*, in which case* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$*.*

iii. *If* $\mathfrak{z} = 0$ *and* $\mathfrak{u} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$*, then* $T = \ker(\beta)$*, and* H *is a Cartan-decomposition subgroup.*

5. *Suppose* $y_u = 0$*,* $\mathsf{y}_u = 0$*, and* $|x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u}) \neq 0$ *for every* $u \in U \backslash U_{2\alpha+2\beta}$*.*

   (a) *If* $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$*, then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

   (b) *If* $\mathfrak{u} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$*, but* $\mathfrak{u} \neq (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$*, then* $T = \ker(\alpha+\beta)$*, and* H *is a Cartan-decomposition subgroup.*

   (c) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$*, but* $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$*, then* $T = \ker(\beta)$*, and* $\rho(h) \asymp h$ *for every* $h \in H$*.*

6. *Suppose* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta})$*,* $T = \ker(\alpha - 2\beta)$*,* $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+\beta}$*, and there is some nonzero* $\phi_0 \in \mathbb{C}$*, such that* $\phi_u = \phi_0 \mathsf{y}_u$ *for every* $u \in U$*. Then* $\mu(H) \approx \big[\|h\|, \|h\|^{4/3}\big]$*, unless* $\dim H = 2$*, in which case,* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$*.*

7. *Suppose* $\phi_u = 0$ *and* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *for every* $u \in U$*, and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$*, for every nonzero* $z \in \mathfrak{z}$*.*

   (a) *If* $\mathfrak{u} \subset \mathfrak{u}_{\alpha+2\beta}$*, then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

   (b) *If* $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+2\beta}$*, and* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$*, then* $T = \ker(\alpha)$*, and* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$*.*

8. *Suppose* $\phi_u = 0$ *for every* $u \in U$*, there exist nonzero* $v_1, v_2 \in \mathfrak{u}$*, such that* $\dim_{\mathbb{C}}\langle x_{v_1}, y_{v_1}\rangle \neq 1$ *and* $\dim_{\mathbb{C}}\langle x_{v_2}, y_{v_2}\rangle = 1$*, and we have* $\mathsf{x}_{v_2}|y_{v_2}|^2 + \mathsf{y}_{v_2}|x_{v_2}|^2 + 2\operatorname{Im}(x_{v_2} y_{v_2}^\dagger \overline{\eta_{v_2}}) \neq 0$ *for every such* $v_2 \in \mathfrak{u}$*, and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$*, for every nonzero* $z \in \mathfrak{z}$*.*

   (a) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+2\beta})$*, then* $T = \ker(\alpha - \beta)$ *and* $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$*.*

   (b) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+2\beta})$*, then* $T = \ker(2\alpha+\beta)$ *and* $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$*.*

9. *Suppose* $\dim \mathfrak{u} \leq 3$*,* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big)$*,* $\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta) \neq 0$*, and we have* $\phi_u \asymp y_u$ *and* $x_u \asymp \mathsf{y}_u$ *for* $u \in U$*. Then* $T = \ker(\alpha - \beta)$*, and* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$*.*

10. *Suppose* $\dim \mathfrak{u} = 2$*,* $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$*,* $\phi_u \neq 0$ *and* $y_u \neq 0$ *for every* $u \in U \setminus Z$*. If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \mathfrak{z}$*, then* $T = \ker(\alpha - \beta)$*, and* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$*.*

11. *Suppose* $\dim \mathfrak{u} \leq 2$ *and* $\phi_u \neq 0$*,* $y_u = 0$*,* $\mathsf{y}_u = 0$*, and* $|x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u}) = 0$ *for every nontrivial* $u \in U$*.*

    (a) *If* $\mathfrak{u} \subset \mathfrak{u}_\alpha$*, then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

    (b) *If* $\mathfrak{u} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$*, but* $\mathfrak{u} \not\subset \mathfrak{u}_\alpha$*, then* $T = \ker(\beta)$*, and* H *is a Cartan-decomposition subgroup.*

(c) *If* $\mathfrak{u} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{u} \not\subset \mathfrak{u}_\alpha$, *then* $T = \ker(\alpha+2\beta)$, *and* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

Proof. For $h \in H$, we wish to approximately calculate $\|\rho(h)\|$. We write $h = au$ with $a \in T$ and $u \in U$. Writing $a = \operatorname{diag}(a_1, a_2, \ldots, a_{n+2})$, we always assume either that $a_1 > 1$ or that $a_1 = 1$ and $a_2 \geq 1$ (perhaps replacing h with h^{-1}—because $\|\rho(h)\| = \|\rho(h^{-1})\|$, this causes no harm).

Because T normalizes U, we know that U is a subgroup that is listed in Corollary 6.2, and we have $T \subset N_G(U)$. This leads to the various cases listed in the statement of the theorem.

(1b) We have $\rho(u) \asymp u$ for $u \in U$ and $\rho(a) \asymp \|a\|^2$ for $a \in T$, so H is a Cartan-decomposition subgroup.

(2) We have $|\phi_u|+|y_u|+|\eta_u|+|\mathsf{x}_u| = 0$ and $\mathsf{y}_u = O(x_u)$, so $u_{i,j} = O\big(1+|x_u|\big)$ whenever $(i,j) \neq (1, n+2)$. Then, because $a_1 = a_2^2$, we see that

$$u_{i,j} = O\big[a_2\big(1+|x_u|\big)\big] = O\big(|h_{1,1}|^{1/2} + |h_{1,n+2}|^{1/2}\big) = O\big(\|h\|^{1/2}\big)$$

whenever $i > 1$. Therefore $\rho(h) = O\big(\|h\|^{3/2}\big)$. This completes the proof if $\dim H > 2$ (that is, if $\dim U > 1$).

If $\dim U = 1$, then $\mathsf{y}_u \asymp x_u$ and $\mathsf{x}_u = 0$. We have $\|h\| = a_1\big(1+|x_u|^2\big)$,

$$\Delta(h) = a_1 a_2 \left[i \left(\frac{1}{2} |x_u|^2 \mathsf{y}_u \right) \right] \asymp \big(a_1 |x_u|^2\big)^{3/2}$$

and

$$\det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} = a_1 a_2 = a_1^{3/2}.$$

Thus, $\|h\|^{3/2} = O\big(\rho(h)\big)$. We conclude that $\rho(h) \asymp \|h\|^{3/2}$.

(3) Replacing H by a conjugate under U_α, we may replace H with a similar subgroup H' with $\lambda = 0$. Thus, $H' = T \ltimes U'$ with $U' \subset U_\beta U_{2\beta}$. Then [14, Prop. 3.17] implies H is a Cartan-decomposition subgroup.

(4(b)i) The Weyl reflection corresponding to the root α conjugates H to a subgroup of type (5c).

(4(b)ii) The Weyl reflection corresponding to the root α conjugates H to a subgroup of type (2).

(4(b)iii) [14, Prop. 3.17] implies H is a Cartan-decomposition subgroup.

(5b) [14, Prop. 3.17] implies H is a Cartan-decomposition subgroup.

(5c) We have

$$h_{i,j} = \begin{cases} O(1) & \text{if } i \neq 1 \text{ and } j \neq n+2 \\ O(a_1 x) & \text{if } i = 1 \text{ and } j \neq n+2 \\ O(x) & \text{if } i \neq 1 \text{ and } j = n+2 \end{cases}$$

and $h_{1,n+2} \asymp a_1\big(|x|^2 + |\mathsf{x}|\big)$. We conclude that $\rho(h) \asymp h$.

(6) From the proof of 4.3(5), we know that $u \asymp u_{1,n+2}$, that $u_{i,j} = O\big(\|u\|^{3/2}\big)$ whenever $(i,j) \neq (1, n+2)$, and that $u_{i,j} = O\big(\|u\|^{1/3}\big)$ whenever $i \neq 1$ and $j \neq n+2$. (In particular, $h \asymp a_1\big(1 + u_{1,n+2}\big)$.) Furthermore, we have $a_1 = a_2^3$. Therefore

$$\rho(h) \asymp a_1 a_2 \rho(u) \asymp a_1^{4/3} \rho(u).$$

The desired conclusion follows.

(7b) From Lemma 5.4, we know $\|u\|^2 = O\big(1 + |\Delta(u)|\big)$. Then, because

$$\det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} = a_1 a_2 = a_1^2$$

and $\Delta(au) = a_1^2 \Delta(u)$, we have

$$\|h\|^2 = O\big(a_1^2 \|u\|^2\big) = O\big(a_1^2 + |\Delta(h)|\big) = O\big(\rho(h)\big).$$

(8) Assume (8a). (The other case, (8b), is conjugate to this one by the Weyl reflection corresponding to the root α.) From Lemma 5.4, we have $\|u\|^{3/2} = O\big(1 + |\Delta(u)|\big)$. Then, because $a_1 = a_2^2$, we have

$$\|h\|^{3/2} = a_1 a_2 \|u\|^{3/2} = O\big(a_1 a_2 + |\Delta(h)|\big) = O\big(\rho(h)\big).$$

(9) From the proof of 4.3(6), we know $\rho(u) \asymp 1 + \Delta(u) \asymp \|u\|^{3/2}$. The proof is completed as in (8).

(10) Because $\phi_u \asymp y_u$, it is easy to see that

$$h \asymp a_1\big(1 + |\phi_u|^2 |y_u|^2 + |\mathsf{x}_u|\big) \asymp a_1 \big(1 + |\phi_u|^4 + |\mathsf{x}_u|\big)$$

and

$$\Delta(h) \asymp a_1 a_2 \big(|y_u|^4 |\phi_u|^2 + |\mathsf{x}_u||y_u|^2\big) \asymp a_1^{3/2} \big(|\phi_u|^6 + |\mathsf{x}_u||\phi_u|^2\big) = O\big(\|h\|^{3/2}\big).$$

Then it is not difficult to see that $\rho(h) = O\big(\|h\|^{3/2}\big)$ for every $h \in H$. So $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.

(11b) We have $\rho(a) \asymp a$ for $a \in T$ and $\rho(u) \asymp \|u\|^2$ for $u \in U$, so H is a Cartan-decomposition subgroup.

(11c) H is conjugate (via an element of $U_{\alpha+2\beta}$) to $T \ltimes U_\alpha$. From [14, Prop. 3.18], we have $\rho(h) \asymp \|h\|^2$ for every $h \in T \ltimes U_\alpha$. Therefore $\rho(h) \asymp \|h\|^2$ for every $h \in H$. ∎

Lemma 7.5. [14, Lem. 2.4] *Assume $G = \mathrm{SU}(2,n)$, and let H be a closed, connected subgroup of AN that is compatible with A. Then either*

1. *$H = (H \cap A) \ltimes (H \cap N)$; or*
2. *there is a positive root ω, a nontrivial group homomorphism $\psi\colon \ker\omega \to U_\omega U_{2\omega}$, and a closed, connected subgroup U of N, such that*
   - *(a) $H = \{\, a\psi(a) \mid a \in \ker\omega \,\}U$;*
   - *(b) $U \cap \psi(\ker\omega) = e$; and*
   - *(c) U is normalized by both $\ker\omega$ and $\psi(\ker\omega)$.*

Proposition 7.6. *Assume that $G = \mathrm{SU}(2,n)$. Let H be a closed, connected, nontrivial subgroup of AN, that is compatible with A, such that*

- *$H \cap N$ is not a Cartan-decomposition subgroup;*
- *$H \neq (H \cap A)(H \cap N)$; and*
- $\dim H > 1$.

Then there are positive roots ω and σ, and a one-dimensional subspace $\mathfrak{x}$ of $(\ker \omega) + \mathfrak{u}_\omega + \mathfrak{u}_{2\omega}$, such that $\mathfrak{h} = \mathfrak{x} + (\mathfrak{h} \cap \mathfrak{n})$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$, and either:

1. *$\omega = \alpha$, $\sigma = \alpha + \beta$, and $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log\|h\|)\big]$; or*
2. *$\omega = \alpha$ $\sigma = \alpha + 2\beta$, and $\mu(H) \approx \big[\|h\|^2/(\log\|h\|)^2, \|h\|^2\big]$; or*
3. *$\omega = \beta$, $\sigma = \alpha + 2\beta$, and $\mu(H) \approx \big[\|h\|\big(\log\|h\|\big)^{r/2}, \|h\|^2\big]$, where*
$$r = \begin{cases} 1 & \text{if } \mathfrak{x} \subset \mathfrak{u}_{2\beta} \\ 2 & \text{otherwise} \end{cases}$$
or
4. *$\omega = \beta$, $\sigma = \alpha + \beta$, and $\mu(H) \approx \big[\|h\|, \|h\|\big(\log\|h\|\big)^r\big]$, where r is defined as above; or*
5. *$\mathfrak{u} \cap (\mathfrak{u}_\omega + \mathfrak{u}_{2\omega}) \neq 0$, in which case H is a Cartan-decomposition subgroup.*

Proof. We use the notation of Lemma 7.5: $T = \ker\omega$, $U = H \cap N$, $\psi\colon T \to U_\omega U_{2\omega}$, and $H = \{a\psi(a)\} \ltimes U$.

We need only consider the cases in Corollary 6.2 for which H (now called U) is normalized by the kernel of some (reduced) positive root. Here is a list of them.

1. $N_A(U) = \ker(\beta)$: 6.2(4(b)iii), 6.2(5c), and 6.2(11b).
2. $N_A(U) = \ker(\alpha+\beta)$: 6.2(4(b)i) and 6.2(5b).
3. $N_A(U) = \ker(\alpha)$: 6.2(1b), 6.2(3), and 6.2(7b).
4. $N_A(U) = \ker(\alpha+2\beta)$: 6.2(11c).
5. $N_A(U) = A$: 6.2(1a), 6.2(4a), 6.2(5a), 6.2(7a), and 6.2(11a).

Note that in each of the cases with $N_A(U) = A$, there is a (reduced) positive root σ, such that $\mathfrak{u} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$.

Case 1. Assume $\omega = \beta$.

Subcase 1.1. Assume 6.2(4(b)iii). From (7.7), we know that H is a Cartan-decomposition subgroup.

Subcase 1.2. Assume 6.2(5c). There is some $u \in U$, such that $\phi_u \neq 0$. Then, because $\psi(T) \subset U_\beta U_{2\beta}$ normalizes U, we must have $U \cap U_{\alpha+2\beta} \neq e$. This is a contradiction.

Subcase 1.3. Assume 6.2(11b). Let $u \in \mathfrak{u}$. Because U is normalized by $\psi(T)$, there is some nonzero $v \in \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$, such that v normalizes $\mathfrak{u}$; thus, $[u,v] \in \mathfrak{u}$. Then, because $\phi_{[u,v]} = 0$, but $\phi_h \neq 0$ for every nontrivial $h \in U$, we conclude that $[u,v] = 0$. However, $\phi_u \neq 0$, and either $y_v \neq 0$ or $\mathsf{y}_v \neq 0$, so either $x_{[u,v]} \neq 0$ or $\eta_{[u,v]} \neq 0$. This is a contradiction.

Subcase 1.4. Assume $N_A(U) = A$. There is a positive root σ, such that $\mathfrak{u} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$.

If $\sigma = \beta$, then, from (7.7), we know that H is a Cartan-decomposition subgroup.

Suppose $\sigma = \alpha + 2\beta$. Clearly $\|h\| \asymp a_1 |\eta_u|$. Also,

$$\rho(h) \asymp a_1 |\eta_u|^2 + a_1 (\log a_1)^r,$$

where $r = 1$ if $\psi(T) \subset U_{2\beta}$ (i.e., if $y_h = 0$ for every $h \in H$) and $r = 2$ if $\psi(T) \not\subset U_{2\beta}$. The smallest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\eta_u \asymp (\log a_1)^{r/2}$, resulting in $\rho(h) \asymp \|h\| \big(\log \|h\|\big)^{r/2}$. Then, since $\rho(u) \asymp \|u\|^2$ for $u \in U$, we conclude that $\mu(H) \approx \big[\|h\| \big(\log \|h\|\big)^{r/2}, \|h\|^2\big]$.

Because U is normalized by the nontrivial subgroup $\psi(T)$ of $U_\beta U_{2\beta}$, we know that $\sigma \neq \alpha$. Therefore, we may now assume $\sigma = \alpha + \beta$. We show that $\mu(H) \approx \big[\|h\|, \|h\| (\log \|h\|)^r\big]$. For $u \in U$, we have $\rho(u) \asymp u$. For $a \in T$, we have

$$\rho\big(a\psi(a)\big) \asymp \|a\| \big(\log \|a\|\big)^r \asymp \|a\psi(a)\| \big(\log \|a\psi(a)\|\big)^r.$$

All that remains is to show that $\rho(h) = O\big[\|h\| \big(\log \|h\|\big)^r\big]$ for every $h \in H$. Because $\rho(au) \asymp au$ for every $au \in TU$ (see [14, Cor. 3.18]) and $\|\psi(a)\| \asymp \|\psi(a)^{-1}\| \asymp (\log \|h\|)^r$, we have

$$\begin{aligned} \rho(h) &= \rho\big(\psi(a)\big)\rho(au) = O\big[\|\rho\big(\psi(a)\big)\| \|\rho(au)\|\big] \\ &= O\big[\big(\log \|a\|\big)^r \|au\|\big] = O\big[\big(\log \|h\|\big)^r \|h\|\big]. \end{aligned}$$

Case 2. Assume $\omega = \alpha + \beta$. The Weyl reflection corresponding to the root α conjugates each of 6.2(4(b)i) and 6.2(5b) to a subgroup with $\omega = \beta$.

Thus, we may now assume $N_A(U) = A$. If $\sigma \neq \alpha$, then the Weyl reflection corresponding to the root α conjugates H to a subgroup with $\omega = \beta$. If $\sigma = \alpha$, then the Weyl reflection corresponding to the root β does not change ω, but conjugates H to a subgroup H_1 with $\sigma = \alpha + 2\beta$. Then (as we already observed) the Weyl reflection corresponding to the root α conjugates H_1 to a subgroup with $\omega = \beta$.

Case 3. Assume $\omega = \alpha$. Because U must be normalized by the nontrivial subgroup $\psi(T)$ of U_α, we see that U cannot be of type 6.2(1b) or 6.2(3).

Subcase 3.1. Assume 6.2(7b). Because U must be normalized by the nontrivial subgroup $\psi(T)$ of U_α, we see that $y_u = 0$ for every $u \in U$, so $\mathfrak{u} = \mathfrak{z}$. Thus, again using the fact that U is normalized by $\psi(T)$, we see that $\mathfrak{u} \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$, and the projection of $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+2\beta}$ is one-dimensional. For every $z \in \mathfrak{u}$, we see that $\eta_z \neq 0$ (because $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$). Thus, we conclude that $\dim \mathfrak{u} = 1$. Therefore H is conjugate under U_α to a subgroup of type 6.2(7a) (considered in Subsubcase 3.2.2 below).

Subcase 3.2. Assume $N_A(U) = A$. If $\sigma = \alpha$, then (7.7) implies that H is a Cartan-decomposition subgroup. Because U is normalized by the nontrivial subgroup $\psi(T)$ of U_α, we know that $\sigma \neq \beta$.

Subsubcase 3.2.1. Assume $\sigma = \alpha + \beta$. We have

$$h = a\psi(a)u = \begin{pmatrix} a_1 & a_1\phi_{\psi(a)} & a_1 x_u & 0 & -\frac{1}{2}a_1|x_u|^2 + i a_1 \mathsf{x}_u \\ & a_1 & 0 & 0 & 0 \\ & & \cdots & & \end{pmatrix}.$$

We have $\|h\| \asymp a_1 \log a_1 + a_1|x_u|^2 + a_1|\mathsf{x}_u|$ and, for $i > 1$, we have $h_{i,j} = O\bigl(a_1 + |x_u|\bigr)$. The largest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\log a_1 \asymp |x_u|^2$ (and x_u small), which yields $\rho(h) \asymp a_1^2 \log a_1 \asymp \|h\|^2/\log\|h\|$. Because $\rho(u) \asymp u$ for $u \in U$, we conclude that $\mu(H) \approx \bigl[\|h\|, \|h\|^2/\log\|h\|\bigr]$.

Subsubcase 3.2.2. Assume $\sigma = \alpha + 2\beta$. We have

$$h = a\psi(a)u = \begin{pmatrix} a_1 & a_1\phi_{\psi(a)} & 0 & a_1\eta_u & -a_1\phi_{\psi(a)}\overline{\eta_u} \\ & a_1 & 0 & 0 & -a_1\overline{\eta_u} \\ & & \cdots & & \end{pmatrix}.$$

We have $h \asymp \bigl(1 + a_1\|\psi(a)\|\bigr)\bigl(1 + |\eta_u|\bigr)$ and $\rho(h) \asymp a_1^2\bigl(1 + |\eta_u|^2\bigr)$ (note that $\det\begin{pmatrix} h_{1,2} & h_{1,n+2} \\ h_{2,2} & h_{2,n+2}\end{pmatrix} = 0$). The smallest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\eta_u = O(1)$, which results in $\rho(h) \asymp a_1^2 \asymp \|h\|^2/\bigl(\log\|h\|\bigr)^2$. Because $\rho(u) \asymp \|u\|^2$ for $u \in U$, we conclude that $\mu(H) \approx \bigl[\|h\|^2/\bigl(\log\|h\|\bigr)^2, \|h\|^2\bigr]$.

Case 4. Assume $\omega = \alpha + 2\beta$. The Weyl reflection corresponding to the root β conjugates 6.2(11c) to a subgroup H' with $\omega = \alpha$ (of type 6.2(7b) with $\mathfrak{h}' = \mathfrak{z}' \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$).

Thus, we may now assume $N_A(U) = A$. If $\sigma \neq \beta$, then the Weyl reflection corresponding to the root β conjugates H to a subgroup with $\omega = \alpha$. Now assume $\sigma = \beta$. The Weyl reflection corresponding to the root α does not change ω, but conjugates H to a subgroup H_1 with $\sigma = \alpha + 2\beta$. Then (as we already observed) the Weyl reflection corresponding to the root β conjugates H_1 to a subgroup with $\omega = \alpha$. ∎

Lemma 7.7. *Assume G is a connected, almost simple, linear, real Lie group of real rank two. Let H be a closed, connected, nontrivial subgroup of AN, such that H is compatible with A, and $H \neq (H \cap A)(H \cap N)$. We use the notation of [14, Lem. 2.4]: $T = \ker\omega$, $H = T \ltimes U$, $\psi\colon T \to U_\omega U_{2\omega}$, and $H = \{a\psi(a)\} \ltimes U$.*

If $\mathfrak{u} \cap (\mathfrak{u}_\omega + \mathfrak{u}_{2\omega}) \neq 0$, then H is a Cartan-decomposition subgroup.

Proof. By passing to a subgroup of H, there is no harm in assuming $\mathfrak{u} \subset \mathfrak{u}_\omega + \mathfrak{u}_{2\omega}$. We use the notation of the proof of [14, Prop. 3.17]. For each $a \in T$, clearly $\mu_{MA}\bigl(a\psi(a)U\bigr) \supset \mu_{MA}\bigl(a\psi(a)\bigr)A_\omega^+$, so $\mu_{MA}(H) \supset \mu_{MA}\bigl(\{a\psi(a)\}\bigr)A_\omega^+$. Beause $\mu_{MA}(T) = T$ is a line perpendicular to A_ω, and $\mu_{MA}\bigl(a\psi(a)\bigr)$ is logarithmically close to this line, it is clear that $\mu_{MA}\bigl(a\psi(a)\bigr)A_\omega^+$ contains all but a bounded subset of the region $\mathcal{C}$. Therefore $\mu(H)$ contains all but a bounded subset of A^+, so H is a Cartan-decomposition subgroup. ∎

reference	Cartan projection	maximum dimension
6.1(1)	$\rho(h) \asymp h$	1
6.1(2)	$\mu(H) \approx \bigl[\|h\|, \|h\|^{3/2}\bigr]$	$2n-3$
6.1(2)*	$\rho(h) \asymp \|h\|^{3/2}$	1
6.1(3a)	$\mu(H) \approx \bigl[\|h\|, \|h\|^{3/2}\bigr]$	$2n-3$
6.1(3a)*	$\rho(h) \asymp \|h\|^{3/2}$	1
6.1(3b)	$\rho(h) \asymp h$	$2n-3$
6.1(4)	$\rho(h) \asymp h$	$2n-1$
6.1(5)	$\mu(H) \approx \bigl[\|h\|, \|h\|^{4/3}\bigr]$	$2n-3$
6.1(5)*	$\rho(h) \asymp \|h\|^{4/3}$	1
6.1(6)	$\rho(h) \asymp \|h\|^2$	$\begin{cases} 2n-1 & n \text{ even} \\ 2n-3 & n \text{ odd} \end{cases}$
6.1(7)	$\mu(H) \approx \bigl[\|h\|^{3/2}, \|h\|^2\bigr]$	$\begin{cases} n+1 & n \geq 4 \\ 3 & n = 3 \end{cases}$
6.1(8)	$\rho(h) \asymp \|h\|^{3/2}$	$\begin{cases} 3 & n \geq 4 \\ 2 & n = 3 \end{cases}$
6.1(9)	$\mu(H) \approx \bigl[\|h\|, \|h\|^{3/2}\bigr]$	2
6.1(10)	$\rho(h) \asymp \|h\|^2$	2
6.1(11)	$\mu(H) \approx \bigl[\|h\|^{5/4}, \|h\|^2\bigr]$	2

Table 1: The subgroups of N that are not Cartan-decomposition subgroups.

Lemma 7.8. (cf. [14, Prop. 3.16(3)]) *Assume that $G = \mathrm{SU}(2,n)$, and let H be a nontrivial one-parameter subgroup of AN, such that H is compatible with A, but $H \neq (H \cap A)(H \cap N)$.*

Then there is a ray R in A^+, a ray R' in A that is perpendicular to R, and a positive number k, such that

$$\mu(H) \approx \{\, rs \mid r \in R,\ s \in R',\ \|s\| = (\log \|r\|)^k \,\}.$$

8 Maximum dimensions of the subgroups

For convenience of reference, Tables 1, 2 and 3 list the (approximate) Cartan projection of each subgroup of AN that is not a Cartan-decomposition subgroup. The maximum possible dimension for a subgroup of each type is also listed. (These dimensions are used in applications to the existence of tessellations.)

Remark 8.1. Here are brief justifications of the dimensions listed in Tables 1, 2 and 3.

6.1(1) By assumption, we have $\dim H = 1$.

6.1(2) Let $p\colon \mathfrak{h} \to \mathfrak{u}_{\alpha+\beta}$ be the natural projection. Then $\ker p = \mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, so

$$\dim \mathfrak{h} \leq (\dim \mathfrak{u}_{\alpha+\beta}) + (\dim \mathfrak{u}_{2\alpha+2\beta}) = 2(n-2) + 1 = 2n-3.$$

6.1(3) We may assume $\lambda = 0$. Then $\mathfrak{h} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$. So

$$\dim \mathfrak{h} \leq (\dim \mathfrak{u}_\beta) + (\dim \mathfrak{u}_{2\beta}) = 2(n-2) + 1 = 2n-3.$$

reference	Cartan projection	maximum dimension
7.4(1a)	$\mu(H) \approx [\|h\|, \|h\|^s]$	2
7.4(2)	$\mu(H) \approx [\|h\|, \|h\|^{3/2}]$	$2n-2$
7.4(2)*	$\rho(h) \asymp \|h\|^{3/2}$	2
7.4(4a)	$\mu(H) \approx [\|h\|, \|h\|^s]$	$2n-2$
7.4(4(b)i)	$\rho(h) \asymp h$	$2n-2$
7.4(4(b)ii)	$\mu(H) \approx [\|h\|, \|h\|^{3/2}]$	$2n-2$
7.4(4(b)ii)*	$\rho(h) \asymp \|h\|^{3/2}$	2
7.4(5a)	$\mu(H) \approx [\|h\|, \|h\|^s]$	$2n-2$
7.4(5c)	$\rho(h) \asymp h$	$2n$
7.4(6)	$\mu(H) \approx [\|h\|, \|h\|^{4/3}]$	$2n-2$
7.4(6)*	$\rho(h) \asymp \|h\|^{4/3}$	2
7.4(7a)	$\mu(H) \approx [\|h\|^s, \|h\|^2]$	3
7.4(7b)	$\rho(h) \asymp \|h\|^2$	$\begin{cases} 2n & n \text{ even} \\ 2n-2 & n \text{ odd} \end{cases}$
7.4(8a)	$\mu(H) \approx [\|h\|^{3/2}, \|h\|^2]$	4
7.4(8b)	$\mu(H) \approx [\|h\|^{3/2}, \|h\|^2]$	4
7.4(9)	$\rho(h) \asymp \|h\|^{3/2}$	$\begin{cases} 4 & n \geq 4 \\ 3 & n = 3 \end{cases}$
7.4(10)	$\mu(H) \approx [\|h\|, \|h\|^{3/2}]$	3
7.4(11a)	$\mu(H) \approx [\|h\|^s, \|h\|^2]$	3
7.4(11c)	$\rho(h) \asymp \|h\|^2$	3

Table 2: The subgroups of AN that are not Cartan-decomposition subgroups, and are a nontrivial semidirect product $T \ltimes U$.

reference	Cartan projection	maximum dimension
7.6(1)	$\mu(H) \approx [\|h\|, \|h\|^2/(\log\|h\|)]$	$2n-2$
7.6(2)	$\mu(H) \approx [\|h\|^2/(\log\|h\|)^2, \|h\|^2]$	2
7.6(3) $(r=1)$	$\mu(H) \approx [\|h\|(\log\|h\|)^{1/2}, \|h\|^2]$	3
7.6(3) $(r=2)$	$\mu(H) \approx [\|h\|(\log\|h\|), \|h\|^2]$	3
7.6(4) $(r=1)$	$\mu(H) \approx [\|h\|, \|h\|(\log\|h\|)]$	$2n-2$
7.6(4) $(r=2)$	$\mu(H) \approx [\|h\|, \|h\|(\log\|h\|)^2]$	$2n-3$
7.8	$\rho(h) \asymp \|h\|^s(\log\|h\|)^{\pm k}$	1

Table 3: The subgroups of AN that are not Cartan-decomposition subgroups **and** are not a semidirect product of a torus and a unipotent subgroup.

It is easy to construct an algebra of this dimension, with or without an element u as described in (3a).

6.1(4) Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. Because $\phi\overline{\eta}$ is a form of signature $(2,2)$ on $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+2\beta}$, we know that $\dim\big(V \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+2\beta})\big) \leq 2$. Thus we have

$$\begin{aligned}\dim \mathfrak{h} &\leq \dim V + \dim \mathfrak{u}_{2\alpha+2\beta} \leq \big(\dim \mathfrak{u}_{\alpha+\beta} + 2\big) + \dim \mathfrak{u}_{2\alpha+2\beta} \\ &= \big(2(n-2)+2\big) + 1 = 2n-1.\end{aligned}$$

6.1(5) Consider $p\colon \mathfrak{h} \to \mathfrak{u}_\alpha$. Because $\mathfrak{z} = 0$, we have $\dim \ker p \leq \dim \mathfrak{u}_{\alpha+\beta} = 2n-4$. Because $p(\mathfrak{h}) \subset \mathbb{R}\phi_0$, we have $\dim p(\mathfrak{h}) \leq 1$. Thus, $\dim \mathfrak{h} \leq 2n-3$.

6.1(6) See Lemma 8.2 below.

6.1(7) See Lemma 8.3 below.

6.1(8) See Lemma 8.4 below.

6.1(9), 6.1(10), 6.1(11) are obvious from the statements.

7.4(1a) Because $\dim \mathfrak{u} = 1$, we have $\dim \mathfrak{h} = \dim \mathfrak{t} + \dim \mathfrak{u} = 2$.

7.4(2) The kernel of the projection from $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+\beta}$ is $\mathfrak{z}$, so $\dim \mathfrak{h} = 1 + \dim U \leq 1 + (1 + \dim \mathfrak{u}_{\alpha+\beta}) = 2n-2$.

7.4(4) $\dim \mathfrak{h} = 1 + \dim \mathfrak{u} \leq 1 + \big(\dim \mathfrak{u}_\beta + \dim \mathfrak{z}\big) = 2n-2$.

7.4(5a) $\dim \mathfrak{h} \leq \dim \mathfrak{t} + \dim \mathfrak{u}_{\alpha+\beta} + \dim \mathfrak{z} \leq 1 + (2n-4) + 1 = 2n-2$.

7.4(5c) Add 1 (the dimension of T) to the bound in 6.1(4).

7.4(6) Add 1 (the dimension of T) to the bound in 6.1(5).

7.4(7a) $\dim \mathfrak{h} \leq \dim \mathfrak{t} + \dim \mathfrak{u}_{\alpha+2\beta} = 1 + 2 = 3$.

7.4(7b) Add 1 (the dimension of T) to the bound in 6.1(6).

7.4(8a) Because $\mathsf{y}_u \neq 0$ for every nonzero $u \in \mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})$, we have $\dim\big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) \leq 1$. Therefore $\dim \mathfrak{h} \leq \dim \mathfrak{t} + 1 + \dim \mathfrak{u}_{\alpha+2\beta} = 4$.

7.4(8b) This is conjugate to 7.4(8a), via the Weyl reflection corresponding to the root α.

7.4(9) Add 1 (the dimension of T) to the bound in 6.1(8). (To achieve this bound for $n \geq 4$, choose $u, \tilde{u} \in \mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)$ in the proof of Lemma 8.4.

7.4(10) and 7.4(11) are obvious from the statements.

7.6(1) $\dim \mathfrak{h} \leq 1 + \dim(\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}) = 2n-2$.

7.6(2) Because $\psi(T)$ normalizes (hence centralizes) U, the subgroup U cannot be all of $U_{\alpha+2\beta}$, so $\dim U \leq 1$. Therefore $\dim H = 1 + \dim U \leq 2$.

7.6(3) $\dim \mathfrak{h} \leq 1 + \dim \mathfrak{u}_{\alpha+2\beta} = 3$.

7.6(4) Because $\psi(T)$ normalizes (hence centralizes) U, the projection of $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+\beta}$ cannot be all of $\mathfrak{u}_{\alpha+\beta}$ if $\psi(T) \not\subset U_{2\beta}$, that is, if $r = 2$. Therefore $\dim U \leq \dim(\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}) - (r-1) = 2n-2-r$. Therefore $\dim \mathfrak{h} = 1 + \dim U \leq 2n-1-r$.

Lemma 8.2. *The maximum dimension of a subalgebra of type* 6.1(6) *is as stated in Table* 1.

Proof. We begin by showing that $\dim \mathfrak{h} \leq 2n-1$ (cf. [16, Lem. 5.8]). Let V be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. Because $\dim \mathfrak{z} \leq 3$, we just need to show that $\dim V \leq 2n-4$. Because V does not intersect $\mathfrak{u}_\beta$ (or $\mathfrak{u}_{\alpha+\beta}$, either, for that matter), and $\mathfrak{u}_\beta$ has codimension $2n-4$ in $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$, this is immediate.

When n is even, there is a subgroup of dimension $2n-1$. (For example, the N subgroup of $\mathrm{Sp}(1,n/2)$. More general examples are constructed in [15, §4].)

Let us show that if n is odd, then $\dim H \leq 2n-3$. (Our proof is topological; we do not know an algebraic proof.) Suppose that $\dim H \geq 2n-2$ (this will lead to a contradiction). Because $\dim \mathfrak{z} \leq 3$, we have $\dim \mathfrak{h}/\mathfrak{z} \geq 2n-5$. Thus, there is a $(2n-5)$-dimensional real subspace X of $\mathbb{C}^{n-2}$ and a real linear transformation $T\colon X \to \mathbb{C}^{n-2}$, such that x and Tx are linearly independent over $\mathbb{C}$, for every nonzero $x \in X$ (cf. [16, Cor. 5.9]). Thus, if we define $U\colon X \to \mathbb{C}^{n-2}$ by $Ux = ix$; then x, Tx, and Ux are linearly independent over $\mathbb{R}$, for every nonzero $x \in X$. Thus (writing $n = 2k+3$): there is a $(4k+1)$-dimensional real subspace X of $\mathbb{R}^{4k+2}$ and real linear transformations $T, U\colon X \to \mathbb{R}^{4k+2}$, such that x, Tx, and Ux are linearly independent over $\mathbb{R}$, for every nonzero $x \in X$. There is no harm in assuming $X = \mathbb{R}^{4k+1}$ (under its natural embedding in $\mathbb{R}^{4k+2}$).

Let $E = (S^{4k} \times \mathbb{R}^{4k+2})/\!\sim$, where $(x,v) \sim (-x,-v)$, and define a continuous map $\zeta\colon E \to \mathbb{R}P^{4k}$ by $\zeta(x,v) = [x]$, so (E,ζ) is a vector bundle over $\mathbb{R}P^{4k}$. Then $(E,\zeta) \cong \tau \oplus \epsilon^1 \oplus \gamma^1_{4k}$, where τ is the tangent bundle of $\mathbb{R}P^{4k}$, ϵ^1 is a trivial line bundle, and γ^1_{4k} is the canonical bundle of $\mathbb{R}P^{4k}$. (To see this, note that the subbundle

$$\{\,(x,v) \in S^{4k} \times \mathbb{R}^{4k+1} \mid v \perp x\,\}/\!\sim$$

is the total space of τ [13, pf. of Lem. 4.4, pp. 43–44], the subbundle

$$\{\,(x,v) \in S^{4k} \times \mathbb{R}^{4k+1} \mid v \in \mathbb{R}x\,\}/\!\sim$$

has the obvious section $x \mapsto (x,x)$, and the subbundle $\big(S^{4k} \times (0 \times \mathbb{R})\big)/\!\sim$ is isomorphic to γ^1_{4k} via the bundle map $\big(x,(0,t)\big) \mapsto (x,tx)$.) Therefore, letting a be a generator of the cohomology ring $H^*(\mathbb{R}P^{4k};\mathbb{Z}_2)$, we see that the total Stiefel-Whitney class of (E,ζ) is $w = (1+a)^{4k+1}(1)(1+a) = (1+a)^{4k+2}$ [13, Eg. 2, p. 43, and Thm. 4.5, p. 45], so

$$w_{(4k+2)-3+1} = w_{4k} = \binom{4k+2}{4k} a^{4k} = (2k+1)(4k+1)a^{4k} \neq 0$$

(because $(2k+1)(4k+1)$ is odd). Therefore, there do not exist three pointwise linearly independent sections of (E,ζ) [13, Prop. 4, p. 39].

Any linear transformation $Q\colon \mathbb{R}^{4k+1} \to \mathbb{R}^{4k+2}$ induces a continuous function $\hat{Q}\colon S^{4k} \to \mathbb{R}^{4k+2}$, such that $\hat{Q}(-x) = -\hat{Q}(x)$ for all $x \in S^{4k}$; that is, a section of (E,ζ). Thus, Id, T, and U each define a section of (E,ζ). Furthermore, these three sections are pointwise linearly independent, because x, Tx, and Ux are linearly independent over $\mathbb{R}$, for every $x \in S^{4k}$. This contradicts the conclusion of the preceding paragraph. ∎

Lemma 8.3. *The maximum dimension of a subalgebra of type* 6.1(7) *is as stated in Table* 1.

Proof. Replacing H by a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume $x_v = 0$. Therefore $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$. (Thus, in particular, we have $\dim \mathfrak{z} \leq 2$.)

For the projection $p\colon \mathfrak{h} \to \mathfrak{u}_{\alpha+\beta}$, we have $\ker p = \mathbb{R}v + \mathfrak{z}$. (There cannot exist a linearly independent v'; otherwise, replacing v' by some linear combination with v, we could assume $\mathsf{x}_{v'} = 0$, which is impossible.) Thus, $\dim \ker p \leq 3$.

Because $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$, $p(\mathfrak{h})$ must be a totally isotropic subspace for the symplectic form $\operatorname{Im}(x\tilde{x}^\dagger)$, so $\dim p(\mathfrak{h}) \le n-2$. Therefore $\dim \mathfrak{h} \le (n-2)+3 = n+1$.

For $n \ge 4$, here is an example that achieves this bound:

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 & x_1 & x_2 & \cdots & x_{n-2} & \eta & i\mathsf{x} \\ 0 & 0 & i\mathsf{x} & x_1 & \cdots & x_{n-3} & ix_{n-2} & -\overline{\eta} \\ & & & \cdots & & & & \end{pmatrix} \;\middle|\; \begin{array}{c} \mathsf{x}, x_1, \ldots, x_{n-2} \in \mathbb{R}, \\ \eta \in \mathbb{C} \end{array} \right\}.$$

For $v \in \mathfrak{h}$, we claim that $\dim_{\mathbb{C}}\langle x_v, y_v \rangle = 1$ only if either $x_v = 0$ or $y_v = 0$. (In either case, it is clear from the definition of $\mathfrak{h}$ that either $\mathsf{x}_u|y_u|^2 \ne 0$ or $\mathsf{y}_u|x_u|^2 \ne 0$, respectively.) Suppose $\dim_{\mathbb{C}}\langle x_v, y_v \rangle = 1$, with $x_v \ne 0$ and $y_v \ne 0$. There is some nonzero $\lambda \in \mathbb{C}$, such that $y_v = \lambda x_v$. We must have $x_1 \ne 0$. (Otherwise, let $i \in \{1, 2, \ldots, n-2\}$ be minimal with $x_i \ne 0$. Then $x_{i-1} = y_i = \lambda x_i \ne 0$, contradicting the minimality of i.) Because $y_1 = i\mathsf{x}$ is pure imaginary, but x_1 is real, we see that λ is pure imaginary. On the other hand, $y_2 = x_1$ is real (and nonzero), and x_2 is also real, so λ is real. Because $\lambda \ne 0$, this is a contradiction.

Now let $n = 3$, and suppose $\dim \mathfrak{h} = 4$. (This will lead to a contradiction.) Because equality is attained in the proof above, we must have $\dim p(\mathfrak{h}) = n-2 = 1$ and $\dim \mathfrak{z} = 2$. In particular, there exists $w \in \mathfrak{h}$ with $x_w \ne 0$. For $t \in \mathbb{R}$, let $w_t = w + tv$. Then

$$\mathsf{x}_{w_t}|y_{w_t}|^2 + \mathsf{y}_{w_t}|x_{w_t}|^2 + 2\operatorname{Im}(x_{w_t} y_{w_t}^\dagger \eta_{w_t}) = t^3 \mathsf{x}_v |y_v|^2 + O(t^2) \to \begin{cases} +\mathsf{x}_v \infty & \text{as } t \to \infty \\ -\mathsf{x}_v \infty & \text{as } t \to -\infty. \end{cases}$$

Thus, this expression changes sign, so it must vanish for some t. This is a contradiction, because $\dim_{\mathbb{C}}\langle x_{w_t}, y_{w_t} \rangle = 1$ for every t. ∎

Lemma 8.4. *The maximum dimension of a subalgebra of type* 6.1(8) *is as stated in Table* 1.

Proof. For $n \ge 4$, here is the construction of 3-dimensional subalgebras of $\mathfrak{n}$ of this type. Let $\phi = 1$ and $\tilde{\phi} = i$. Choose $y, \tilde{y}, x, \tilde{x} \in \mathbb{C}^{n-2}$, $\eta, \tilde{\eta} \in \mathbb{C}$, and $\mathsf{x}, \tilde{\mathsf{x}} \in \mathbb{R}$, such that

$$|y|^2 = |\tilde{y}|^2 = 3iy\tilde{y}^\dagger \ne 0. \tag{8.1}$$

Now, choose $\mathsf{y}, \tilde{\mathsf{y}} \in \mathbb{R}$, such that

$$\operatorname{Im}(\tilde{y}x^\dagger - iyx^\dagger + \tilde{y}x^\dagger - y\tilde{x}^\dagger + i\tilde{\mathsf{y}}) = 0 \tag{8.2}$$

and

$$\operatorname{Im}(\tilde{y}\tilde{x}^\dagger - iy\tilde{x}^\dagger + i\tilde{y}x^\dagger - iy\tilde{x}^\dagger + i\mathsf{y}) = 0. \tag{8.3}$$

Define $u, \tilde{u}$ as in Eq. (2.3), and let $v = [u, \tilde{u}]$. Then $\mathsf{y}_v \ne 0$ and $x_v \ne 0$, but, from Eq. (8.1), Eq. (8.2) and Eq. (8.3), we have $[v, u] = [v, \tilde{u}] = 0$. Thus, we may let $\mathfrak{h}$ be the subalgebra generated by u and $\tilde{u}$. (So $\{u, \tilde{u}, v\}$ is a basis of $\mathfrak{h}$ over $\mathbb{R}$.)

Note that, because $|y\tilde{y}^\dagger| = |y|^2/3 \ne |y|^2$, we know that y and $\tilde{y}$ must be linearly independent over $\mathbb{C}$. Thus, these 3-dimensional examples do not exist when $n = 3$. ∎

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Received July 27, 2000
and in final form February 2, 2001