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On Kazhdan's Property (T) for $\mathrm{Sp}_2(\mathbf{k})$

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Abstract. The aim of this note is to give a new and elementary proof of Kazhdan's Property (T) for $\mathrm{Sp}_2(\mathbf{k})$, the symplectic group on 4 variables, for any local field $\mathbf{k}$. The crucial step is the proof that the Dirac measure δ_0 at 0 is the unique mean on the Borel subsets of the second symmetric power $S^2(\mathbf{k}^2)$ of $\mathbf{k}^2$ which is invariant under the natural action of $\mathrm{SL}_2(\mathbf{k})$. In the case where $\mathbf{k}$ has characteristic 2, we observe that this is no longer true if $S^2(\mathbf{k}^2)$ is replaced by its dual, the space of the symmetric bilinear forms on $\mathbf{k}^2$.

1. Introduction

A locally compact group G has *Kazhdan's Property (T)* ([5]) if whenever a strongly continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ has almost invariant vectors, then it actually has a nonzero fixed vector. Recall that π has almost invariant vectors if, for any $\varepsilon > 0$ and any compact subset K of G, there exists a unit vector ξ in $\mathcal{H}$ with $\|\pi(g)\xi - \xi\| < \varepsilon$ for all g in K. Kazhdan's Property (T) is a powerful tool and has remarkable applications. For an excellent account, see [3].

Let $\mathbf{k}$ be a local field (that is, a non-discrete locally compact field), and let $\mathbf{G}$ be a connected almost simple algebraic $\mathbf{k}$-group with $\mathbf{k}$-rank ≥ 2. Then, as is well known, $\mathbf{G}(\mathbf{k})$, the group of the $\mathbf{k}$-rational points in $\mathbf{G}$, has Property (T). This central result is quickly deduced, using Howe-Moore's theorem and root theory, from the fact that $\mathrm{SL}_3(\mathbf{k})$ and $\mathrm{Sp}_2(\mathbf{k})$, the symplectic group on 4 variables, have Property (T) (see [6, Chap. III, (5.3) Theorem]). The aim of this note is to give an elementary proof of Property (T) for $\mathrm{Sp}_2(\mathbf{k})$. A proof in this spirit was given in [1] for $\mathrm{SL}_3(\mathbf{k})$. Property (T) for $\mathrm{Sp}_2(\mathbf{k})$ was first established in [2] and [8].

Theorem 1.1. ([2], [8]) $\mathrm{Sp}_2(\mathbf{k})$ *has Property (T), for any local field* $\mathbf{k}$.

The group $\mathrm{Sp}_2(\mathbf{k})$ contains a copy of the semi-direct product $\mathrm{SL}_2(\mathbf{k}) \ltimes S^{2*}(\mathbf{k}^2)$, for the standard action of $\mathrm{SL}_2(\mathbf{k})$ on the space $S^{2*}(\mathbf{k}^2)$ of the symmetric bilinear forms on $\mathbf{k}^2$ (see Section 4).

Theorem 1.1 is a consequence of the following fact.

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Theorem 1.2. *The pair* $(\mathrm{SL}_2(\mathbf{k}) \ltimes S^{2*}(\mathbf{k}^2), S^{2*}(\mathbf{k}^2))$ *has the relative Property (T), for any local field* $\mathbf{k}$.

Recall that a pair (G, H) of a locally compact group G and a closed subgroup H has the relative Property (T), if whenever a unitary representation π of G on a Hilbert space $\mathcal{H}$ has almost invariant vectors, then it has a nonzero H-fixed vector.

Our proof of Theorem 1.2 is based on the next result and avoids the usual Mackey type analysis of the irreducible unitary representations of $\mathrm{SL}_2(\mathbf{k}) \ltimes S^{2*}(\mathbf{k}^2)$, used in [6, Chap. III, (5.1) Lemma], in [10, 7.4.2 Theorem] or in [2] and [8] (see also [3] for a sketch of proof of Theorem 1.2 based on the so-called Furstenberg Lemma). The dual group of the abelian group $S^{2*}(\mathbf{k}^2)$ may be identified with the second symmetric power $S^2(\mathbf{k}^2)$ of $\mathbf{k}^2$.

Theorem 1.3. *Let* $\mathbf{k}$ *be a local field. The Dirac measure* δ_0 *at* 0 *is the unique mean on the Borel subsets of* $S^2(\mathbf{k}^2)$ *which is invariant under the natural action of* $\mathrm{SL}_2(\mathbf{k})$ *on* $S^2(\mathbf{k}^2)$.

Recall that a *mean* on a ring of subsets of a set X is a finitely additive positive measure m on this ring with total mass 1 (that is, with $m(X) = 1$). Theorem 1.3 has a completely elementary proof. Theorem 1.2 is a consequence of Theorem 1.3, via the following general fact which was observed in [7, Theorem 5.5] and for which we give a short and different proof.

Proposition 1.4. *([7]) Let* G *be a locally compact group and* N *a normal abelian subgroup. Let* $\widehat{N}$ *be the dual group of* N. *Assume that the Dirac measure at the trivial character of* N *is the unique mean on the Borel subsets of* $\widehat{N}$ *which is invariant under the dual action of* G *on* $\widehat{N}$ *given by conjugation. Then* (G, N) *has the relative Property (T).*

The two natural representations of $\mathrm{SL}_2(\mathbf{k})$ on $S^2(\mathbf{k}^2)$ and $S^{2*}(\mathbf{k}^2)$ (which are contragredient to each other) are equivalent representations, but only if $\mathrm{char}(\mathbf{k})$, the characteristic of $\mathbf{k}$, is different from 2. In fact, in case $\mathrm{char}(\mathbf{k}) = 2$, one has the following somewhat surprising result.

Theorem 1.5. *The pair* $(\mathrm{SL}_2(\mathbf{k}) \ltimes S^2(\mathbf{k}^2), S^2(\mathbf{k}^2))$ *does* not *have the relative Property (T) if* $\mathbf{k}$ *is a local field of characteristic 2.*

The paper is organized as follows. In Section 2, we give the proof of Theorem 1.3. Proposition 1.4 and Theorem 1.2 are proved in Section 3 and Theorem 1.1 and Theorem 1.5 in Section 4.

2. Proof of Theorem 1.3

Denote by ρ the standard representation of $\mathrm{SL}_2(\mathbf{k})$ on the second symmetric power $S^2(\mathbf{k}^2)$ of $\mathbf{k}^2$. Fix a basis $\{e_1, e_2\}$ of $\mathbf{k}^2$. Identify $S^2(\mathbf{k}^2)$ with $\mathbf{k}^3$ by means of the basis $\{e_1^2, e_1 \otimes e_2, e_2^2\}$. The matrix of $\rho(g)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{k})$ is then

$$\rho(g) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

So, for $u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, one has

$$\rho(u_b) = \begin{pmatrix} 1 & b & b^2 \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall b \in \mathbf{k}.$$

Hence,

$$\rho(u_b) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + by + b^2 z \\ y + 2bz \\ z \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3.$$

Assume first that $\mathrm{char}(\mathbf{k}) \neq 2$. For every $c \in \mathbf{R}$, $c > 0$, consider the following Borel subset of $\mathbf{k}^3$

$$\Omega_c = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 : |z| > c\,|y| \right\},$$

where $|\cdot|$ denotes the absolute value of $\mathbf{k}$. One has, for every $b \in \mathbf{k}$ and every $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Omega_c$,

$$\left(|2b| - \frac{1}{c} \right) |z| < |y + 2bz| < \left(|2b| + \frac{1}{c} \right) |z| \,. \tag{1}$$

Let μ be an invariant mean on $\mathcal{B}(\mathbf{k}^3)$, the Borel subsets of $\mathbf{k}^3$. For every fixed $c > 0$, choose a sequence $(b_n)_n$ in $\mathbf{k}$ such that

$$|b_{n+1}| > |b_n| + \frac{2}{|2|\,c}, \quad \forall n \in \mathbf{N}.$$

Then, by (1),

$$\rho(u_{b_n})\Omega_c \cap \rho(u_{b_m})\Omega_c = \emptyset,$$

for all $n \neq m$. Since $\mu(\mathbf{k}^3) < \infty$, this implies that

$$\mu\left(\Omega_c\right) = 0, \quad \forall c > 0.$$

For $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2\left(\mathbf{k}\right)$, one has

$$\rho(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

So,

$$\rho(\omega) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3,$$

and hence,

$$\rho(\omega)\Omega_c = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 : |x| > c\,|y| \right\}$$

Observe that $\mu\,(\rho(\omega)\Omega_c) = 0$, by invariance of μ. For $r = \begin{pmatrix} 2^{-1} & 2^{-1} \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}_2\,(\mathbf{k})$, one has

$$\rho(r) = \begin{pmatrix} 4^{-1} & 4^{-1} & 4^{-1} \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

So,

$$\rho(r) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4^{-1}\,(x+y+z) \\ -x+z \\ x-y+z \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3.$$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3$ with $|y| \geq |x|$ and $|y| \geq |z|$ and $(x,z) \neq (0,0)$. Then

$$\begin{aligned} |x+y+z| + |x-y+z| &= |x+y+z| + |-x+y-z| \geq |2y| = |2|\,|y| \\ &\geq \frac{|2|}{2}\,(|x|+|z|) \\ &> \frac{|2|}{4}\,(|-x+z|)\,. \end{aligned}$$

Hence, for $c_0 = |2|\,/4\,(|4|+1)$, one has

$$\text{either } \left|4^{-1}\,(x+y+z)\right| > c_0\,(|-x+z|) \text{ or } |x-y+z| > c_0\,(|-x+z|)\,.$$

Therefore,

$$\rho(r) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Omega_{c_0} \cup \rho(\omega)\Omega_{c_0},$$

for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 \setminus (\Omega_1 \cup \rho(\omega)\Omega_1)$ with $(x,z) \neq (0,0)$. Since

$$\rho(r) \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ -y \end{pmatrix},$$

this also holds if $(x,z) = (0,0)$ and $y \neq 0$. The above shows that

$$\mathbf{k}^3 \setminus \{0\} = \Omega_1 \cup \rho(\omega)\Omega_1 \cup \rho(r^{-1})\Omega_{c_0} \cup \rho(r^{-1}\omega)\Omega_{c_0}.$$

Since

$$\mu\,(\Omega_1) = \mu\left(\rho(r^{-1})\Omega_{c_0}\right) = \mu\left(\rho(r^{-1}\omega)\Omega_{c_0}\right) = \mu\,(\rho(\omega)\Omega_1) = 0,$$

this implies that $\mu\,(\mathbf{k}^3 \setminus \{0\}) = 0$ and concludes the proof of the theorem in the case where $\mathrm{char}(\mathbf{k}) \neq 2$.

Assume now that $\mathrm{char}(\mathbf{k}) = 2$. Let

$$\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 \setminus \{0\} : |x| \leq |z| \quad \text{and} \quad |y| \leq |z| \right\}.$$

One has, with the above notation,

$$\rho(u_b) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + by + b^2 z \\ y \\ z \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3$$

and $|x| < |b^2 z|$ and $|by| < |b^2 z|$ if $|b| > 1$. Choose a sequence $(b_n)_n$ in $\mathbf{k}$ with $|b_{n+1}| > |b_n| > 1$. Then

$$\rho(u_{b_n})\Omega \cap \rho(u_{b_m})\Omega = \emptyset \quad \forall n \neq m,$$

since $|x + by + b^2 z| = |b^2 z|$, if $|x| < |b^2 z|$ and $|by| < |b^2 z|$. Indeed, the absolute value on $\mathbf{k}$ being non-archimedean satisfies

$$|x + by| \leq \max\{|x|, |by|\} < \left|b^2 z\right|$$

and, hence, $|x + by + b^2 z| \leq |b^2 z|$ as well as

$$\left|b^2 z\right| \leq \max\left\{\left|x + by + b^2 z\right|, |x + by|\right\} \leq \left|b^2 z\right|$$

Let μ be an invariant mean on $\mathcal{B}(\mathbf{k}^3)$. By the above, $\mu(\Omega) = 0$.

Clearly,

$$\rho(\omega)\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 \setminus \{0\} : |z| \leq |x| \quad \text{and} \quad |y| \leq |x| \right\}.$$

So, $\mathbf{k}^3 \setminus \{0\} = \Omega \cup \rho(\omega)\Omega \cup \Omega'$, where

$$\Omega' = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{k}^3 : |x| < |y| \quad \text{and} \quad |z| < |y| \right\}.$$

Moreover,

$$\rho(u_1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ y \\ z \end{pmatrix},$$

and $|x + y + z| = |y|$ when $|x| < |y|$ and $|z| < |y|$. Hence, $\rho(u_1)\Omega'$ is contained in $\rho(\omega)\Omega$. Therefore, $\mu(\Omega') = 0$. This implies $\mu(\mathbf{k}^3 \setminus \{0\}) = 0$. So, μ has to be δ_0.

3. Proofs of Proposition 1.4 and Theorem 1.2

The proof of Proposition 1.4 is based on the following elementary lemma from [1, Lemma 2.1]

Lemma 3.1. *Let G be a locally compact group and let $(\pi, \mathcal{H})$ be a unitary representation of G with almost invariant vectors. Then there exists an* $\operatorname{Ad} G$*-invariant state on the C^*-algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, that is, a positive linear form φ on $\mathcal{L}(\mathcal{H})$ with $\varphi(\mathrm{Id}) = 1$ and $\varphi(\pi(x)T\pi(x)^{-1}) = \varphi(T)$ for all $x \in G$ and $T \in \mathcal{L}(\mathcal{H})$.*

Proof of Proposition 1.4. Let $(\pi, \mathcal{H})$ be a unitary representation of G on $\mathcal{H}$, with almost invariant vectors. Let

$$P : \mathcal{B}(\widehat{N}) \to \mathcal{L}(\mathcal{H}), \quad E \mapsto P(E)$$

be the projection valued measure on $\widehat{N}$ associated with the unitary representation $\pi|_N$ of the abelian group N. One has

$$\pi(g)P(E)\pi(g)^{-1} = P(gE), \quad \forall E \in \mathcal{B}(\widehat{N}), g \in G,$$

where

$$g\lambda(n) = \lambda(g^{-1}ng), \quad n \in N, \lambda \in \widehat{N}$$

is the dual action of $g \in G$ on $\widehat{N}$. By the above lemma, there exists an $\operatorname{Ad} G$-invariant state φ on the C*-algebra of all bounded linear operators on $\mathcal{H}$. Define

$$\mu(E) = \varphi(P(E)), \quad \forall E \in \mathcal{B}(\widehat{N}).$$

Then μ is a G-invariant mean on $\mathcal{B}(\widehat{N})$. So, μ is the Dirac measure at the trivial character λ_0 of N. Hence, $P(\{\lambda_0\}) \neq 0$. This shows that $\pi|_N$ has a non-zero fixed vector and (G, N) has the relative Property (T). ∎

Proof of Theorem 1.2. The dual vector space of the space $S^{2*}(\mathbf{k}^2)$ of the bilinear forms on $\mathbf{k}^2$, may be identified with the second symmetric power $S^2(\mathbf{k}^2)$ of $\mathbf{k}^2$, by means of the duality formula

$$(B, \sum_i x_i \otimes y_i) \mapsto \sum_i B(x_i, y_i), \quad \forall B \in S^{2*}(\mathbf{k}^2) \text{ and } \sum_i x_i \otimes y_i \in S^2(\mathbf{k}^2).$$

Under this identification, the dual action of the natural action of $\mathrm{SL}_2(\mathbf{k})$ on $S^{2*}(\mathbf{k}^2)$ corresponds to the inverse transpose of the representation ρ of $\mathrm{SL}_2(\mathbf{k})$ on $S^2(\mathbf{k}^2)$ considered in Section 2, that is, to the representation $\widetilde{\rho}$ of $\mathrm{SL}_2(\mathbf{k})$ on $S^2(\mathbf{k}^2)$ defined by $\widetilde{\rho}(g) = \rho({}^t g^{-1})$, where ${}^t g$ is the transpose of g.

Observe that ρ and $\widetilde{\rho}$ are equivalent (for any value of $\mathrm{char}(\mathbf{k})$). Indeed, for $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, one has ${}^t g^{-1} = \omega g \omega^{-1}$ for $g \in \mathrm{SL}_2(\mathbf{k})$.

The dual group of $S^{2*}(\mathbf{k}^2)$ may be identified with $S^2(\mathbf{k}^2)$ as follows (see [9, Chap. II, section 5, Theorem 3]). Fix a non-trivial character λ of the additive group of $\mathbf{k}$. Then, for any $X \in S^2(\mathbf{k}^2)$, the formula

$$\lambda_X(Y) = \lambda(Y(X)), \quad \forall Y \in S^{2*}(\mathbf{k}^2)$$

defines a character on $S^{2*}(\mathbf{k}^2)$ and the mapping $X \to \lambda_X$ is an isomorphism between the additive group $S^2(\mathbf{k}^2)$ and the dual group of $S^{2*}(\mathbf{k}^2)$. Theorem 1.2 follows now from Theorem 1.3. ∎

4. Proofs of Theorem 1.1 and Theorem 1.5

For the proof of Theorem 1.1, we shall need the two following well-known lemmas, the first of which is called Mautner's lemma (for the elementary proofs, see [6, Chap. II, (3.2) Lemma and (3.4) Lemma]).

Lemma 4.1. *Let G be a locally compact group, $(\pi, \mathcal{H})$ a unitary representation of G and $x, y \in G$ such that $\lim_{n\to\infty} x^n y x^{-n} = e$. If $\pi(x)\xi = \xi$ for some $\xi \in \mathcal{H}$, then $\pi(y)\xi = \xi$.*

Lemma 4.2. *Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathrm{SL}_2(\mathbf{k})$. Let ξ be a vector in $\mathcal{H}$ which is invariant under the subgroup*

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{k} \right\}.$$

Then ξ is invariant under $\mathrm{SL}_2(\mathbf{k})$.

Recall that $\mathrm{Sp}_2(\mathbf{k})$ is the group of all matrices $g \in \mathrm{GL}_4(\mathbf{k})$ with ${}^t g J g = J$, where ${}^t g$ is the transpose of g and

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Proof of Theorem 1.1. Consider the following subgroups of $\mathrm{Sp}_2(\mathbf{k})$:

$$\begin{aligned} G &= \left\{ \begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in \mathrm{SL}_2(\mathbf{k}),\ A^t B = B^t A \right\} \\ N &= \left\{ \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} : {}^t B = B \right\}. \end{aligned}$$

Since

$$\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} I_2 & AB^t A \\ 0 & I_2 \end{pmatrix}$$

and $N \cong S^{2*}(\mathbf{k}^2)$, the group G is isomorphic to $\mathrm{SL}_2(\mathbf{k}) \ltimes S^{2*}(\mathbf{k}^2)$.

Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathrm{Sp}_2(\mathbf{k})$ with almost invariant vectors. Then, by Theorem 1.2, there exists a vector $\xi \neq 0$ in $\mathcal{H}$ which is fixed by N.

Consider the following subgroups of $\mathrm{Sp}_2(\mathbf{k})$:

$$H_1 = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : ad - bc = 1 \right\} \cong \mathrm{SL}_2(\mathbf{k})$$

and

$$H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} : ad - bc = 1 \right\} \cong \mathrm{SL}_2(\mathbf{k}).$$

Since ξ is fixed by the subgroups

$$N_1 = \left\{ \begin{pmatrix} I_2 & \begin{matrix} b & 0 \\ 0 & 0 \end{matrix} \\ 0 & I_2 \end{pmatrix} : b \in \mathbf{k} \right\} \text{ and } N_2 = \left\{ \begin{pmatrix} I_2 & \begin{matrix} 0 & 0 \\ 0 & b \end{matrix} \\ 0 & I_2 \end{pmatrix} : b \in \mathbf{k} \right\}$$

of H_1 and H_2, it is fixed by H_1 and H_2 (by Lemma 4.2). In particular, ξ is fixed by the matrices

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbf{k}.$$

Now choosing $\lambda \in \mathbf{k}$ with $|\lambda| < 1$ or $|\lambda^{-1}| < 1$, it follows from Mautner's Lemma above that ξ is fixed by the subgroups

$$\begin{aligned} L_1 &= \left\{ \begin{pmatrix} I_2 & 0 \\ B & I_2 \end{pmatrix} : {}^tB = B \right\}, \\ L_2 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in \mathrm{GL}_2(\mathbf{k}) \right\}. \end{aligned}$$

Since $L_1 \cup L_2 \cup N$ generates $\mathrm{Sp}_4(\mathbf{k})$ (see [4, Section 6.9]), the vector ξ is fixed under $\mathrm{Sp}_2(\mathbf{k})$. ■

Proof of Theorem 1.5. Suppose that the characteristic of $\mathbf{k}$ is 2. Set $N = S^2(\mathbf{k}^2)$. The dual group of N may be identified with the space $S^{2*}(\mathbf{k}^2)$ of symmetric bilinear forms on $\mathbf{k}^2$, with compatible $\mathrm{SL}_2(\mathbf{k})$-actions (see proof of Theorem 1.2 above). Now, $\mathrm{SL}_2(\mathbf{k})$ has non-zero fixed points in $S^{2*}(\mathbf{k}^2)$. Indeed, since $\mathrm{char}(\mathbf{k}) = 2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} {}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{k}),$$

and the bilinear forms B_s defined by the symmetric matrices $\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$, $s \in \mathbf{k}$, are fixed under $\mathrm{SL}_2(\mathbf{k})$. The characters λ_s of N corresponding to the B_s's extend to characters $\widetilde{\lambda_s}$ of $G = \mathrm{SL}_2(\mathbf{k}) \ltimes N$ defined by

$$\widetilde{\lambda_s}(A, X) = \lambda_s(X), \quad \forall A \in \mathrm{SL}_2(\mathbf{k}), X \in N.$$

Now, $\lim_{s \to 0} \widetilde{\lambda_s} = 1_G$ uniformly on compact subsets of G and $\lambda_s \neq 1_N$ for $s \neq 0$. Hence, (G, N) does not have the relative Property (T). ■

References

[1] Bekka, M. E. B., and M. Mayer, *On Kazhdan's property (T) and Kazhdan constants associated to a Laplacian for SL(3,R)*, J. Lie Theory **10** (2000), 93–105.

[2] Delaroche, C., and A. Kirillov, *Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés*, Séminaire Bourbaki **343** (1967/68).

[3] de la Harpe, P., and A. Valette, "La propriété (T) de Kazhdan pour les groupes localement compacts," Astérisque **175**, Soc. Math. de France, Paris, 1989.

[4] Jacobson, N., "Basic algebra I," Freeman, New York, 1985.

[5] Kazhdan, D., *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Anal. Appl. **1** (1967), 63–65.

[6] Margulis, G. A., "Discrete subgroups of semisimple Lie groups," Springer-Verlag, New York, 1991.

[7] Shalom, Y., *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's property (T)*, Trans. Amer. Math. Soc. **351** (1999), 3387–3412.

[8] Vaserstein, L. N., *On groups possessing property T*, Funct. Anal. Appl. **2** (1968), 174.

[9] Weil, A., "Basic number theory," Springer-Verlag, New York, 1973.

[10] Zimmer, R. J., "Ergodic theory and semisimple groups," Birkhäuser, Boston, 1984.

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