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# On the Group of Isometries on a Locally Compact Metric Space

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**Abstract.** In the present paper we study conditions under which the group of isometries on a locally compact metric space is locally compact, or acts properly.

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## 1. Introduction

It is long known from the work of van Dantzig and van der Waerden ([1], cf. also [2, Ch.I, Th.4.7]) that if $(X, d)$ is a connected locally compact metric space then its group of isometries $I(X, d)$, when endowed with the topology of pointwise convergence, is always locally compact and acts properly on $X$. More recently it was shown by one of the authors ([6]) that the pointwise closure of $I(X, d)$ is locally compact if the space $\Sigma(X)$ of the connected components of $X$ is quasicompact (compact but not necessarily Hausdorff) with respect to the quotient topology. The question whether $I(X, d)$ is closed in $C(X, X)$ (the space of all continuous selfmaps of $X$ endowed with the topology of pointwise convergence) remained open. In this note we fill this gap (cf. also [4]), i.e., we show that if $\Sigma(X)$ is quasicompact then $I(X, d)$ coincides with its Ellis' semigroup, completing the proof of the following:

**Theorem.** *Let $(X, d)$ be a locally compact metric space. Denote by $I(X, d)$ its group of isometries, with the topology of pointwise convergence, and by $\Sigma(X)$ the space of the connected components of $X$, endowed with the quotient topology. Then*

1. *If $\Sigma(X)$ is not quasicompact, then $I(X, d)$ need not be locally compact, nor act properly on $X$.*
2. *If $\Sigma(X)$ is quasicompact then*

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(a) *$I(X,d)$ is locally compact,*

(b) *the action $(I(X,d),X)$ is not always proper, and*

(c) *the action $(I(X,d),X)$ is proper if $X$ is connected.*

For the sake of completeness, we give short and slightly improved proofs of some of the previously published partial results of the authors, these are crucial for a unified proof of the above theorem. Our treatment is based on the sets $(x,V_x)=\{g\in I(X,d):\, g(x)\in V_x\}$, where $V_x$ is a neighborhood of $x\in X$. These sets form a neighborhood subbasis at the identity with respect to the topology of pointwise convergence, the natural topology of $I(X,d)$.

## 2. Generalities

**2.1.** The following simple examples establish 1 and 2(b) of the above theorem.

**Example.** Let $X=\mathbb{Z}$ with the discrete metric. Obviously $\Sigma(X)$ is not quasi-compact. It can be easily seen that $I(X,d)$ is the group of all bijections of $\mathbb{Z}$, which is not locally compact with respect to the topology of pointwise convergence, therefore it cannot act properly on a locally compact space.

**Example.** Let $X=Y\cup\{(1,0)\}\subset\mathbb{R}^2$ where $Y=\{(0,y):\ y\in\mathbb{R}\}$, and $d=\min\{1,\delta\}$, where $\delta$ denotes the Euclidean metric. As we shall see in §3, by Theorem 3.7, $I(X,d)$ is locally compact; however the action $(I(X,d),X)$ is not proper, because the isotropy group of $(1,0)$ is not compact, since it contains the translations of $Y$. So, the action of $I(X,d)$ on $X$ is not proper, even if $X$ has two components.

Since the sets $(x,V_x)$ as above form a neighborhood subbasis at the identity in $I(X,d)$, the following condition is necessary for the local compactness of $I(X,d)$:

**(a)** There exist $x_i\in X$, $i=1,\dots,m$ such that $\bigcap_{i=1}^m(x_i,V_{x_i})$ is relatively compact in $C(X,X)$.

This condition becomes also sufficient if, in addition, the following condition is satisfied:

**(b)** $I(X,d)$ is closed in $C(X,X)$.

So, to prove that $I(X,d)$ is locally compact, we have to ensure that both of the above conditions are satisfied.

## 3. The local compactness of $I(X,d)$

The following is crucial for the investigation of the conditions 2.1(a) and (b):

**3.1. Lemma.** *Let $(X,d)$ be a locally compact metric space, $F\subseteq I(X,d)$, and*

$$K(F)=\{x\in X:\, F(x)=\{f(x):f\in F\}\ \textit{is relatively compact}\}.$$

*Then $K(F)$ is an open and closed subset of $X$.*

**Proof**. Since $F$ is an equicontinuous family of selfmaps of $X$ we see that $K(F)$ is open. It remains to prove that $K(F)$ is closed.

We write $S(x,\eta) = \{y \in X \mid d(x,y) < \eta\}$ for any $x \in X$ and $\eta > 0$, and $S(M,\eta) = \bigcup\{S(x,\eta) \mid x \in M\}$ for subsets $M \subseteq X$. Let $x$ be a cluster point of $K(F)$ and let $\eta$ be a positive real such that $S(x,5\eta)$ is relatively compact. Choose a point $k \in K(F) \cap S(x,\eta)$. Then $\overline{F(k)} \subseteq S(F(S(x,\eta)),\eta) = F(S(x,2\eta))$, and by the compactness of $\overline{F(k)}$ we can find a finite subset $L \subseteq F$ such that $\overline{F(k)} \subseteq L(S(x,2\eta))$. We show that $F(x)$ is contained in the relatively compact set $L(S(x,5\eta))$. To see this, pick $f \in F$ and let $g \in L$ such that $f(k) \in g(S(x,2\eta))$. Then

$$\begin{aligned} d(f(x),g(k)) &\leq d(f(x),f(k)) + d(f(k),g(x)) + d(g(x),g(k)) \\ &= d(x,k) + d(f(k),g(x)) + d(x,k) \leq 4\eta \end{aligned}$$

and therefore

$$f(x) \in S(g(k),4\eta) = g(S(k,4\eta)) \subseteq g(S(x,5\eta)) \subseteq L(S(x,5\eta)).$$

Thus $x \in K(F)$ and the proof is finished.

**3.2. Remark.** In the sequel we assume that $\Sigma(X)$ *is quasicompact* in the quotient topology via the natural map $q : X \to \Sigma(X)$. Note that $\Sigma(X)$ is a $T_1$–space, and need not be Hausdorff. Nevertheless

*$X$ is separable, hence second countable; so sequences are adequate in C(X,X).*

The proof is similar to the lengthy one in [5] (see also [2, Appendix 2]).

**3.3. Lemma.** *Let $(X,d)$ be a locally compact metric space with a quasicompact space of connected components $\Sigma(X)$. Then condition 2.1(a) is satisfied.*

**Proof.** Let $V_x$ be a relatively compact neighborhood of $x \in X$. Then

$$(x,V_x) = \{g \in I(X,d) : g(x) \in V_x\}$$

is a neighborhood of the identity in $I(X,d)$. Since $x \in K((x,V_x))$, $K((x,V_x))$ is not empty, and by Lemma 3.1 is open and contains entire components of $X$. Therefore $q(K((x,V_x)))$ is an open subset of $\Sigma(X)$. Since $\Sigma(X)$ is quasicompact, there are $x_i$, $i = 1,\ldots,m$, such that the corresponding $q(K((x_i,V_{x_i})))$' s cover $\Sigma(X)$. This means that $X = \bigcup_{i=1}^m K((x_i,V_{x_i}))$, i.e., the neighborhood $F = \bigcap_{i=1}^m (x_i,V_{x_i})$ of the identity has the property: for every $x \in X$ the set $F(x)$ is relatively compact in $X$. Therefore, by Ascoli' s theorem, $F$ is relatively compact in $C(X,X)$.

**3.4.** Now we prove that if $\Sigma(X)$ is quasicompact then $I(X,d)$ is a closed subspace of $C(X,X)$. Because of Remark 3.2, the elements $f$ of the boundary of $I(X,d)$ in $C(X,X)$ are limits of sequences $\{f_n \in I(X,d), n \in \mathbb{N}\}$. Obviously, such an $f$ preserves $d$; so the question is whether $f$ is surjective. If $\Sigma(X)$ is not quasicompact then this is not always true:

**Example.** Let $X = \mathbb{Z}$ with the discrete metric. If $f_n(z) = z$ for $-n < z < 0$, $f_n(-n) = 0$, and $f_n(z) = z+1$ otherwise, then $f_n \to f$, where $f(z) = z$ for $z < 0$, and $f(z) = z + 1$ for $z \geq 0$. Hence each $f_n$ is an isometry, but $f$ is not surjective since $0 \notin f(\mathbb{Z})$.

**3.5. Lemma.** *If $\Sigma(X)$ is quasicompact and $\{(f_n) : f_n \in I(X,d)\}$ is a sequence such that $f_n \to f$ for some selfmap $f$ of $X$ with respect to the topology of pointwise convergence, then $f(X)$ is open and closed in $X$.*

**Proof.** By Lemma 3.1, it suffices to show that $f(X) = K(F)$, where $F = \{f_n^{-1},\, n \in \mathbb{N}\}$. Indeed, since $d(f_n(x), f(x)) = d(x, f_n^{-1}(f(x))$, we have $f_n^{-1}(f(x)) \to x$, so (since $X$ is locally compact) $f(x) \in K(F)$, for every $x \in X$. Now, if $y \in K(F)$, we may assume $f_{n_k}^{-1}(y) \to x$ for some $x \in X$, because $F(y)$ is relatively compact in $X$, hence $f(x) = y$.

**3.6. Proposition.** *If $(X,d)$ is a locally compact metric space, and $\Sigma(X)$ is quasicompact, then $I(X,d)$ is closed in $C(X,X)$.*

**Proof.** Let $\{(f_n) : f_n \in I(X,d)\}$ be a sequence such that $f_n \to f$ for some selfmap $f$ of $X$ with respect to the topology of pointwise convergence. We prove that $f$ is surjective. Let $y \in X$. We denote by $S_x$ the connected component containing $x \in X$, and by $S_n$ the component of $f_n^{-1}(y)$. If $\{S_n, n \in \mathbb{N}\}$ has a constant subnet $\{S_{n_i}, i \in I\}$, then $S_{n_i} = S_0$, for some $S_0 \in \Sigma(X)$. Hence $S_{f_{n_i}^{-1}(y)} = S_0$, so $f_{n_i}(S_0) = S_y$, for every $i \in I$. Pick an $x \in S_0$, then $f_{n_i}(x) \in S_y$. But $f_{n_i}(x) \to f(x)$, so $f(x) \in S_y$. By Lemma 3.5 $S_y \subseteq f(X)$, hence $y \in f(X)$.

Suppose that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet. By the quasicompactness of $\Sigma(X)$, there exists a subnet $\{S_{n_i}, i \in I\}$ of $\{S_n, n \in \mathbb{N}\}$ such that $S_{n_i} \to S$, for some $S \in \Sigma(X)$. With the above notation, the following is true:

**Claim.** *There exists a subsequence $\{S_k, k \in \mathbb{N}\}$ of $\{S_n,\ n \in \mathbb{N}\}$ such that there are $x_k \in S_k$ with $x_k \to x_0$, for some $x_0 \in X$.*

*Proof.* If not, $R = (\bigcup_{n=1}^{\infty} S_n) \setminus S$ is closed in $X$. Indeed, let $\{(y_m) : y_m \in R\}$ be a sequence such that $y_m \to y \in X$. If $y_m \in (\bigcup_{n=1}^{n_0} S_n) \setminus S$ for $m > m_0$, then a subsequence of $\{y_m,\, m \in \mathbb{N}\}$ is contained in some $S_i$ for some $i \in \{1, \ldots, n_0\}$, therefore $y \in S_i \subseteq R$, as required. If this is not the case, we construct a subsequence $\{y_{m_p}, p \in \mathbb{N}\}$ of $\{y_m, m \in \mathbb{N}\}$ in the following way: For $S_1$ we choose a point $y_{m_1} \in S_{n_1}$ with $n_1 > 1$ and $d(y_{m_1}, y) < 1$, for $(\bigcup_{n=1}^{n_1} S_n) \setminus S$ a point $y_{m_2} \in S_{n_2}$ with $n_2 > n_1$ and $d(y_{m_2}, y) < \frac{1}{2}$, and so on. Obviously, $y_{m_p} \in S_{n_p}$ and $y_{m_p} \to y$, a contradiction.

Since $S$ does not meet $R$, then $S \subseteq X \setminus R$. On the other hand $X \setminus R$ is open (since $R$ is closed in $X$) and contains entire components (recall that $R$ is a union of components), so $S_{n_i} \subseteq X \setminus R$, eventually. Therefore $S_{n_i} = S$, a contradiction, since we have assumed that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet.

According to the Claim, there exists a sequence $\{(x_k) : x_k \in S_k\}$ such that $x_k \to x_0 \in X$, where $S_k = S_{f_k^{-1}(y)} = f_k^{-1}(S_y)$, from which follows $x_k = f_k^{-1}(y_k)$ for

some $y_k \in S_y$. Then

$$\begin{aligned} d(y_k, f(x_0)) &\leq d(y_k, f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &= d(f_k^{-1}(y_k), x_0) + d(f_k(x_0), f(x_0)) \to 0, \end{aligned}$$

therefore $f(x_0) \in S_y$, which means that $S_y \cap f(X) \neq \emptyset$ and, by Lemma 3.5, $S_y \subseteq f(X)$, hence $y \in f(X)$, and $f$ is surjective.

**3.7. Theorem.** *If $\Sigma(X)$ is quasicompact, then $I(X,d)$ is locally compact.*

**Proof.** This assertion follows from Lemma 3.3 and Proposition 3.6, since both conditions 2.1(a) and (b) are satisfied.

## 4. The properness of the action $(I(X,d), X)$

In this short section, applying the methods used previously, we give a complete proof of the following:

**Proposition.** *If $(X,d)$ is locally compact and connected, then $I(X,d)$ is locally compact and the action $(I(X,d), X)$ is proper.*

**Proof.** Since $X$ is connected $G = I(X,d)$ is locally compact by Theorem 3.7. So, we have to show that, for every $x,\ y \in X$, there are neighborhoods $U_x$, $U_y$ of $x$ and $y$ respectively such that

$$(U_x, U_y) := \{g \in G : (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in $G$. Let $U_x = S(x,\varepsilon)$ and $U_y = S(y,\varepsilon)$ be such that $S(y,2\varepsilon)$ is relatively compact. Then, for $g \in (U_x, U_y)$ and $z \in U_x$ with $g(z) \in U_y$, we have

$$d(g(x),(y)) \leq d(g(x),g(z)) + d(g(z),y) = d(x,z) + d(g(z),y) < 2\varepsilon,$$

therefore $g \in F = \{g \in G : g(x) \in S(y,2\varepsilon)\}$. Then $x \in K(F)$, and, according to Lemma 3.1, $K(F)$ coincides with the connected space $X$. From this and Ascoli's theorem it follows that $F$ is relatively compact in $C(X,X)$. So $(U_x, U_y) \subseteq F$ is relatively compact in $C(X,X)$, hence in $G$, because $G$ is closed (cf. Proposition 3.6).

This proves the Proposition and completes the proof of the Theorem in the Introduction.

## 5. Final Remark

Using the same arguments we can prove that if $X$ is a locally compact metrizable space, then $I(X,d)$ is locally compact for all admissible metrics $d$, provided that the space $Q(X)$ of the quasicomponents of $X$ is compact with respect to the quotient topology (note that $Q(X)$ is always Hausdorff) (cf. [3]). Recall that the

quasicomponent of a point is the intersection of all open and closed sets which contain it. Our exposition is given via $\Sigma(X)$ because we regard the condition "$\Sigma(X)$ is quasicompact" as a topologically more natural condition than "$Q(X)$ is compact", although it is more restrictive: There are locally compact metric spaces with compact $Q(X)$ and non quasicompact $\Sigma(X)$ as the following example shows:

**Example.** The space of all connected components of the locally compact space

$$X = \left( \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : y \in [-1, 1]\} \right) \cup \{(0, y) : y \in [-1, 0)\} \cup \left( \bigcup_{k=1}^{\infty} I_k \right) \subseteq \mathbb{R}^2,$$

where

$$I_k = \{(0, y) : y \in (\frac{1}{k+1}, \frac{1}{k})\}, \quad k \in \mathbb{N}^*,$$

is not quasicompact, because the sequence $\{I_k\} \subseteq \Sigma(X)$ does not have a convergent subsequence in $\Sigma(X)$. On the contrary, $Q(X)$ is compact, because the quasicomponent of the point $(0, -1)$ consists of the set $\{(0, y) : y \in [-1, 0)\}$ and the intervals $I_k$ , $k \in \mathbb{N}^*$ .

So the quasicompactness of $\Sigma(X)$ is not necessary for the local compactness of $I(X, d)$.

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