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Variationality of Four-Dimensional Lie Group Connections

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Abstract. This paper gives a comprehensive analysis of the inverse problem of Lagrangian dynamics for the geodesic equations of the canonical linear connection on Lie groups of dimension four. Starting from the Lie algebra, in every case a faithful four-dimensional representation of the algebra is given as well as one in terms of vector fields and a representation of the linear group of which the given algebra is its Lie algebra. In each case the geodesic equations are calculated as a starting point for the inverse problem. Some results about first integrals of the geodesics are obtained. It is found that in three classes of algebra, there are algebraic obstructions to the existence of a Lagrangian, which can be determined directly from the Lie algebra without the need for any representation. In all other cases there are Lagrangians and indeed whole families of them. In many cases a formula for the most general Hessian of a Lagrangian is obtained.
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1. Introduction

The inverse problem of Lagrangian dynamics consists of finding necessary and sufficient conditions for a system of second order ordinary differential equations to be the Euler-Lagrange equations of a regular Lagrangian function and in case they are, to describe all possible such Lagrangians. By far the most important contribution in the area was the 1941 article of Douglas [1]. Douglas' analysis of the two degrees of freedom case turned out to be so involved that work on the problem was effectively stalled for more that thirty years. Three important contributions were the papers of Crampin [2], Henneaux and Shepley [3] and [4]. An excellent and comprehensive analysis of the state of the art in 1990 is given in the article by Morandi et al [5]. In the 1990's investigations advanced on three fronts. In [6] Anderson and Thompson presented an algorithm for solving the inverse problem in a concrete situation and it is essentially that procedure that will be adopted here. In Section 3 we give a very brief outline of the algorithm but refer the reader to [6] for complete details and worked examples. Meanwhile Martinez, Sarlet and Crampin and others developed a powerful calculus associated to any second order ODE system [7, 8]. Finally, Muzsnay and Grifone took a different approach to

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the problem and completely by-passed the Helmholtz conditions. They worked directly with the Euler-Lagrange operator and employed the techniques of Spencer cohomology [10, 11].

One aspect of the inverse problem which seems to remain little explored is the very special case of the geodesic equations of the canonical symmetric connection, that we shall denote by ∇, belonging to any Lie group G. One of the present authors has investigated the situation for Lie groups of dimension two and three [12]. It was found in [12] that in all these cases the geodesics were the Euler -Lagrange equations of a suitable Lagrangian defined on an open subset of the tangent bundle TG. The canonical connection ∇ was introduced by Cartan and Schouten in [13]. In Section 2 we review the main properties of ∇. In the case where G is semi-simple ∇ is the Levi-Civita connection of the Killing form but ∇ does not seem to have been studied much in the more general context.

In this paper we shall be concerned with the inverse problem for the canonical connection ∇ in the case of Lie groups of dimension four; our primary concern is to ascertain whether or not a particular connection is derivable from a Lagrangian function and, if so, to give at least one such Lagrangian. ¿From the group representation it is straightforward to obtain a local coordinate description of the geodesic equations. In most cases we are even able to give a closed form solution for the most general Hessian. On the other hand most of our Lagrangians are singular on the zero section of the tangent bundle TG and the nature of these singularities is another issue which is under investigation. We have found it convenient in several cases to modify the procedure givin in [6], for example, by simplifying the system of geodesics before implementing the algorithm. In Section 3 we review very briefly the inverse problem in general and in Section 4 we specialize to the case of the geodesic flow of a linear connection and also consider the problem of determining when such a connection is the Levi-Civita connection of some pseudo-Riemannian metric.

In Section 5 we make some general comments about Lie groups and Lie algebras as they relate to the inverse problem for the canonical connection. We prove several results about first integrals and obtain a normal form for a connection in dimension n for which the Lie algebra has a representation in which $n-1$ basis elements are coordinate vector fields.

In Section 6 we investigate each of the Lie algebras listed in [14]. Finally we make some comments about our notation. The summation convention on repeated indices applies throughout the text. In Section 6 we use x, y, z and w as local coordinates on $\mathbf{R}^4$ to describe our connections. In order to avoid having an excessive number of dots, the corresponding derivative or velocity variables will be denoted by u, v, s, and t, respectively.

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2. The canonical connection on a Lie group

In this section we shall outline the main properties of the canonical symmetric connection ∇ on a Lie group G. In fact ∇ is defined on left invariant vector fields X and Y by $\nabla_X Y = \frac{1}{2}\,[X, Y]$, and then extended to arbitrary vector fields by making ∇ tensorial in the X argument and satisfy the Leibnitz rule in the Y argument. Following the conventions of [15] a left invariant vector field associated to an element X in T_IG is denoted by $\tilde{X}$; that is, $\tilde{X}(S) = L_{S*}X$, where S and I denote a typical and identity group elements, respectively, and L_S denotes left translation. Likewise using R_S for right translation the right invariant vector field induced by X is denoted by $\tilde{X}^{R(S)}$ so that $\tilde{X}^{R(S)} = R_{S_*}X$.

Lemma 2.1. *In the definition of ∇ we can equally assume that X and Y are right invariant vector fields and hence ∇ is also right invariant.*

Proof: Let E_i be a basis for the Lie algebra of G, that is T_IG. According to the conventions introduced above and letting S denote a generic element of G, there must exist a matrix of functions $f_{ij}(S)$ such that $\tilde{E}_i^{R(S)} = f_{ik}(S)\tilde{E}_k$, where here and for the rest of the lemma the summation convention on repeated indices applies. If we calculate the quantity $\nabla_{\tilde{E}_i^{R(S)}}\tilde{E}_j^{R(S)} - \frac{1}{2}[\tilde{E}_i^{R(S)}, \tilde{E}_j^{R(S)}]$ using the definition of ∇ and the Koszul axioms, we find that it is zero if and only if $f_{ik}(\tilde{E}_k f_{jl})\tilde{E}_l + f_{jl}(\tilde{E}_l f_{ik})\tilde{E}_k = 0$, where the point S in the group has been suppressed. If we interchange k and l in the second term above we find that the latter condition is equivalent to

$$f_{ik}(\tilde{E}_k f_{jl}) + f_{jk}(\tilde{E}_k f_{il}) = 0. \tag{1}$$

Starting from the condition above that relates left and right invariant fields and using the fact that the left invariant $\tilde{E}_l$ and right invariant vector fields $\tilde{E}_i^{R(S)}$ commute we find that $\tilde{E}_l f_{ik} + f_{im}C^k_{lm} = 0$. However, because of the skew-symmetry in C^k_{lm} the latter condition implies 1 and hence in the definition of ∇ we can equally use right invariant vector fields. □

Clearly ∇ is symmetric, bi-invariant and the curvature tensor on left invariant vector fields is given by

$$R(X,Y)Z = \frac{1}{4}\,[Z,[X,Y]]. \tag{2}$$

Furthermore, G is a symmetric space in the sense that R is a parallel tensor field. Indeed suppose that W, X, Y and Z are left-invariant vector fields. Then from the definition of ∇ and (2) we have that

$$\begin{aligned}
4\nabla_W R(X,Y)Z &= 1/2[W,[Z,[X,Y]]] - 4R(\nabla_W X,Y)Z - 4R(X,\nabla_W Y)Z \\
&\quad -4R(X,Y)\nabla_W Z \\
&= 1/2[W,[Z,[X,Y]]] - [Z,[\nabla_W X,Y]] - [Z,[X,\nabla_W Y]] - [\nabla_W Z,[X,Y]] \\
&= 1/2[W,[Z,[X,Y]]] - 1/2[Z,[[W,X],Y]] \\
&\quad -1/2[Z,[X,[W,Y]]] - 1/2[[W,Z],[X,Y]] \\
&= 1/2([Z,[W,[X,Y]]] - [Z,[[W,X],Y]] - [Z,[X,[W,Y]]]) = 0.
\end{aligned}$$

It follows from (2) that ∇ is flat if and only if the Lie algebra $\mathbf{g}$ of G is nilpotent of order two. Clearly left and right invariant vector fields are auto-parallel. Hence the geodesics of ∇ are translates either to the left or right of one-parameter subgroups of G, that is of the form $S(\exp(tX))$ or $(\exp(tX))S$, where X and S are in $\mathbf{g}$ and G, respectively. The Ricci tensor R_{ij} of ∇ is symmetric and bi-invariant. In fact, if $\{E_i\}$ is a basis of left invariant vector fields then

$$[E_i, E_j] = C^k_{ij}E_k \tag{3}$$

where C^k_{ij} are the structure constants and relative to this basis the Ricci tensor R_{ij} is given by

$$R_{ij} = \frac{1}{4}\,C^l_{jm}C^m_{il} \tag{4}$$

from which the symmetry of R_{ij} becomes apparent. Ricci gives rise to a quadratic Lagrangian which may, however, not be regular. We shall assume that G is indecomposable in the sense that the Lie algebra $\mathbf{g}$ of G is not a direct sum of lower dimensional algebras. It should be noted that generally, in solving the inverse problem, it is not sufficient to restrict to indecomposable algebras. However, in the case of dimension four, a decomposable algebra will be a sum of algebras each of which possesses a variational connection according to the results of [12]. Hence a decomposable connection is always variational and so we shall assume henceforth that our Lie algebras are indecomposable.

Since our starting point is the Lie algebra $\mathbf{g}$ of a Lie group it is of interest to ask how the ideals of $\mathbf{g}$ are related to ∇. We quote the following result [16].

Proposition 2.2. *Let ∇ denote a symmetric connection on a smooth manifold M. Necessary and sufficient conditions that there exist a submersion from M to a quotient space Q such that ∇ is projectable to Q are that there exists an integrable distribution D on M that satisfies:*

(i) $\nabla_X Y$ belongs to D whenever Y belongs to D and X is arbitrary.

(ii) $R(Z,X)Y$ belongs to D whenever Z belongs to D and X and Y are arbitrary, where R denotes the curvature of ∇. □

In the case of the canonical connection on G we deduce:

Proposition 2.3. *Every ideal $\mathbf{h}$ of $\mathbf{g}$ gives rise to a quotient space Q consisting of the leaf space of the integrable distribution determined by $\mathbf{h}$ and ∇ on G projects to Q.* □

The center of $\mathbf{g}$ is of course an ideal and it has the property that any element of it gives rise to a parallel vector field on G. A very interesting situation occurs where $\mathbf{g}$ possesses two ideals $\mathbf{h_1}$ and $\mathbf{h_2}$ such that $\mathbf{h_1} \cap \mathbf{h_2}$ is zero. Denote the corresponding distributions on G by D_1 and D_2, respectively. Since we are always assuming that $\mathbf{g}$ is indecomposable, $\mathbf{g}$ cannot be the direct sum of $\mathbf{h_1}$ and $\mathbf{h_2}$ and hence $D_1 \cap D_2$ is non-zero. In fact $D_1 \cap D_2$ is the integrable distribution on G that corresponds to the ideal $\mathbf{h_1} + \mathbf{h_2}$ of $\mathbf{g}$ and simliarly $D_1 + D_2$ corresponds to the ideal $\mathbf{h_1} \cap \mathbf{h_2}$.

3. The inverse problem for second order ODE's

In this Section we shall outline the method given in [6] for solving the inverse problem for a system of second order ODE of the form

$$\ddot{x}^i = f^i(x^j, \dot{x}^j). \tag{5}$$

In fact, we shall denote $\dot{x}^i$ by u^i. The first step in the method is to construct the $n \times n$ matrix of functions Φ defined by see [7]

$$\Phi^i_j = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial f^i}{\partial u^j}\right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4}\frac{\partial f^i}{\partial u^k}\frac{\partial f^k}{\partial u^j}, \tag{6}$$

one now finds the algebraic solution for g of the equation

$$g\Phi = (g\Phi)^t \tag{7}$$

which expresses the self-adjointness of Φ relative to g. The symmetric matrix g will represent the Hessian with respect to the u^i variables of a putative Lagrangian L. Since there is just a single matrix Φ, one can always find non-degenerate solutions to (7), whatever the algebraic normal form of Φ may be. In fact, (7) imposes at most $\begin{pmatrix} n \\ 2 \end{pmatrix}$ conditions on the $\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ components of g.

In the general theory there is, in fact, a hierarchy $\overset{n}{\Phi}$ of matrices defined recursively by

$$\overset{n+1}{\Phi} = \frac{d}{dt}\left(\overset{n}{\Phi}\right) + \frac{1}{2}\left[\frac{\partial f}{\partial u}, \overset{n}{\Phi}\right]. \tag{8}$$

Since R is parallel we need only consider the first term of the hierarchy. There is, in general, a second hierarchy of algebraic conditions that must be satisfied by g. Define functions Ψ^i_{jk} by

$$\Psi^i_{jk} = \frac{1}{3}\left(\frac{\partial \Phi^i_j}{\partial u^k} - \frac{\partial \Phi^i_k}{\partial u^j}\right). \tag{9}$$

The Ψ^i_{jk} are, in fact, the principal components of the curvature of the linear connection associated to the ODE system 5 (see [7] for further details). Again only the first set of conditions in the hierarchy need be considered, namely,

$$g_{mi}\Psi^m_{jk} + g_{mk}\Psi^m_{ij} + g_{mj}\Psi^m_{ki} = 0. \tag{10}$$

According to the general theory we now assume that we have a basis of solutions to the double hierarchy of algebraic conditions. If we cannot find a non-singular solution then we can be sure at this stage that no regular Lagrangian exists for the problem under consideration. The problem is that such a two-form need not be closed. One of the auxiliary conditions that must be satisfied by g if the corresponding two-form is to be closed is

$$\frac{dg_{ij}}{dt} + \frac{1}{2}\frac{\partial f^k}{\partial u^i}g_{kj} + \frac{1}{2}\frac{\partial f^k}{\partial u^j}g_{ki} = 0. \tag{11}$$

Now (11) is a system of ODE's and it is possible, in principle, to scale basis elements which are solutions to (7) by first integrals of (5) so as to satisfy (11). When (11) are integrated the "arbitrary constants" that enter in the solution are just first integrals of the geodesics.

After we have obtained a basis of solutions for (7), each of which satisfies (11), the final step is to impose the so-called closure conditions

$$\frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j} = 0. \tag{12}$$

This step is accomplished by looking for linear combinations of the basis elements over the ring of first integrals for (5) so that (12) is satisfied. Then (7) and (11) still hold and the resulting closed two-forms, if indeed they exist, will be Cartan two-forms, albeit possibly degenerate. We remark that (7), (11) and (12) together with the symmetry and non-degeneracy of g constitute the Helmholtz conditions for the inverse problem for (5).

4. The inverse problem for linear connections

In the case of a linear connection the matrix Φ is of the form

$$\Phi^i_j = R^i_{kjl}u^k u^l \tag{13}$$

where R^i_{kjl} are the components of the curvature R of the connection relative to a coordinate system (x^i). The higher order Φ-tensors in this case just correspond to covariant derivatives of the curvature so that, for example,

$$\overset{1}{\Phi}{}^i_j = R^i_{kjl;m} u^k u^l u^m. \tag{14}$$

In particular if R is parallel then all the higher order Φ-tensors vanish. For the case of a linear connection, one finds that

$$\Psi^i_{jk} = R^i_{ljk} u^l \tag{15}$$

and again the higher order Ψ's correspond to covariant derivatives of R. Thus, for example,

$$\overset{1}{\Psi}{}^i_{jk} = R^i_{ljk;m} u^l u^m. \tag{16}$$

Again if R is parallel the higher order Ψ-tensors vanish. The condition coming from Φ is

$$\left(g_{mi} R^i_{pjq} - g_{ji} R^i_{pmq}\right) u^p u^q = 0, \tag{17}$$

while the condition coming from Ψ is

$$(g_{mi} R^i_{pjq} + g_{qi} R^i_{pmj} + g_{ji} R^i_{pqm}) u^p = 0. \tag{18}$$

If we contract u^q into (17) we find from (16) that

$$g_{qi} R^i_{pmj} u^p u^q = 0. \tag{19}$$

Thus, for the special case of a linear connection, we can use (18) and (19) as the first and only algebraic conditions in the double hierarchy.

We take up next the question of when ∇ is the Levi-Civita connection of some metric. It is known that in the case where the metric is Riemannian the necessary and sufficient conditions for a group G to admit a metric is that G be the product of a compact and an abelian group [18]. More generally one can pose the question of whether a given connection, not necessarily the canonical connection, is the Levi-Civita connection of some metric. The answer is provided by the following Theorem[19,20].

Theorem 4.1. *The necessary and sufficient conditions for a connection to be a Levi-Civita connection are that the following system of linear equations for unknown g stabilize and that it admit a non-singular solution, R denoting the curvature tensor of the connection:*

$$gR + (gR)^t = 0 \tag{20}$$

$$g\nabla R + (g\nabla R)^t = 0 \tag{21}$$

$$g\nabla^2 R + (g\nabla^2 R)^t = 0 \tag{22}$$

$$\dots$$

□

Let us now apply Theorem 4.1 to the case of the canonical connection ∇ on G. Since in this case R is parallel, only the first condition in Theorem 4.1 applies. By applying equation (2) we obtain immediately the following result.

Theorem 4.2. *The canonical connection ∇ on G is the Levi-Civita connection of some metric if and only if there is a non-degenerate solution to*

$$g([Z,[X,Y]],W)+g(Z,[W,[X,Y]])=0 \tag{23}$$

where X,Y,Z and W are arbitrary left or right-invariant vector fields. □

If we denote the Killing form of G by K we know that K is "ad-invariant", that is,

$$K([Z,X],W)+K(Z,[W,X])=0. \tag{24}$$

Clearly then K satisfies (23) and in the case where G is semi-simple we obtain a bi-invariant metric as noted earlier. In fact, if G is semi-simple, then (23) and (24) are equivalent since the derived algebra $[\mathbf{g},\mathbf{g}]=\mathbf{g}$ and since R is parallel.

5. Generalities on Lie groups and Lie algebras

In practice we begin our investigations at the Lie algebra rather the Lie group level which leads to a number of interesting complications. It is apparent that conditions (18) and (19) can be formulated for any Lie algebra. Thus the first step in our procedure will be to solve equations (18) and (19) for a given Lie algebra $\mathbf{g}$ where the curvature tensor is defined by (2). In some cases we find that, even at that level, the matrix g_{ij} is forced to be singular. In such a case we can be sure that there will be no Lagrangian corresponding to the geodesic equations of any Lie group that has $\mathbf{g}$ as its Lie algebra. Suppose, however, that conditions (18) and (19) do not entail that g_{ij} should be singular. One is now faced with the problem of finding a Lie group G so that $\mathbf{g}$ is its Lie algebra. An answer of sorts is furnished by Ado's theorem [21], which asserts, in the first instance, that any finite dimensional Lie algebra over $\mathbf{R}$ or $\mathbf{C}$ has a faithful finite -dimensional linear representation. If $\mathbf{g}$ has only a trivial center then the adjoint representation is faithful. If the center is non-trivial then there is no obvious representation available. It is very convenient, if not essential for our purposes, to work with linear representations of order n for algebras and groups of order n. The cases of dimensions 2 and 3 have been discussed in [12]. As for dimension 4, we have in every case been able to find a faithful linear representation by 4×4 matrices without recourse to Ado's theorem, as the reader will see in the next section. It seems to be worthwhile to record this result.

Theorem 5.1. *Every Lie algebra in dimensions two, three and four has a faithful representation by matrices of order two, three and four, respectively.* □

Let us assume now that we have a Lie algebra $\mathbf{g}$, that conditions (18) and (19) do not entail that the matrix g_{ij} is singular and that we have a matrix representation and that we are able to determine a corresponding Lie group G by exponentiation. On G we construct the right invariant Maurer-Cartan one-form and then by dualizing, we obtain a basis for the right invariant vector fields. We obtain thereby a representation of $\mathbf{g}$ by vector fields. From Section 4 we see that no further algebraic conditions can arise and we proceed to formulate conditions (11).

We now state and prove several propositions about first integrals.

Proposition 5.2. *Any left or right invariant one-form on G gives rise to a linear first integral on TG.*

Proof: Let α be a one-form on G that is right-invariant and let X and Y be right -invariant vector fields. The function $\langle Y, \alpha \rangle$ is right invariant and so is constant. Hence

$$X\langle Y, \alpha\rangle = \langle \nabla_X Y, \alpha\rangle + \langle Y, \nabla_X, \alpha\rangle = 0. \tag{25}$$

Now interchange X and Y and add the resulting equations and use the definition of ∇. One finds that

$$\langle X, \nabla_Y \alpha\rangle + \langle Y, \nabla_X \alpha\rangle = 0. \tag{26}$$

Since the last condition is tensorial in X and Y it follows that α satisfies Killing's equation and hence gives a first integral on TG. □

Proposition 5.3. *Consider the following conditions for a one-form α on G:*

(i) α is right-invariant and closed.

(ii) α is left-invariant and closed.

(iii) α is bi-invariant.

(iv) α is parallel.

Then we have the following implications: (iii) implies (i);(iii) implies (ii);each of (i),(ii) or (iii) implies (iv).

Proof:(i) implies (iv): The fact that α is closed implies that for any two vector fields X and Y, in particular right invariant ones, that

$$\langle X, \nabla_Y \alpha\rangle - \langle Y, \nabla_X \alpha\rangle = 0. \tag{27}$$

On the other hand according to Proposition 5.2

$$\langle X, \nabla_Y \alpha\rangle + \langle Y, \nabla_X \alpha\rangle = 0. \tag{28}$$

Hence α is parallel. The proof that (ii) implies (iv) is similar to the proof that (i) implies (iv). (iii) implies (i): A lemma of Helgason [15] states that if a one-form is bi-invariant then it is closed. Hence we are reduced to proving that (i) implies (iv), which has already been done. □

Proposition 5.4. *Suppose that a basis for a Lie algebra* $\mathbf{g}$ *of a Lie group G consists of*

$$X_i = \frac{\partial}{\partial x^i}, \quad W = \frac{\partial}{\partial w} + a^k_j x^j \frac{\partial}{\partial x^k} \tag{29}$$

where a^k_j is a constant $n \times n$ matrix. Then the geodesic equations for the canonical connection on G are given by

$$\ddot{x}^i = a^i_j \dot{x}^j \dot{w}, \quad \ddot{w} = 0. \tag{30}$$

Proof: The proof is a straightforward calculation. □

In the next Theorem, we come back to the idea introduced in Section 2, where the Lie algebra $\mathbf{g}$ has two ideals that intersect trivially. It is very useful for building up a list of Lagrangians for variational connections in successive dimensions. However, the result is valid much more generally for Lagrangian systems that have two submersions and so we introduce it in that context. We shall be content to give a proof that uses local coordinates. Referring to (7) we let d denote the dimension of the algebraic solution for the g_{ij}. Recall [6] that the set of such solutions is a finite dimensional module over the ring of first integrals.

Theorem 5.5. *(i) Given a system of second order ODE of the form*

$$\ddot{x}^i = f^i(x^j, z^B, \dot{x}^j, \dot{z}^B), \quad \ddot{y}^a = g^a(y^b, z^B, \dot{y}^b, \dot{z}^B), \quad \ddot{z}^A = h^A(z^B, \dot{z}^B), \tag{31}$$

suppose that

$$L_1 = L_1(x^j, z^B, \dot{x}^j, \dot{z}^B), \quad L_2 = L_2(y^b, z^B, \dot{y}^b, \dot{z}^B) \tag{32}$$

are Lagrangians for the f,h pair and g,h pair of equations, respectively; then

$$L := L_1 + L_2 \tag{33}$$

is a Lagrangian for the full f, g, h system.
(ii) Suppose we are given the ODE system in (i). We let d, d_1, d_2 and d_{12} denote the dimension of the algebraic solution space to (7) for the (f, g, h),(f, h),(g, h) and (h) subsystems, respectively. Suppose further that the (f, h) and (g, h) and (h) subsystems are of Euler-Lagrange type and that $d = d_1 + d_2 - d_{12}$. Then the f, g, h system is of Euler-Lagrange type and the Hessian of the (f, g, h) system is the sum (as a vector space) of the Hessians of the (f, h) and (g, h) systems; the part common to the Hessians of the (f, h) and (g, h) systems is precisely the Hessian of the h system.

Proof:(i) The proof follows immediately from the definition of the Euler-Lagrange equations.
(ii) Consider first of all the algebraic solutions for the (f, h) and (g, h) subsystems. The algebraic solution for the h system is clearly a solution for each of these systems but the hypothesis entails that a basis for the full system is obtained by extracting a basis from the (f, h) and (g, h) subsystems. We can illustrate the situation by saying that the algebraic solution of the full system corresponds to the sum of the matrices

$$g_{ij} = \begin{bmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B^t & 0 & C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & E \\ 0 & E^t & F \end{bmatrix}.$$

The closure conditions (12) on the full system now imply that A and B are independent of $\dot{y}^b$ and that D and E are independent of $\dot{x}^i$, respectively. Next we use the "horizontal closure conditions", which are conditions analogous to (12) but in which the $\frac{\partial}{\partial u^i}$ "vertical" derivatives are replaced by H_i "horizontal" derivatives; explicitly H_i is given in this special case, by $\frac{\partial}{\partial x^i} + \frac{1}{2}\frac{\partial f^j}{\partial u^i}\frac{\partial}{\partial u^j}$ and there are similar formulas for H_a and H_A. For a further discussion of these conditions we refer to [8]. They are integrability conditions that are consequences of the Helmholtz conditions. In the present context they imply that A and B are independent of y^b and that D and E are independent of x^i, respectively. By differentiating the closure conditions we can easily obtain four second order conditions which imply that the matrix C_{AB} is of the form

$$C = C_1(x^i, z^A, u^i, t^A) + C_2(y^a, z^A, v^a, t^A).$$

It follows that the Hessian of the full system projects both onto the (x^i, z^A, u^i, t^A) and (y^a, z^A, v^a, t^A) coordinate systems and that on these spaces we recover the Hessians of the (f, h) and (g, h) systems. □

Next in this Section we shall obtain a formula for the connection components Γ^i_{jk} of ∇ in a coordinate system (x^i). Suppose that the right-invariant Maurer-Cartan forms of G are α^i. Then there must exist a matrix Y^i_j of functions such that

$$\alpha^i = Y^i_j dx^j. \tag{34}$$

The fact that such a matrix Y^i_j exists is the content of Lie's third theorem [15]. We denote the right -invariant vector fields dual to the α^i by E_j. It follows that

$$E_i = X^k_i \frac{\partial}{\partial x^k} \tag{35}$$

where X^k_i is the inverse of Y^i_j. We denote the structure constants of $\mathbf{g}$ relative to the basis E_i by C^i_{jk}. Then by definition

$$\nabla_{E_i} E_j = \frac{1}{2} C^k_{ij} E_k. \tag{36}$$

Using (36) we find the following condition relating C^i_{jk} and Γ^i_{jk}:

$$X^k_i (X^m_{j,k} + X^l_j \Gamma^m_{kl}) = \frac{1}{2} C^k_{ij} X^m_k. \tag{37}$$

Taking the symmetric part of (38) we obtain

$$\Gamma^m_{pq} = -\frac{1}{2}(Y^j_q X^m_{j,p} + Y^j_p X^m_{j,q}). \tag{38}$$

To conclude this Section we shall revisit the inverse problem for $E(2)$, the Euclidean group of the plane[12]. The corresponding Lie algebra is denoted by $A_{3,6}$ in [14] and has basis e_1, e_2, e_3 with non-zero brackets, $[e_1, e_3] = -e_2$ and $[e_2, e_3] = e_1$. As in [12] the geodesics are given by

$$\dot{u} = tv, \quad \dot{v} = -tu, \quad \dot{t} = 0, \tag{39}$$

where u, v and t denote $\dot{x}, \dot{y}$ and $\dot{w}$, respectively. The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & -dw & -dy \\ dw & 0 & dx \\ 0 & 0 & 0 \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & dxdw \\ 0 & 0 & dydw \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we see that the curvature tensor has essentially only the following non-zero components

$$4R^1_{313} = 1, \quad 4R^2_{323} = 1. \tag{40}$$

Conditions (18) and (19) entail that g_{ij} satisfies the conditions

$$g_{1q}u^q = g_{2q}u^q = 0$$

and the solution of the ODE conditions (11) imply that g_{ij} is given by

$$\begin{aligned} g_{ij} &= M \begin{bmatrix} t^2 u & t^2 v & -t(u^2+v^2) \\ t^2 v & -t^2 u & 0 \\ -t(u^2+v^2) & 0 & u(u^2+v^2) \end{bmatrix} + P \begin{bmatrix} t^2 & 0 & -tu \\ 0 & t^2 & -tv \\ -tu & -tv & u^2+v^2 \end{bmatrix} \\ &+ N \begin{bmatrix} -t^2 v & t^2 u & 0 \\ t^2 u & t^2 v & -t(u^2+v^2) \\ 0 & -t(u^2+v^2) & v(u^2+v^2) \end{bmatrix} + H \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

where H, M, N and P are arbitrary first integrals.

The closure conditions (12) turn out to be

$$uM_u + vM_v + tM_t + 4M = 0, uN_u + vN_v + tN_t + 4N = 0, F_u = 0, F_v = 0 \tag{41}$$

$$uP_u+vP_v+tP_t+3P = 0, P_v+2N+tN_t+uM_v-vM_u = 0, P_u+2M+tM_t-uN_v+vN_u = 0 \tag{42}$$

and can be solved by introducing the following first integrals: $\alpha = \frac{u}{t} - y$, $\beta = \frac{v}{t} + x$, $\gamma = \frac{\cos(w)u - \sin(w)v}{t}$, $\delta = \frac{\cos(w)v + \sin(w)u}{t}$. Thus

$$F = F(t), M = \frac{m(\alpha, \beta)}{t^4}, N = \frac{n(\alpha, \beta)}{t^4}, P = \frac{(2n + \delta m_\gamma - \gamma m_\delta)\beta + (2m - \delta n_\gamma + \gamma n_\delta)\alpha}{t^3} \tag{43}$$

where m, n and F are arbitrary smooth functions.

A very simple Lagrangian in this case is given by

$$L = \frac{(u^2 + v^2)}{t} + xv - yu + t^2. \tag{44}$$

6. Case by case analysis of 4-dimensional Lie groups

According to [14] there are 12 classes of Lie algebras in dimension 4 and they are listed as $A_{4,n}$ where n varies between 1 and 12. The generators of the algebra are listed as e_1, e_2, e_3, e_4 and in each case we list the non-zero Lie brackets. The first three algebras are such that the matrix g_{ij} appearing in (18) and (19) is singular.
$A_{4,7}$: The brackets are:

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3. \tag{45}$$

Using equation (2) we find that the non-zero components of the curvature tensor are given by:

$$R^1_{441} = 1, R^1_{432} = \frac{1}{2}, R^1_{342} = \frac{1}{4}, R^2_{442} = \frac{1}{4}, R^1_{234} = \frac{1}{4}, R^1_{343} = \frac{1}{4}, R^2_{443} = \frac{1}{2}, R^3_{443} = \frac{1}{4}.$$

Conditions (19) give

$$u^q g_{q1} = u^q g_{q2} = u^q g_{q3} = 0 \tag{46}$$

whereas (18) yield

$$g_{11} = g_{12} = g_{13} = g_{22} = 0. \tag{47}$$

It follows that g_{ij} must be singular and hence there can be no Lagrangian.

We shall give the group representation and geodesics for the sake of completeness. Note that the algebra has trivial center and therefore the adjoint representation is faithful. A typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^{2w} & -ze^w & ye^w & x \\ 0 & e^w & we^w & y + zw \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using (39) the corresponding system of geodesic equations is found to be

$$\dot{t} = 0, \quad \dot{u} = 2ut + zvt - (y + z)st + z^2t^2, \quad \dot{v} = vt - st + zt^2, \quad \dot{s} = st \tag{48}$$

where s, t, u and v denote $\dot{z}, \dot{w}, \dot{x}$ and $\dot{y}$, respectively.

$A_{4,9b}$: The brackets are:

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = (1+b)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = be_3(-1 < b < 1) \tag{49}$$

and the non-zero components of the curvature tensor are given by:

$$4R^1_{441} = (1+b)^2, 4R^1_{432} = 1+b, 4R^1_{342} = 1, 4R^2_{442} = 1, 4R^1_{234} = b, 4R^3_{443} = -b^2$$

Conditions (18) and (19) imply that g_{ij} is singular unless $b = 0, -\frac{1}{2}$. A typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^{(b+1)w} & -xe^w & ye^{bw} & z \\ 0 & e^w & 0 & y \\ 0 & 0 & e^{bw} & bx \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A basis for the right-invariant Maurer-Cartan one-form dSS^{-1} is given by $dw, dx - bxdw, dy - ydw, dz - (b+1)zdw + bxdy - ydx$. The corresponding right-invariant frame of vector fields is given by

$$W = \frac{\partial}{\partial w} + bx\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (b+1)z\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - bx\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

The corresponding system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = btu, \quad \dot{v} = tv, \quad \dot{s} = (1-b)uv + byut - bxvt + (b+1)st. \tag{50}$$

We shall return later to the two exceptional cases, which are among the most difficult.

$A_{4,11a}$: The brackets are:

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2ae_1, \quad [e_2, e_4] = ae_2 - e_3, \quad [e_3, e_4] = e_2 + ae_3(0 < a) \tag{51}$$

and the non-zero components of the curvature tensor are given by:

$$R^1_{441} = a^2, \quad 2R^1_{432} = a, \quad 4R^1_{342} = a, \quad 4R^1_{242} = 1, \quad 4R^1_{234} = a, 4R^3_{443} = a^2 - 1,$$
$$4R^1_{343} = 1, \quad 2R^2_{443} = a, \quad 4R^2_{442} = a^2 - 1, \quad 2R^3_{424} = a.$$

Just as in case $A_{4,7}$ conditions (18) and (19) entail that g_{ij} is singular whatever the value of a. Indeed (19) implies the following conditions:

$$u^q g_{q1} = u^q g_{q2} = u^q g_{q3} = 0 \tag{52}$$

whereas (18) implies

$$2ag_{12} - (3a^2+1)g_{13} = (3a^2+1)g_{13} = g_{11} = g_{22} + g_{33} = 0. \tag{53}$$

It easily follows from (53) that $g_{12} = g_{13}$ are zero whatever the value of a and hence also g_{11} and g_{14} are zero. A typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^{2aw} & -e^{aw}(x\sin w + y\cos w) & e^{aw}(x\cos w - y\sin w) & z \\ 0 & e^{aw}\cos w & e^{aw}\sin w & ax+y \\ 0 & -e^{aw}\sin w & e^{aw}\cos w & ay-x \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and a basis for the Maurer-Cartan one-form dSS^{-1} is given by $dw, dx-(ax+y)dw, dy+(x-ay)dw, dz+(x-ay)dx+(ax+y)dy-2azdw$. The corresponding vector fields are given by

$$W=\frac{\partial}{\partial w}+(ax+y)\frac{\partial}{\partial x}+(ay-x)\frac{\partial}{\partial y}+2az\frac{\partial}{\partial z},\quad X=\frac{\partial}{\partial x}+(ay-x)\frac{\partial}{\partial z},$$

$$Y=\frac{\partial}{\partial y}-(ax+y)\frac{\partial}{\partial z},\quad Z=\frac{\partial}{\partial z}.$$

The corresponding system of geodesic equations is given by

$$\dot{t}=0,\quad \dot{u}=(au+v)t,\quad \dot{v}=(av-u)t,\quad \dot{s}=-u^2-v^2+(a^2+1)(yu-xv)t+2ast. \tag{54}$$

The next class of algebras that we consider are such that conditions (18) and (19) do not entail that the matrix g_{ij} is singular and that in addition have trivial center.
$A_{4,2a}$: The Lie algebra relations are:

$$[e_1,e_4]=ae_1,\quad [e_2,e_4]=e_2,\quad [e_3,e_4]=e_2+e_3(a\neq 0). \tag{55}$$

A typical element S of the Lie group associated to the Lie algebra is given by

$$S=\begin{bmatrix} e^{aw} & 0 & 0 & x\\ 0 & e^w & we^w & y\\ 0 & 0 & e^w & z\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

A basis for right-invariant Maurer-Cartan one-form dSS^{-1} is given by $dw, dx-axdw, dy-(y+z)dw, dz-zdw$. The corresponding vector fields are given by

$$W=\frac{\partial}{\partial w}+ax\frac{\partial}{\partial x}+(y+z)\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\quad X=\frac{\partial}{\partial x},\quad Y=\frac{\partial}{\partial y},\quad Z=\frac{\partial}{\partial z}.$$

According to Proposition 5.4 the corresponding system of geodesic equations is given by

$$\dot{u}=atu,\quad \dot{v}=t(v+s),\quad \dot{s}=st,\quad \dot{t}=0. \tag{56}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta=\begin{bmatrix} adw & 0 & 0 & adx\\ 0 & dw & dw & dy+dz\\ 0 & 0 & dw & dz\\ 0 & 0 & 0 & 0\end{bmatrix},\quad 4\Omega=\begin{bmatrix} 0 & 0 & 0 & a^2dwdx\\ 0 & 0 & 0 & dw(dy+2dz)\\ 0 & 0 & 0 & dwdz\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

We see that the curvature tensor has essentially only the following non-zero components

$$4R^1_{441}=a^2,\quad 4R^2_{442}=1,\quad 2R^2_{443}=1,\quad 4R^3_{443}=1. \tag{57}$$

Conditions (18) and (19) entail that g_{ij} satisfies the conditions

$$g_{1q}u^q=g_{2q}u^q=g_{3q}u^q=g_{22}=(a^2-1)g_{12}=(a^2-1)g_{13}-2g_{12}=0.$$

We now distinguish three cases according as a is $1,-1$ or any other non-zero value. In the latter case the solution to (18) and (19) is four-dimensional. We can now invoke Theorem 5.5 using the ideals generated by e_1 and e_2, e_3 corresponding to the systems in the variables w, y, z and w, x, respectively. We define the following first integrals: $\alpha = \frac{e^{-w}(v-ws)}{t}, \beta = \frac{v}{t} - (y+z), \gamma = \frac{s}{t} - z, \delta = \frac{se^{-w}}{t}, \rho = \frac{u}{t} - ax, \sigma = \frac{ue^{-aw}}{t}$. Then the general Hessian for system $A4.2a(a^2 \neq 1)$ is given by

$$\begin{aligned} g_{ij} &= F \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & -\frac{1}{t} \\ 0 & \frac{1}{s} & -\frac{v}{s^2} & 0 \\ 0 & -\frac{1}{t} & 0 & \frac{v}{t^2} \end{bmatrix} + (\alpha F_\delta + \beta F_\gamma + G) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & -\frac{1}{t} \\ 0 & 0 & -\frac{1}{t} & \frac{s}{t^2} \end{bmatrix} \\ &+ H \begin{bmatrix} \frac{1}{u} & 0 & 0 & -\frac{1}{t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{t} & 0 & 0 & \frac{u}{t^2} \end{bmatrix} + K \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where F and G are arbitrary functions of γ and δ, H is an arbitrary function of ρ and σ and K is an arbitrary function of t, respectively. We now consider the exceptional cases where a is 1 or -1. When $a = -1$, the algebraic solution for g_{ij} is given by

$$g_{ij} = \lambda \begin{bmatrix} 1 & 0 & 0 & -\frac{u}{t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{u}{t} & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{t}{s} & 1 \\ 0 & -\frac{t}{s} & 0 & \frac{v}{s} \\ 0 & 1 & \frac{v}{s} & 0 \end{bmatrix} + \rho \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{s}{t} \\ 0 & 0 & -\frac{s}{t} & 0 \end{bmatrix}$$

$$+ \sigma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \tau \begin{bmatrix} 0 & -s/t & 1 & -\frac{s}{t} \\ -s/t & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ -\frac{s}{t} & 1 & -\frac{u}{t} & 0 \end{bmatrix}.$$

The ODE conditions (11) are given by

$$\dot{\lambda} - t\lambda = 0, \quad \dot{\mu} = 0, \quad \dot{\rho} - \frac{t^2}{s}\mu + t\rho = 0, \quad \dot{\sigma} - \frac{u^2}{t}\lambda + (2v+s)\mu - \frac{s^2}{t}\rho = 0, \quad \dot{\tau} = 0.$$

The solution to the above system is:

$$\lambda = Ke^w, \quad \mu = L, \quad \rho = \frac{wt}{s}L + Me^{-w}, \quad \tau = R,$$

$$\sigma = -\frac{xue^w}{t}K - (2y + 2z - \frac{sw}{t})L + \frac{zs}{t}e^{-w}M + N,$$

and so g becomes:

$$\begin{aligned} g_{ij} &= K \begin{bmatrix} \frac{t^2}{u} & 0 & 0 & -t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t & 0 & 0 & u \end{bmatrix} + R \begin{bmatrix} 0 & 0 & 1 & -s/t \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ -s/t & 0 & -\frac{u}{t} & 0 \end{bmatrix} + M \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{t^2}{s} & -t \\ 0 & 0 & -t & s \end{bmatrix} \\ &+ N \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + L \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & & t \\ 0 & 0 & \frac{t^2 v}{s^2} & -\frac{t^2}{s} \\ 0 & t & -\frac{t^2}{s} & -v \end{bmatrix}. \end{aligned}$$

The closure conditions are:

$$tK_t + uK_u + sK_s + 2K = 0, tR_t + uR_u + sR_s + R = 0, tL_t + sL_s + 2L = 0$$

$$tM_t + uM_u + vM_v + sM_s + 2M = 0, L_v = 0, L_u = 0, K_v = 0, R_v = 0, M_v = -L_s, N_v = 0$$

$$N_u = \frac{2s}{t^2}(uR_u + R), N_s = \frac{2u}{t^2}(sR_s + R)R_u = \frac{t^2}{u}K_s, R_s = \frac{t^2}{s}M_u.$$

For the moment we leave N to one side, noting that the integrability condition on N arising from the last two conditions above is identically satisfied by virtue of the condition satisfied by R. We use the first integrals introduced earlier but now with $a = -1$ so that we may write:

$$K = \frac{F(\gamma,\delta,\rho,\sigma)}{t^2}, \quad R = \frac{G(\gamma,\delta,\rho,\sigma)}{t}, \quad L = \frac{H(\gamma,\delta)}{t^2} \tag{58}$$

$$M = \frac{\alpha C(\gamma,\delta,\rho,\sigma) + \beta E(\gamma,\delta,\rho,\sigma) + B(\gamma,\delta,\rho,\sigma)}{t^2} \tag{59}$$

and besides the conditions on N there are three conditions involving K, M and R that remain to be satisfied. When they are imposed, one finds:

$$C + H_\gamma = E + H_\delta = 0, \sigma G_\sigma - F_\gamma = \sigma G_\rho - F_\delta = 0, \delta G_\gamma - B_\sigma = \delta G_\delta - B_\rho = 0. \tag{60}$$

Thus we see from (73) and (74) that G must satisfy the single integrability condition

$$G_{\gamma\rho} = G_{\delta\sigma}. \tag{61}$$

Finally we go back to N which is evidently seen to be of the form,

$$N = \frac{2suG}{t^3} + n(t) \tag{62}$$

noting that su is a first integral. In conclusion, the general Hessian is determined by the arbitrary functions L and n and also G which is subject to the single condition (61). We shall not discuss the case $a = 1$ since it is very similar to the preceeding one.
We end the discussion of this subcase with a specific Lagrangian that covers the three cases above, namely,

$$L = v(-w + ln(\frac{s}{t})) + \frac{e^{-w}s^2}{t} + \frac{e^{-aw}u^2}{t} + zt + t^2. \tag{63}$$

$A_{4,4}$: The Lie algebra relations are:

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3. \tag{64}$$

A typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^w & e^w w & e^w w^2/2 & x \\ 0 & e^w & e^w w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The one-forms $dw, dx - (x+y)dw, dy - (y+z)dw, dz - zdw)$ comprise a right-invariant coframe. The corresponding right-invariant frame of vector fields is given by

$$W = \frac{\partial}{\partial w} + (x+y)\frac{\partial}{\partial x} + (y+z)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \tag{65}$$

The corresponding system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = tu, \quad \dot{v} = t(u+v), \quad \dot{s} = t(v+s). \tag{66}$$

For later use we define the following first integrals of the geodesics: $\alpha = \frac{e^{-x}t}{u}$,

$$\beta = \frac{t-wu}{u}, \gamma = \frac{e^{-x}(s-xt)}{u}, \delta = \frac{s}{u} - (z+w), \zeta = \frac{v}{u} - (y+z), \eta = \frac{e^{-x}(v - xs + \frac{x^2 t}{2})}{u}.$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dy+dz & dx & dx & 0 \\ dz+dw & 0 & dx & dx \\ dw & 0 & 0 & dx \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dx(dy+2dz+dw) & 0 & 0 & 0 \\ dx(dz+2dw) & 0 & 0 & 0 \\ dxdw & 0 & 0 & 0 \end{bmatrix}.$$

We see that the curvature tensor has essentially only the following non-zero components

$$4R^2_{112} = 1, \quad 2R^2_{113} = 1, \quad 4R^3_{113} = 1, \quad 4R^2_{114} = 1, \quad 2R^3_{114} = 1, \quad 4R^4_{114} = 1. \tag{67}$$

Conditions (18) and (19) entail that g_{ij} satisfies the following condition:

$$g_{ij} = \begin{bmatrix} \lambda & t^3\mu & t^2\sigma & t\rho \\ t^3\mu & 0 & 0 & -t^2\mu \\ \sigma t^2 & 0 & -\mu t^2 & -\sigma tu + \mu stu \\ t\rho & -t^2\mu & -\sigma tu + \mu stu & \mu(tuv - s^2u) + \sigma su - \rho u \end{bmatrix}$$

The ODE conditions (11) are given by

$$\dot{\lambda} + (s+v)t^3\mu + (s+t)t^2\sigma + t^2\rho = 0, \quad \dot{\mu}t + 2\mu\dot{t} + ut\mu = 0, \quad t\dot{\rho} + \rho\dot{t} = 0, \quad t\dot{\sigma} + 2\dot{t}\sigma = 0. \tag{68}$$

The solution to these ODE conditions is given by $g_{ij} =$

$$M \begin{bmatrix} -\frac{v}{u} & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ 0 & 0 & -\frac{u}{t} & -\frac{su}{t^2} \\ 0 & -\frac{u}{t} & -\frac{su}{t^2} & \frac{tuv - s^2u}{t^3} \end{bmatrix} + S \begin{bmatrix} -\frac{s}{u} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \\ 0 & 0 & -\frac{u}{t} & \frac{su}{t^2} \end{bmatrix} + R \begin{bmatrix} -\frac{t}{u} + \frac{L}{R} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{u}{t} \end{bmatrix}$$

where L, M, R, S are first integrals.

After some rearrangement, the closure conditions turn out to be equivalent to:

$$L_s = L_t = L_v = 0, \quad uM_u + tM_t + M = 0, \quad sS_s + tS_t + uS_u + S = 0 \tag{69}$$

$$sR_s + tR_t + uR_u + vR_v + R = 0, \quad S_s = M_t, M_t = R_v, S_t = R_s \tag{70}$$

Using the first integrals introduced earlier we can write the solutions for L, M, R, S as

$$L = L(u), uM = a(\alpha, \beta), \quad uS = b(\alpha, \beta), \quad uR = c((\alpha, \beta, \gamma, \delta, \eta, \zeta)). \tag{71}$$

There are three closure conditions that remain to be satisfied. However, they already imply that

$$R_{vv} = 0, R_{sv} = 0, R_{sss} = 0.$$

It follows that we may write

$$uS = A(\alpha, \beta)\gamma + B(\alpha, \beta)\delta \tag{72}$$

$$uR = H(\alpha, \beta)\eta + J(\alpha, \beta)\zeta + C(\alpha, \beta)(\delta)^2 + D(\alpha, \beta)\delta + E(\alpha, \beta)(\gamma)^2 + F(\alpha, \beta)\gamma + G(\alpha, \beta). \tag{73}$$

In (72) and (73) $A, B, \ldots, J$ are arbitrary smooth functions of their arguments. If we substitute (72) and (173) into (69) these functions may be identified via the following equations:

$$uS = a_\alpha \gamma + a_\beta \delta, uR = a_\alpha \eta + a_\beta \zeta + a_{\beta,\beta}\frac{\delta^2}{2} + a_{\alpha,\alpha}\frac{\gamma^2}{2} + G(\alpha, \beta). \tag{74}$$

Thus the Hessian is parametrized by the functions a, G and L. A concrete Lagrangian is given by

$$L = (2s + u - v)ln(\frac{u}{t}) - 2ws - wv + \frac{v^2}{u} + t^2. \tag{75}$$

$A_{4,9}$ $(b = 0)$:We resume from the discussion above where we considered a generic value of b. The system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = 0, \quad \dot{v} = tv, \quad \dot{s} = uv + st. \tag{76}$$

The curvature tensor has essentially only the following non-zero components

$$4R^4_{123} = 1, \quad 4R^3_{113} = 1, \quad 4R^4_{213} = 1, \quad 4R^4_{114} = 1. \tag{77}$$

Conditions (18) and (19) entail that g_{ij} satisfies the following conditions:

$$u^q g_{q3} = u^q g_{q4} = g_{24} = g_{44} = g_{14} - g_{23} = 0. \tag{78}$$

The algebraic solution for g may be written as

$$g = \begin{bmatrix} \lambda & \mu & \sigma & -\frac{v\tau}{t} \\ \mu & \rho & -\frac{v\tau}{t} & 0 \\ \sigma & -\frac{v\tau}{t} & \frac{(uv-st)\tau}{tv} & \tau \\ -\frac{v\tau}{t} & 0 & \tau & 0 \end{bmatrix}.$$

The ODE conditions (11) are given by

$$\dot{\lambda} + v\sigma - \frac{sv\tau}{t} = 0, \quad \dot{\rho} = 0, \quad \dot{\mu} - \frac{v^2\tau}{t} = 0, \quad \dot{\sigma} = 0, \quad \dot{\tau} + t\tau = 0. \tag{79}$$

When (79) are integrated we find that a new solution for g is given by

$$g = P\begin{bmatrix} \frac{s}{t^2} - \frac{2uv}{t^3} & \frac{v}{t^2} & \frac{u}{t^2} & -\frac{1}{t} \\ \frac{v}{t^2} & 0 & -\frac{1}{t} & 0 \\ \frac{u}{t^2} & -\frac{1}{t} & -\frac{s}{v^2} & \frac{1}{v} \\ -\frac{1}{t} & 0 & \frac{1}{v} & 0 \end{bmatrix} + Ne^{-w}\begin{bmatrix} \frac{2v^2}{t^3} & 0- & \frac{2v}{t^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2v}{t^2} & 0 & \frac{2}{t} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} K & L & 0 & 0 \\ L & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The closure conditions (12) turn out to be

$$K = K(t, u), \quad L = L(t, u), \quad M = M(t, u), \quad M_t - L_u = 0, \quad K_u - L_t = 0 \tag{80}$$

$$tN_t + uN_u + vN_v + sN_s = 0, N_s - P_v = 0, P_t - N_u = 0, vP_v + tP_t = 0, P_u = P_s = 0. \tag{81}$$

We note that the following seven functions constitute a maximally functionally independent set of linear first integrals: $t, u, e^{-w}v, xu - wu, e^{-w}(s - vx), yt - v, s - yu - zt$. For P we note that it is annihilated by $\frac{\partial}{\partial u}, \frac{\partial}{\partial s}$, $t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$ and the geodesic vector field Γ where

$$\Gamma = t\frac{\partial}{\partial w} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + s\frac{\partial}{\partial z} + vt\frac{\partial}{\partial v} + (uv + st)\frac{\partial}{\partial s}. \tag{82}$$

The general solution for P is given by

$$P = P(\alpha, \beta) \tag{83}$$

where

$$\alpha = y - \frac{v}{t}, \beta = e^{-w}\frac{v}{t}. \tag{84}$$

As for N, at the outset it is subject to three conditions. Two of them are embodied in (81) and the third arises from the fact that N is a first integral and so is annihilated by the geodesic vector field Γ. In addition to Γ and $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial s}$ define the following two differential operators:

$$\Delta = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + s\frac{\partial}{\partial s}, T = t\frac{\partial}{\partial u} + v\frac{\partial}{\partial s}. \tag{85}$$

The general solution for N involves four arbitrary functions. Finally there remain the two conditions relating N and P in (81) and when they are imposed one finds the following general solution:

$$N = (z + \frac{(uv - st)}{t^2})\frac{\partial P}{\partial \alpha} + \frac{e^{-w}(uv - st + xtv)}{t^2}\frac{\partial P}{\partial \beta} + R(\alpha, \beta) \tag{86}$$

where R is an arbitrary function. Together (83) and (86) furnish a complete solution for the Hessian of the Lagrangian that we are seeking. A particular Lagrangian is given by

$$L = sln(\frac{v}{t}) - \frac{uv}{t} + tu + zt - xv. \tag{87}$$

$A_{4,9}$ $(b = -\frac{1}{2})$:As in the preceding example we resume from the discussion above where we considered a generic value of b. We choose for the parametrization of the group

$$S = \begin{bmatrix} e^w & xe^w & ye^w & z \\ 0 & e^{2w} & 0 & 2ye^w \\ 0 & 0 & e^{-w} & xe^{-w} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the order (x, y, z, w) the geodesic equations are:

$$\dot{u} = tu, \quad \dot{v} = -2tv, \quad \dot{s} = e^w(3uv + 4yut - xvt) + st, \quad \dot{t} = 0. \tag{88}$$

We record for later use the following first integrals: $\alpha = \frac{e^{-w}u}{t}, \beta = \frac{u-xt}{t}, \gamma = \frac{e^{2w}v}{t}$, $\delta = \frac{v+2ty}{t}, \rho = \frac{s-zt-e^w(2yu+xv)}{t}, \sigma = \frac{e^{-w}s-yu-2xv-3xyt}{t}$. We see that the curvature tensor has essentially only the following non-zero components

$$2R^3_{124} = -e^w, 4R^3_{241} = e^w, R^2_{442} = 1, 4R^1_{441} = 1, 4R^3_{412} = e^w, 4R^3_{442} = 3xe^w, 4R^3_{443} = 1.$$

Conditions (18) and (19) entail that g_{ij} satisfies the following conditions:

$$u^q g_{q1} = u^q g_{q2} = u^q g_{q3} = g_{23} = g_{33} = 3tg_{12} - e^w tg_{34} + e^w(2u + 3xt)g_{13} = 0. \tag{89}$$

The algebraic solution for g may be written as

$$g = \lambda \begin{bmatrix} t & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & 0 & 0 & \frac{u^2}{t} \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 0 & -v \\ 0 & 0 & 0 & 0 \\ 0 & -v & 0 & \frac{v^2}{t} \end{bmatrix}$$

$$
+\rho\begin{bmatrix} 0 & -tv(u+xt) & e^{-w}t^2v & v(v(u+xt)-e^{-w}st) \\ -tv(u+xt & tu(u+xt) & 0 & 0 \\ e^{-w}t^2v & 0 & 0 & -e^{-w}tu \\ v(v(u+xt)-e^{-w}st) & 0 & -e^{-w}tu & 0 \end{bmatrix}.
$$

The ODE conditions (11) are given by

$$
\begin{aligned} &\dot{\lambda}+t\lambda+(3tv^2+4yt^2v)\rho=0, v\dot{\rho}+\rho\dot{v}=0, \dot{\mu}-2t\mu=0, \\ &\dot{\sigma}+4\rho(xtuv^2-4ytu^2v-u^2v^2-stuve^{-w})=0. \end{aligned} \tag{90}
$$

When (90) are integrated we find that a new solution for g is given by

$$
g=Le^{-w}\begin{bmatrix} t & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & 0 & 0 & \frac{u^2}{t} \end{bmatrix}+S\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}+Me^{-w}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 0 & -v \\ 0 & 0 & 0 & 0 \\ 0 & -v & 0 & \frac{v^2}{t} \end{bmatrix}
$$

$$
+R\begin{bmatrix} t^2(4yt+v) & t^2(u+xt) & \frac{-t^3}{e^w} & -t(2uv+xtv+4ytu-\frac{st}{e^w}) \\ t^2(u+xt) & 0 & 0 & -t(xut+u^2) \\ \frac{-t^3}{e^w} & 0 & 0 & \frac{t^2u}{e^w} \\ -t(2uv+xtv+4ytu-\frac{st}{e^w}) & -t(xut+u^2) & \frac{t^2u}{e^w} & 4ytu^2+3u^2v+2xtuv-\frac{2stu}{e^w} \end{bmatrix}.
$$

The closure conditions turn out to be equivalent to:

$$
S_u=0, S_v=0, S_s=0, R_s=0, R_v=0, K_s=0, K_u=0, L_s+R_u=0, \frac{tL_v}{e^w}-(u+xt)R_u=0 \tag{91}
$$

$$
tL_t+uL_u+vL_v+sL_s=0, vK_v+tK_t+2K=0, uR_u+tR_t+2R=0. \tag{92}
$$

S is a first integral and is a function of t only. The functions K, L and R have to be of the following form, where we make use of the first integrals introduced earlier:

$$
K=\frac{F(\gamma,\delta)}{t^2}, L=\frac{G(\alpha,\beta,\gamma,\delta,\sigma)}{t^2}, R=\frac{H(\alpha,\beta)}{t^2} \tag{93}
$$

where F, G and H are arbitrary functions. It remains to satisfy the two conditions that relate L and R. A short calculation shows that H must be constant, that G can depend only on α and β and that F involves only γ and δ and that these conditions furnish the most general Hessian. A concrete Lagrangian is given by

$$
L=\frac{e^{-w}u^2}{t}+\frac{e^{2w}v^2}{t}-\frac{2e^{-w}su}{t}+\frac{(4yt+v)u^2}{t^2}+\frac{2xuv}{t}+t^2+2xyu+ws+zt. \tag{94}
$$

$A_{4,5ab}$: The Lie algebra relations are:

$$
[e_1,e_4]=e_1, \quad [e_2,e_4]=ae_2, \quad [e_3,e_4]=be_3 (ab\neq 0, -1\leq a,b\leq 1). \tag{95}
$$

A typical element S of the Lie group associated to the Lie algebra is given by

$$
S=\begin{bmatrix} e^w & 0 & 0 & x \\ 0 & e^aw & 0 & y \\ 0 & 0 & e^bw & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The one-forms $dw, dx - xdw, dy - aydw, dz - bzdw$ comprise a right-invariant coframe. The corresponding right-invariant frame of vector fields is given by

$$W = \frac{\partial}{\partial w} + x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + bz\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}, \tag{96}$$

and the system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = tu, \quad \dot{v} = atv, \quad \dot{s} = bst. \tag{97}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dx & dw & 0 & 0 \\ ady & 0 & adw & 0 \\ bdz & 0 & 0 & bdw \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dwdx & 0 & 0 & 0 \\ a^2dwdy & 0 & 0 & 0 \\ b^2dwdz & 0 & 0 & 0 \end{bmatrix}.$$

The curvature tensor has essentially only the following non-zero components:

$$4R^2_{112} = 1, \quad 4R^3_{113} = a^2, \quad 4R^4_{114} = b^2. \tag{98}$$

There is a variety of cases for the solution to conditions (18) and (19) depending on the values of a and b. Indeed it is possible to reduce to the following five cases: $(1) a^2 \neq 1, b^2 \neq 1, a^2 \neq b^2$, $(2) a = 1, b \neq \pm 1$, $(3) a = -1, b \neq \pm 1$ $(4) a = 1, b = 1$,$(5) a = 1, b = -1$. The first three of these cases can be handled by applying Theorem 5.5. In the first case we use the extension of Theorem 5.5 in which case there are three "overlapping" submersions. In the second and third cases we have submersions onto three and two-dimensional systems and quote the results from [12]. The Hessians are respectively:

$$\begin{aligned} g_{ij} &= \frac{K(\frac{e^{-w}u}{t}, x-\frac{u}{t})}{tu} \begin{bmatrix} \frac{u^2}{t} & -u & 0 & 0 \\ -u & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{L(\frac{e^{-w}v}{t}, y-\frac{v}{t})}{tv} \begin{bmatrix} \frac{v^2}{t} & 0 & -v & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & t & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \frac{M(\frac{e^{-w}s}{t}, z-\frac{s}{t})}{ts} \begin{bmatrix} 2 & 0 & 0 & -s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -s & 0 & 0 & \frac{s^2}{t} \end{bmatrix} + \begin{bmatrix} N(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

K, L, M and N being arbitrary functions of their arguments;

$$\begin{aligned} g_{ij} &= \frac{e^{-w}k(\bar{x},\bar{y},\bar{u},\bar{v})}{t^2} \begin{bmatrix} \frac{u^2}{t} & -u & 0 & 0 \\ -u & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{e^{-w}l(\bar{x},\bar{y},\bar{u},\bar{v})}{t^2} \begin{bmatrix} \frac{v^2}{t} & 0 & -v & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & t & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \frac{e^{-w}m(\bar{x},\bar{y},\bar{u},\bar{v})}{t^2} \begin{bmatrix} \frac{2uv}{t} & -v & -u & 0 \\ -v & 0 & t & 0 \\ -u & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{M(\frac{e^{-w}s}{t}, z-\frac{s}{t})}{ts} \begin{bmatrix} 2 + \frac{tsN}{M} & 0 & 0 & -s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -s & 0 & 0 & \frac{s^2}{t} \end{bmatrix} \end{aligned}$$

where

$$\bar{x} = x - \frac{u}{t}, \quad \bar{y} = y - \frac{v}{t}, \quad \bar{u} = \frac{e^{-w}u}{t}, \quad \bar{v} = \frac{e^{-w}v}{t}, \tag{99}$$

M and N are arbitrary functions and k, l and m satisfy the following system of involutive PDE $m_{\bar{x}} = k_{\bar{y}}, m_{\bar{y}} = l_{\bar{x}}, m_{\bar{u}} = k_{\bar{x}}, m_{\bar{v}} = l_{\bar{u}}$.

In the fourth case we may as well consider the following system in dimension $n+1$,

$$\dot{u}^i = u^i t, \quad \dot{t} = 0 \tag{100}$$

$n = 3$ being the value of particular interest. We find that the connection and curvature matrices are, respectively,

$$2\Theta = \begin{bmatrix} dt & 0 & 0 & ... & dx^1 \\ 0 & dt & 0 & ... & dx^2 \\ \\ 0 & 0 & ... & dt & dx^n \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & ... & dtdx^1 \\ 0 & dt & 0 & ... & dtdx^2 \\ \\ 0 & 0 & ... & 0 & dtdx^n \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}.$$

The non-zero components of R^i_{jkl} are given by $R^i_{n+1,n+1,i} = \frac{1}{4}$ where $1 \leq i \leq n$. The solution for g_{ij} is

$$g_{ij} = \begin{bmatrix} h_{ij} & -\frac{u^k}{t} h_{ik} \\ -\frac{u^k}{t} h_{jk} & a \end{bmatrix}$$

where h_{ij} is an $n \times n$ symmetric matrix and a is a function. The ODE conditions (11) on g_{ij} are

$$\dot{g}_{ij} + t g_{ij} = 0, \quad t\dot{a} = g_{km} u^k u^m. \tag{101}$$

Using the fact that $u^i - x^i t$ and $e^{-w} u^i$ are first integrals the solution to (132) is given by

$$g_{ij} = e^{-w} \begin{bmatrix} tC_{ij} & -u^k C_{ik} \\ -u^k C_{jk} & \frac{C_{ij} u^i u^j}{t} + e^w K \end{bmatrix}$$

where C_{ij} is a matrix of first integrals and K is a first integral. The closure conditions are given by

$$\frac{\partial K}{\partial u^i} = 0, \quad \frac{\partial C_{ij}}{\partial u^k} = \frac{\partial C_{ik}}{\partial u^j} u^k \frac{\partial C_{ij}}{\partial u^k} + t \frac{\partial C_{ij}}{\partial t} + 2C_{ij} = 0. \tag{102}$$

It follows that K is a function of t only. To continue, define the first integrals α^i and β^i to be $\frac{u^i - x^i t}{t}$ and $\frac{e^{-w} u^i}{t}$, respectively, and define c_{ij} to be $\frac{C_{ij}}{t^2}$. The conditions on C_{ij} turn into

$$\frac{\partial c_{ij}}{\partial \alpha^k} = \frac{\partial c_{ik}}{\partial \alpha^j}, \quad \frac{\partial c_{ij}}{\partial \beta^k} = \frac{\partial c_{ik}}{\partial \beta^j} \tag{103}$$

It appears to be impossible to take the calculation further in explicit terms. It is natural to ask therefore if the system comprised of (102-103) is involutive. The system (102-103) as it stands is not in involution. For $n = 3$ the weighted character count in Cartan's test comes out to 50, whereas the number of independent second order conditions is given by $126 - 81 = 45$. We must therefore repeat the test on the prolonged system, which involves working also with third order derivatives, of which there are 336. It would seem therefore to be practically impossible to count the number of independent conditions in such large dimensions. Of course the Hessians that we found for the reduced systems prior to (99) are valid in the two special cases where $a^2 = 1$, so we have an abundance of such Hessians. We shall not discuss the case where $a = -1$ which quite similar to the case $a = 1$, except to give the geodesics which are

$$\dot{u}^i = u^i t, \quad \dot{s} = -st, \quad \dot{t} = 0. \tag{104}$$

In all these various cases we obtain a concrete Lagrangian as follows:

$$L = \frac{e^{-w}u^2 + e^{-aw}v^2 + e^{-bw}s^2}{2t} + t^2. \tag{105}$$

$A_{4,6ab}$: The Lie algebra relations are:

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = be_2 - e_3, \quad [e_3, e_4] = e_2 + be_3 (a \neq 0, b \geq 0). \tag{106}$$

A typical element S of the Lie group associated to the Lie algebra is given by

$$S = \begin{bmatrix} e^{aw} & 0 & 0 & x \\ 0 & e^{bw}cos(t) & e^{bw}sin(t) & y \\ 0 & -e^{bw}sin(t) & e^{bw}cos(t) & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The one-forms $dw, dx - axdw, dy - (by + z)dw, dz + (y - bz)dw$ comprise a right-invariant coframe. The corresponding right-invariant frame of vector fields is given by

$$W = \frac{\partial}{\partial w} + ax\frac{\partial}{\partial x} + (by + z)\frac{\partial}{\partial y} + (bz - y)\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \tag{107}$$

According to Proposition 5.4 the corresponding system of geodesic equations is given by

$$\dot{t} = 0, \ \dot{u} = atu, \ \dot{v} = t(bv + s), \ \dot{s} = t(bs - v). \tag{108}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ adx & adw & 0 & 0 \\ bdy + dz & 0 & bdw & dw \\ bdz - dy & 0 & -dw & bdw \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a^2dwdx & 0 & 0 & 0 \\ (b^2 - 1)dwdy + 2bdwdz & 0 & 0 & 0 \\ (b^2 - 1)dwdz - 2bdwdy & 0 & 0 & 0 \end{bmatrix}.$$

The curvature tensor has essentially only the following non-zero components

$$4R^2_{112} = a^2, \quad 4R^3_{113} = b^2 - 1, \quad 2R^3_{114} = b, \quad 4R^4_{114} = b^2 - 1, \quad 2R^4_{113} = -2b. \tag{109}$$

After making a permutation of coordinates, the geodesics have the form:

$$\dot{u} = t(bu + v), \quad \dot{v} = t(bv - u), \quad \dot{s} = ats, \quad \dot{t} = 0. \tag{110}$$

We suppose first of all that $b = 0$, in which case g_{ij} is given by

$$\begin{aligned} g_{ij} \ &= M \begin{bmatrix} t^2u & t^2v & 0 & -t(u^2 + v^2) \\ t^2v & -t^2u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t(u^2 + v^2) & 0 & 0 & u(u^2 + v^2) \end{bmatrix} + P \begin{bmatrix} t^2 & 0 & 0 & -tu \\ 0 & t^2 & 0 & -tv \\ 0 & 0 & 0 & 0 \\ -tu & -tv & 0 & u^2 + v^2 + \frac{F}{P} \end{bmatrix} \\ &+ N \begin{bmatrix} -t^2v & t^2u & 0 & 0 \\ t^2u & t^2v & 0 & -t(u^2 + v^2) \\ 0 & 0 & 0 & 0 \\ 0 & -t(u^2 + v^2) & 0 & v(u^2 + v^2) \end{bmatrix} + \tfrac{1}{tv}Q(\tfrac{e^{-w}s}{t}, \tfrac{s}{t} - y) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{v^2}{t} & -v \\ 0 & 0 & -v & t \end{bmatrix} \end{aligned}$$

where F, M, N and P are exactly the same as in (43) and Q is an arbitrary smooth function of its arguments. On the other hand if we suppose that $b \neq 0$, then g_{ij} is given by

$$\begin{bmatrix} \frac{(uK+vL)t}{u^2+v^2} & \frac{(vK-uL)t}{u^2+v^2} & 0 & -K \\ \frac{(vK-uL)t}{u^2+v^2} & \frac{(uK+vL)t}{u^2+v^2} & 0 & L \\ 0 & 0 & 0 & 0 \\ -K & L & 0 & \frac{(uK-vL)t}{t} + F(t) \end{bmatrix} + \frac{1}{tv}Q(\frac{e^{-w}s}{t}, \frac{s}{t} - y) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{v^2}{t} & -v \\ 0 & 0 & -v & t \end{bmatrix} \tag{111}$$

where F and Q are arbitrary smooth functions of their arguments and K and L are the real and imaginary parts of an arbitrary complex function in the variables $\frac{u+iv}{w} - (b-i)(x+iy)$ and $\frac{e^{(-b+i)w}(u+iv)}{w}$. We obtain a concrete Lagrangian as

$$L = \frac{e^{-aw}s^2}{t} + \frac{e^{-bw}}{2t}((u^2-v^2)cos(w) - 2sin(w)uv) + t^2 \tag{112}$$

which covers both the cases where b is and is not zero.
$A_{4,12}$: The Lie algebra relations are:

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1. \tag{113}$$

Note that over $\mathbf{C}$ this Lie algebra is decomposable: it is a direct sum of two non-abelian algebras with bases consisting of $e_1 + ie_2, e_3 - ie_4$ and $e_1 - ie_2, e_3 + ie_4$. A typical element S of the Lie group is given by

$$S = \begin{bmatrix} e^x\cos(y) & e^x\sin(y) & z & w \\ -e^x\sin(y) & e^x\cos(y) & w & -z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right-invariant one-forms are given by $dx, dy, dz - zdx - wdy, dw + zdy - wdx$. The corresponding right-invariant frame of vector fields is given by

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \quad Y = \frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w}, \quad Z = \frac{\partial}{\partial z}, \quad W = \frac{\partial}{\partial w}. \tag{114}$$

The corresponding system of geodesic equations is given by

$$\dot{u} = 0, \quad \dot{v} = 0, \quad \dot{s} = us + vt, \quad \dot{t} = ut - sv. \tag{115}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ dz & dw & dx & dy \\ dw & -dz & -dy & dx \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ dxdz + dydw & dxdw - dydz & 0 & 0 \\ dxdw - dydz & -(dxdz + dydw) & 0 & 0 \end{bmatrix}.$$

The curvature tensor has essentially only the following non-zero components:

$$R^3_{113} = R^3_{124} = R^3_{214} = R^3_{232} = R^4_{132} = R^4_{231} = R^4_{231} = R^4_{242} = \frac{1}{4}. \tag{116}$$

In this case one can proceed with the usual algorithm starting from (18) and (19). However, it is simpler by far to introduce complex variables as follows:

$$\zeta = x + iy, \quad \eta = w + iz \tag{117}$$

As such the geodesic equations are then the real and imaginary parts of

$$\ddot{\zeta} = 0, \quad \ddot{\eta} = \dot{\zeta}\dot{\eta}. \tag{118}$$

We solve the algebraic and ODE conditions in the usual way and deduce that the Hessian g_{ij} must be of the form

$$g_{ij} = S\begin{bmatrix} s & t & -u & -v \\ t & -s & v & -u \\ -u & v & \frac{s(u^2-v^2)-2tuv}{s^2+t^2} & \frac{t(u^2-v^2)+2suv}{s^2+t^2} \\ -v & -u & \frac{t(u^2-v^2)+2suv}{s^2+t^2} & -\frac{s(u^2-v^2)-2tuv}{s^2+t^2} \end{bmatrix} + \begin{bmatrix} L & M & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+T\begin{bmatrix} t & -s & v & -u \\ -s & -t & u & v \\ v & u & -\frac{t(u^2-v^2)-2suv}{s^2+t^2} & \frac{s(u^2-v^2)-2tuv}{s^2+t^2} \\ -u & v & \frac{s(u^2-v^2)-2tuv}{s^2+t^2} & \frac{t(u^2-v^2)+2suv}{s^2+t^2} \end{bmatrix}$$

for certain functions S, T, L, M and N. The closure conditions are as follows after performing some manipulation:

$$S_t - T_s = S_s + T_t = S_u - T_v = S_v + T_u = 0 \tag{119}$$

$$uS_u + vS_v + sS_s + tS_t + 2S = uT_u + vT_v + sT_s + tT_t + 2T = 0 \tag{120}$$

$$L_s = L_t = M_s = M_t = N_s = N_t = L_v - M_u = M_v - N_u = 0 \tag{121}$$

Comparing with the real case [22], we may write

$$g_{\alpha\beta} = F(\frac{e^{x+iy}(\ \dot{w} + i\ \dot{z})}{\dot{x} + i\ \dot{y}}, \quad w + iz - \frac{\dot{w} + i\ \dot{z}}{\dot{x} + i\ \dot{y}}). \tag{122}$$

We obtain all possible real Lagrangians by choosing F as an arbitrary holomorphic function in (122). Its real and imaginary parts will then yield real Hessians to which can be added an arbitrary Hessian in the variables u and v. One such example is given by:

$$L = \frac{e^{-x}((scosy - tsiny)(su - vt) - (tcosy + ssiny)(tu + sv)}{u^2 + v^2} + u^2 - v^2. \tag{123}$$

The final class of algebras that we consider are such that the matrix g_{ij} in (18) and (19) is non-singular but whose center is non-trivial.
$A_{4,1}$: The non-zero Lie brackets are given by

$$[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2. \tag{124}$$

We see that e_1 spans the center of $A4,1$. A typical element of the group may be written as

$$S = \begin{bmatrix} 1 & w & w^2/2 & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the right-invariant forms are $dz, dw, dx - ydw, dy - zdw$ and that the dual vector fields are

$$W = \frac{\partial}{\partial w} + y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \tag{125}$$

The corresponding system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = tv, \quad \dot{v} = st, \quad \dot{s} = 0. \tag{126}$$

In this example we shall take a different tack. We shall apply several coordinate transformations in order to simplify the form of the geodesic equations. First of all define

$$w' = w, \quad x' = x, \quad y' = y - \frac{zw}{2}, \quad z' = \frac{z}{2} \tag{127}$$

so that the geodesic equations become

$$\dot{t} = 0, \quad \dot{u} = t(v + ws + zt), \quad \dot{v} = 0, \quad \dot{s} = 0 \tag{128}$$

on dropping the primes. Again define

$$w' = w, \quad x' = x - \frac{wy}{2}, \quad y' = y, \quad z' = z \tag{129}$$

so that the geodesic equations become

$$\dot{t} = 0, \quad \dot{u} = t(ws + zt), \quad \dot{v} = 0, \quad \dot{s} = 0 \tag{130}$$

again on dropping primes. At this point we note that the geodesic system, and hence the connection, factors as a product of one and three-dimensional systems. Thus we define the transformation

$$w' = y, \quad x' = x - \frac{zw^2}{2}, \quad y' = z, \quad z' = w \tag{131}$$

after which the geodesic equations become

$$\dot{t} = 0, \quad \dot{u} = -zvs, \quad \dot{v} = 0, \quad \dot{s} = 0. \tag{132}$$

We now devote attention to the connection (not a canonical connection) in the variables x, y, z. The connection and curvature matrices are given by

$$2\Theta = \begin{bmatrix} 0 & zdz & zdy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 2\Omega = \begin{bmatrix} 0 & 0 & -dydz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The algebraic solution corresponding to (18) and (19) is given by

$$g_{ij} = \begin{bmatrix} 0 & s\lambda & -v\lambda \\ s\lambda & \rho & \mu \\ -v\lambda & \mu & \sigma \end{bmatrix}.$$

The ODE conditions (11) are given by

$$\dot{\lambda} = 0, \quad \dot{\mu} = 0, \quad \dot{\rho} - zs^2\lambda = 0, \quad \dot{\sigma} + zs^2\lambda = 0. \tag{133}$$

The ODE may be integrated to give the following solution for g

$$g_{ij} = P \begin{bmatrix} 0 & s & -v \\ s & -\frac{us}{v} & 0 \\ -v & 0 & \frac{uv}{s} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & K & M \\ 0 & M & N \end{bmatrix}$$

The closure conditions are:

$$P_u = 0, \quad u(sP_s + P) + vM_v - vK_s = 0, \quad s(P_s + P) + s = 0 \tag{134}$$

$$s(vP_v + P) + sN_v - sM_s = 0, \quad sP_s + P - M_u = 0, \quad sP_s + vP_v + 2P = 0. \tag{135}$$

We obtain the following formula for the Hessian g_{ij}

$$g_{ij} = \frac{F(\alpha,\beta)}{v^2}\begin{bmatrix} 0 & s & -v & 0 \\ s & -\frac{2us}{v} & u & 0 \\ -v & u & \frac{uv}{s} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{s}{v^3}(\frac{\partial F}{\partial \alpha} + y\frac{\partial F}{\partial \beta})\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{us}{v} & u & 0 \\ 0 & u & \frac{uv}{s} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_{vv} & \lambda_{vs} & \lambda_{vt} \\ 0 & \lambda_{vs} & \lambda_{ss} & \lambda_{st} \\ 0 & \lambda_{vt} & \lambda_{st} & \lambda_{tt} \end{bmatrix}$$

where λ is an arbitrary smooth function of v, s, t. In order to obtain a concrete Lagrangian we specialize to the case where the quantity $sF' + vF$ is zero and we find

$$L = 2uln(v/s) + z^2v + s^2. \tag{136}$$

We can obtain a Lagrangian for the original system by successively applying the inverses of the various coordinate transformations that changed (159) into (165), which gives:

$$L' = (2u - vw - yt)ln(\frac{s}{t}) + \frac{w^2s}{2} + t^2 + (v - \frac{zt}{2} - \frac{ws}{2})^2. \tag{137}$$

$A_{4,3}$: This case is very similar to the previous one. The non-zero Lie brackets are given by

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2. \tag{138}$$

We see that e_2 spans the center. A typical element of the group may be written as

$$S = \begin{bmatrix} e^w & 0 & 0 & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A basis of right-invariant forms is given by $dz, dw, dx - xdw, dy - zdw$ and the dual vector fields are given by

$$W = \frac{\partial}{\partial w} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \tag{139}$$

The corresponding system of geodesic equations is given by

$$\dot{t} = 0, \quad \dot{u} = tu, \quad \dot{v} = st, \quad \dot{s} = 0. \tag{140}$$

As in the last example we apply a coordinate transformation in order to simplify the form of the geodesic equations. Define

$$w' = w, \quad x' = x, \quad y' = y - \frac{zw}{2}, \quad z' = z \tag{141}$$

so that the geodesic equations become

$$\dot{t} = 0, \quad \dot{u} = tu, \quad \dot{v} = 0, \quad \dot{s} = 0 \tag{142}$$

on dropping the primes. We can now write down a general Hessian in the new coordinates as:

$$g_{ij} = \frac{K(\frac{e^{-w}u}{t}, x - \frac{u}{t})}{tu}\begin{bmatrix} u & -t & 0 & 0 \\ -t & \frac{u^2}{t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_{vv} & \lambda_{vs} & \lambda_{vt} \\ 0 & \lambda_{vs} & \lambda_{ss} & \lambda_{st} \\ 0 & \lambda_{vt} & \lambda_{st} & \lambda_{tt} \end{bmatrix} \tag{143}$$

where λ is an arbitrary smooth function of v, s, t. A particular Lagrangian is a given by

$$L = \frac{e^{-w}u^2}{t} + s^2 + t^2 + v^2. \tag{144}$$

The Lagrangian in the original variables is given by

$$L' = \frac{e^{-w}u^2}{t} + s^2 + t^2 + (v - \frac{ws}{2} - \frac{zt}{2})^2. \tag{145}$$

$A_{4,8}$: The non-zero Lie brackets are given by

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3. \tag{146}$$

A typical element of the group may be written as

$$S = \begin{bmatrix} 1 & 0 & xe^w & y \\ 0 & e^{-w} & 0 & x \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right invariant forms are given by $dw, dx + xdw, dz - zdw$ and $dy - zdx - xzdw$ and the corresponding right invariant vector fields are given by

$$W = \frac{\partial}{\partial w} - x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}. \tag{147}$$

The geodesic equations are

$$\dot{t} = 0, \quad \dot{u} = -tu, \quad \dot{v} = s(u + xt), \quad \dot{s} = st. \tag{148}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} -dw & 0 & 0 & -dx \\ dz & 0 & dx + xdw & xdz \\ 0 & 0 & dw & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & dwdx \\ dwdz & 0 & dwdx & xdwdz \\ 0 & 0 & 0 & dwdz \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The curvature tensor has essentially only the following non-zero components

$$R^2_{143} = R^2_{341} = R^1_{441} = R^3_{443} = \frac{1}{4}, \quad R^2_{443} = \frac{x}{4}. \tag{149}$$

Equations (18) and (19) imply that:

$$sg_{q2}u^q + tg_{q1}u^q = 0, (u + xt)g_{q2}u^q + tg_{q3}u^q = 0, sg_{22} + tg_{12} = 0, (u + xt)g_{22} + tg_{23} = 0$$

and so the algebraic solution for g is given by:

$$\begin{aligned} g_{ij} &= \rho \begin{bmatrix} \frac{s^2(2u+xt)}{ut^2} & -\frac{s}{t} & 0 & 0 \\ -\frac{s}{t} & 1 & -\frac{(u+xt)}{t} & 0 \\ 0 & -\frac{(u+xt)}{t} & \frac{(u+xt)(2u+xt)}{t^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{s}{u} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{u}{s} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \mu \begin{bmatrix} -\frac{t}{u} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \tau \begin{bmatrix} -\frac{s}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(u+xt)}{s} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{t}{s} & 1 \\ 0 & 0 & 1 & \frac{\pi}{\sigma} \end{bmatrix} \end{aligned}$$

The ODE conditions (11) turn out to be

$$\dot{\rho} = 0, \quad \dot{\lambda} - \frac{s(u+xt)}{t}\rho = 0, \quad \dot{\mu} + s\tau + s\lambda - \frac{s^2(u+xt)}{t^2}\rho = 0 \tag{150}$$

$$\dot{\tau} = 0, \quad \dot{\sigma} - u\lambda + \frac{us(u+xt)}{t^2}\rho = 0, \quad \dot{\pi} - u\mu + xs\tau + s\sigma = 0. \tag{151}$$

The solution to the above system is:

$$\rho = K, \quad \lambda = \frac{z(u+xt)}{t}K + L, \quad \tau = M, \quad \mu = \frac{z(u+xt)(s-zt)}{t^2}K - zL - zM + N$$

$$\sigma = -\frac{x(u+xt)(s-zt)}{t^2}K + xL + R, \quad \pi = \frac{xz(u+xt)(s-zt)}{t^2}K - xzL - xzM + xN - zR + Q$$

where K, L, M, N, Q and R are first integrals and so g becomes:

$$\begin{aligned} g_{ij} &= K\begin{bmatrix} \frac{s^2}{t^2} & -s/t & \frac{s(u+zt)}{t^2} & 0 \\ -s/t & 1 & -\frac{(u+xt)}{t} & 0 \\ \frac{s(u+zt)}{t^2} & -\frac{(u+xt)}{t} & \frac{(u+xt)^2}{t^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + L\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & x \\ 0 & -1 & x & \frac{Q}{L} \end{bmatrix} \\ &+M\begin{bmatrix} 0 & 0 & 1 & -\frac{s}{t} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\frac{u}{t} \\ -\frac{s}{t} & 1 & -\frac{u}{t} & 0 \end{bmatrix} + R\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -t/s & 1 \\ 0 & 0 & 1 & -z \end{bmatrix} + N\begin{bmatrix} -t/u & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x \end{bmatrix}. \end{aligned}$$

Finally we turn to the closure conditions. After much calculation and rearrangement we obtain the following PDE system for the unknowns K, L, M, N, Q and R:

$$tK_s + (u+xt)K_v = 0, sK_s - (u+xt)K_u = 0, L_v + K_t = 0, t^2L_s + uK - t(u+xt)K_t = 0 \tag{152}$$

$$t^2L_u + s(K - tK_t) = 0, tM_v + K - tK_t = 0, t^3N_s + t^2uM_u + su(tK_t - 2K) = 0 \tag{153}$$

$$t^3R_u + st^2M_s + st(u+xt)K_t - s(2u+xt)K = 0 \tag{154}$$

$$t^3L_t + t^2(uM_u + vM_v + sM_s + tM_t) + t(s(2u+xt) - tv)K_t + (tv - s(4u+xt))K = 0 \tag{155}$$

$$R_v = 0, uR_u + sR_s + tR_t + R = 0, Q_v + L_t = 0 \tag{156}$$

$$t^2Q_u - t^2N_t - zt^2R_u + xt^2N_u + s(tM_t - M) = 0 \tag{157}$$

$$t^2Q_s - t^2R_t - zt^2R_s + xt^2N_s + u(tM_t - M) = 0 \tag{158}$$

$$uN_u + vN_v + sN_s + tN_t + N = 0. \tag{159}$$

We are unable to to find any nontrivial integrability conditions, but since K is a first integral it must be of the form

$$K = K(t, s(u+xt) - vs). \tag{160}$$

We note that the canonical connection is a Levi-Civita connection and it turns out that a metric is given by

$$g = xdwdz - dydw + dxdz. \tag{161}$$

Notice, however, that g can be re-written in either of the following two equivalent forms:

$$g = dw(xdz - dy) + (e^w dx)(e^{-w}dz) \tag{162}$$

and

$$g = (dz - zdw)(xdw + dx) - dw(dy - zdx - xzdw). \tag{163}$$

The first of these forms has been expressed in terms of left-invariant forms whereas the second in terms of right-invariant forms. It follows that g is bi-invariant.

$A_{4,10}$: The non-zero Lie brackets are given by

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = -e_3, \quad [e_3, e_4] = e_2. \tag{164}$$

The group representation is given by

$$S = \begin{bmatrix} 1 & y\cos(w) + x\sin(w) & -y\sin(w) + x\cos(w) & z \\ 0 & \cos(w) & -\sin(w) & x \\ 0 & \sin(w) & \cos(w) & -y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right invariant forms are given by $dw, dx - ydw, dy + xdw$ and $dz + ydx - xdy - (x^2 + y^2)dw$ and the corresponding right invariant vector fields are given by

$$W = \frac{\partial}{\partial w} - x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \tag{165}$$

The geodesic equations are

$$\dot{u} = tv, \quad \dot{v} = -ut, \quad 2\dot{s} = (xu + yv)t, \quad \dot{t} = 0. \tag{166}$$

The connection form θ and the curvature two-form Ω are given by

$$-2\theta = \begin{bmatrix} 0 & dw & 0 & dy \\ -dw & 0 & 0 & -dx \\ xdw & ydw & 0 & xdx + ydy \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad 4\Omega = \begin{bmatrix} 0 & 0 & 0 & dxdw \\ 0 & 0 & 0 & dydw \\ 2dwdx & 2dwdy & 0 & dw(xdy - ydx) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the curvature tensor has essentially only the following non-zero components

$$R^1_{414} = R^2_{424} = 2R^3_{141} = 2R^3_{242} = \frac{1}{4}, \quad R^3_{414} = \frac{y}{4}, \quad R^3_{424} = -\frac{x}{4}. \tag{167}$$

Conditions (18) and (19) give:

$$\begin{aligned} g_{ij} \;&= \lambda \begin{bmatrix} t^2u & t^2v & 0 & -t(u^2+v^2) \\ t^2v & -t^2u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t(u^2+v^2) & 0 & 0 & u(u^2+v^2) \end{bmatrix} + \rho \begin{bmatrix} t^2 & 0 & 0 & -tu \\ 0 & t^2 & 0 & -tv \\ 0 & 0 & 0 & 0 \\ -tu & -tv & 0 & u^2+v^2 \end{bmatrix} \\ &+\mu \begin{bmatrix} -t^2v & t^2u & 0 & 0 \\ t^2u & t^2v & 0 & -t(u^2+v^2) \\ 0 & 0 & 0 & 0 \\ 0 & -t(u^2+v^2) & 0 & v(u^2+v^2) \end{bmatrix} + +\sigma \begin{bmatrix} 2 & 0 & 0 & -y \\ 0 & 2 & 0 & x \\ 0 & 0 & 0 & 1 \\ -y & x & 1 & \frac{\tau}{\sigma} \end{bmatrix} \\ &+\pi \begin{bmatrix} 0 & 0 & 2u - yt & 0 \\ 0 & 0 & 2v + xt & 0 \\ 2u - yt & 2v + xt & t & -\frac{u^2+v^2-ytu+xvt}{t} \\ 0 & 0 & -\frac{u^2+v^2-ytu+xvt}{t} & 0 \end{bmatrix}. \end{aligned}$$

One ODE condition (11) for g_{13} gives easily that π is a first integral P, say, and then the g_{23} and g_{33} conditions are satisfied identically. The g_{34} condition gives that

$\sigma - \pi\frac{u(2u-yt)+v(2v+xt)}{t}$ is a first integral, say, S. The remaining ODE conditions coming from the g_{11}, g_{22} and g_{12} and g_{44} components are:

$$u\dot{\lambda} - v\dot{\mu} + \frac{\pi(xu - yv) - xyt}{t} = 0, \quad v\dot{\lambda} + u\dot{\mu} + \frac{\pi 2(yu + xv) - xyt}{2t} = 0 \tag{168}$$

$$\dot{\tau} - t(u^2 + v^2)(v\dot{\lambda} - u\dot{\mu} - \frac{(xu + yv)\pi}{t}(u(2u - yt) + v(2v + xt)) = 0. \tag{169}$$

The complete solution of the ODE conditions (11) is similar to the one given for g_{ij} above except that the greek letters $\lambda, \mu, \rho, \sigma$ and τ become the first integrals L, M, R, S and T. In addition π is replaced by the first integral P and its associated matrix is replaced by

$$\begin{bmatrix} \frac{2t(vx-uy)+y^2t^2}{t} & -xyt & 2u - yt & 0 \\ -xyt & \frac{2t(vx-uy)+x^2t^2}{t} & 2v + xt & 0 \\ 2u - yt & 2v + xt & t & -2\frac{(u^2+v^2)}{t} \\ 0 & 0 & -2\frac{(u^2+v^2)}{t} & \frac{(x^2+y^2)(u^2+v^2)}{t} \end{bmatrix}.$$

There are fourteen closure conditions that we shall not write down and the system is similar to that of $A_{4,8}$ only more complicated. We shall be content to note that the solution for the function P is given by:

$$P = P(u^2 + v^2 - 2yut + 2xvt + st, t) \tag{170}$$

and it leads to the bi-invariant metric:

$$g = dx^2 + dy^2 - ydxdw + xdydw + dzdw. \tag{171}$$

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