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A Dynamical Approach to Compactify the Three Dimensional Lorentz Group

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Abstract. The Lorentz group acts on the projectivized light cone in the three dimensional Lorentz space as the group G of Möbius transformations of the circle. We find the closure of G in the space of all measurable functions of the circle into itself, obtaining a compactification of it as an open dense subset of the three-sphere, with a dynamical meaning related to generalized flows.
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The canonical action of the Lorentz group $O_o(1,2)$ on the projectivized light cone in the three dimensional Lorentz space is equivalent to the action of the group G on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, where G consists of the Möbius transformations of the extended plane preserving the circle. The group G is isomorphic to $PSU(1,1)$ and $PSl(2,\mathbb{R})$. In this note we compactify G as an open dense subset of the three-sphere, with a dynamical motivation.

The group G consists of maps of the form uT_α, where $u \in S^1$ and

$$T_\alpha(z) = \frac{z+\alpha}{1+\bar{\alpha}z}$$

for $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and all $z \in S^1$. The map $S^1 \times \Delta \to G$, $(u,\alpha) \mapsto uT_\alpha$ is a diffeomorphism. Although we are interested in the action of G on the circle, we recall that if the unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ carries the canonical Poincaré metric of constant negative curvature -1 and $\alpha \neq 0$, then T_α is the transvection translating the geodesic with end points $\pm\alpha/|\alpha|$, sending 0 to α.

Dynamical motivation. If $t \in \mathbb{R}, |t| < 1$, then T_t fixes $1, -1 \in S^1$ and if $z \in S^1, z \neq -1$, then

$$\lim_{t \to 1^-} T_t(z) = 1.$$

One can imagine that all particles of the circle (except -1) moving according to T_t concentrate in the point 1 at $t = 1$. It is natural to think that a particle coming to the point 1 at $t = 1$ from the upper half of the circle, will continue its way into the lower part of the circle for $t > 1$ (notice that T_t does not make sense for $|t| \geq 1$) and similarly for a particle coming to the point 1 from the lower part of

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the circle. This can be rendered precise with the compactification of G described in Theorem 1.1 below (see Proposition 1.3).

Let $\mathcal{F} = \{f : S^1 \to S^1 \mid f \text{ is measurable}\} / \sim$, where $f \sim g$ if and only if f and g coincide except on a set of measure zero, equipped with the distance

$$D(f,g) = \int_{S^1} d(f(z), g(z))\, ds(z),$$

being s is an arc length parameter and d the associated distance on S^1 (we think of each function as representing its equivalence class). Let S^3 be the three dimensional sphere realized as the Lie group of unit vectors in the quaternions $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. We recall that if q is an imaginary cuaternion with $|q| = 1$, then $\exp(tq) = \cos t + (\sin t)\, q$. For $v \in S^1$, let c_v denote the constant map in $\mathcal{F}$ with value v.

1. The Main Theorem

Theorem 1.1. *The frontier of G in $\mathcal{F}$ consists of the constant functions. Moreover, if one considers on the closure $\overline{G}$ of G the relative topology from $\mathcal{F}$, then the map $F : \overline{G} \to S^3$ defined by*

$$F(uT_\alpha) = u \exp\left(\tfrac{\pi}{2}\alpha j\right), \qquad F(c_v) = vj,$$

is a homeomorphism and $F|_G : G \to S^3$ determines a submanifold.

Proof. Clearly G is a subset of $\mathcal{F}$. If $u \in S^1$, let m_u denote multiplication by u. By abuse of notation we write $T_\alpha m_u = T_\alpha u$. Notice that $uT_\alpha = T_{u\alpha} u$ for any $u \in S^1$, $\alpha \in \Delta$. Let α_n and u_n be sequences in Δ and S^1, respectively. Suppose first that $\alpha_n \to \alpha \in S^1$ as $n \to \infty$. We show that

$$T_{\alpha_n} u_n \to c_\alpha \text{ in } \mathcal{F} \text{ as } n \to \infty. \tag{1}$$

Indeed, since ds is invariant by rotations, then $D(T_{\alpha_n} u_n, c_\alpha) = D(T_{\alpha_n}, c_\alpha)$. This sequence converges to zero as $n \to \infty$ by the Bounded Convergence Theorem, since $\lim_{n\to\infty} T_{\alpha_n}(z) = \alpha$ for any $z \neq -\alpha$ (d and the euclidean distance are equivalent). In particular constant functions are in the frontier of G. On the other hand, if $u_n \to u$ and $\alpha_n \to \alpha \in \Delta$, then $T_{\alpha_n} u_n \to T_\alpha u$ pointwise, and hence in $\mathcal{F}$, again by the Bounded Convergence Theorem. Moreover, by the preceding, if $T_{\alpha_n} u_n$ converges to f in $\mathcal{F}$, then $f \in G$ or is constant, since by the compactness of $\overline{\Delta} \times S^1$ there exists a subsequence of (α_n, u_n) converging in it. Then the frontier consist only of constant functions. Now, F is a bijection since a straightforward computation shows that $F^{-1} : S^3 \to \overline{G}$ is given by

$$F^{-1}(v + wj) = \begin{cases} c_w & \text{if } v = 0, \\ m_v & \text{if } w = 0 \\ T_\alpha u & \text{if } v \neq 0 \neq w, \end{cases} \tag{2}$$

for $v, w \in \mathbb{C}$, $|v|^2 + |w|^2 = 1$, where $u = v/|v|$ and $\alpha = \frac{2}{\pi} \arccos(|v|) \frac{w}{|w|}$.

Hence F^{-1} is smooth at $v+wj \in S^3$ with $v \neq 0 \neq w$. Since $F|_G$ is smooth and injective, to show that $F|_G$ is an embedding it suffices to see that F^{-1} is smooth at $v \in S^1 \subset S^3$. This will follow from the Inverse Function Theorem if we check that

$$dF_{m_v} : T_{m_v}G \to T_v S^3$$

is an isomorphism. We can identify $T_{m_v}G = T_{(v,0)}\left(S^1 \times \Delta\right) = T_v S^1 \oplus T_0 \Delta = \mathbb{R}iv \oplus \mathbb{C}$ and also $T_v S^3 = \mathbb{R}iv \oplus \mathbb{C}j$, the orthogonal complement of v in $\mathbb{H}$. We compute

$$dF_v\left(xiv, z\right) = \left.\frac{d}{dt}\right|_0 F\left(ve^{txi}T_{tz}\right) = \left.\frac{d}{dt}\right|_0 ve^{txi}\exp\left(t\frac{\pi}{2}zj\right) = v\left(xi + \frac{\pi}{2}zj\right).$$

Hence, dF_v is an isomorphism.

In order to verify that F^{-1} is continuous at wj we consider the map $\overline{F} : G \to S^3$, $\overline{F} = R_j \circ F$ (R_j denotes right multiplication by j), which, by the preceding, is a diffeomorphism onto its image $S^3 - S^1$. We have to show that $F^{-1} \circ \overline{F}$ is continuous at $u \in S^1$. Clearly, $\overline{F}\left(m_u\right) = uj$. If $\alpha \neq 0$, we compute $\overline{F}\left(uT_\alpha\right) = v + wj$, where $v = -\frac{u\alpha}{|\alpha|}\sin\left(\frac{\pi}{2}|\alpha|\right)$ and $w = u\cos\left(\frac{\pi}{2}|\alpha|\right)$. Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ for all θ, we have by (2) that

$$F^{-1}\left(\overline{F}\left(uT_\alpha\right)\right) = T_{u(1-|\alpha|)}\left(-u\alpha/|\alpha|\right), \tag{3}$$

which by (1) converges to $c_u = \left(F^{-1} \circ \overline{F}\right)\left(m_u\right)$ as $\alpha \to 0$. Finally, since S^3 is compact and Hausdorff, F^{-1} is a homeomorphism. ■

Remark. If $u_n = e^{2\pi x_n i}$ with $x_n = 1/2, 1/4, 2/4, 3/4, 1/8, 2/8, 3/8, \ldots$, then $T_{1-1/n}m_{u_n}$ converges to c_1 in $\mathcal{F}$ but it does not converge pointwise on a dense subset of S^1. This distinguishes our approach from that of Topological Dynamics.

Proposition 1.2. *The canonical action of $G \times G$ on G, $(g,h).f = gfh^{-1}$, extends to a continuous action of $G \times G$ on S^3 via $F|_G : G \to S^3$. If we call $K = S^1 \subset G$, the restricted action of $K \times K$ on S^3 is given by $A\left(u, v, z_1 + z_2 j\right) = u\left(\bar{v}z_1 + z_2 j\right)$.*

Proof. We define an action $\bar{A}$ of $G \times G$ on $\overline{G}$ by

$$\bar{A}\left(g, h, f\right) = gfh^{-1}, \quad \bar{A}\left(g, h, c_v\right) = c_{gv},$$

for $g, h, f \in G$, $v \in S^1$. Since $F : \overline{G} \to S^3$ is a homeomorphism, we have to show that $\bar{A}$ is continuous. Suppose that $f_n \in G, v_n \in S^1$ are sequences converging to $c_v \in \overline{G}$, and g_n, h_n are sequences in G converging to $g, h \in G$, respectively. By arguments similar to those used in the proof of Theorem 1.1, $g_n f_n h_n^{-1}$ and $c_{g_n v_n}$ both converge to c_{gv} in $\mathcal{F}$.

Next we verify the second assertion. We have to show that the following diagram is commutative.

$$\begin{array}{ccc}
K \times K \times \overline{G} & \xrightarrow{\bar{A}} & \overline{G} \\
\downarrow \left(\mathrm{id}_{K\times K}, F\right) & & \downarrow F \\
K \times K \times S^3 & \xrightarrow{A} & S^3
\end{array}$$

For $u, v, w \in S^1, \alpha \in \Delta,$ we compute

$$\left(F \circ \bar{A}\right)(u, v, c_w) = F(c_{uw}) = uwj = A(u, v, wj) = A(u, v, F(c_w)).$$

Besides, $\left(F \circ \bar{A}\right)(u, v, wT_\alpha) = F(uwT_\alpha \bar{v}) = F(uw\bar{v}T_{v\alpha}) = A(u, v, F(wT_\alpha))$, since $\exp\left(\frac{\pi}{2}\beta j\right) = \cos\left(\frac{\pi}{2}|\beta|\right) + \sin\left(\frac{\pi}{2}|\beta|\right)\frac{\beta}{|\beta|}j$ for any $\beta \in \Delta$. ■

Next we make precise the comment at the beginning of the article concerning moving particles in the circle.

Proposition 1.3. *If $\overline{G}$ is endowed with the differentiable structure and the Riemannian metric induced from S^3 via the homeomorphism F, then the curve $\gamma : \mathbb{R} \to \overline{G}$ defined by*

$$\gamma(s) = \begin{cases} (-1)^k T_{s-2k} & \textit{if } |s - 2k| < 1, k \in \mathbb{Z} \\ c_{(-1)^\ell} & \textit{if } s = 2\ell + 1, \ell \in \mathbb{Z} \end{cases}$$

is a complete geodesic in $\overline{G}$. Moreover, if $z \neq \pm 1$, then the curve $\gamma_z(s) := \gamma(s)(z)$ in S^1, describing the motion of the particle z under $\gamma(s)$, is continuous with period 4 and runs n times around the circle in any interval of time of length $4n$ (clockwise if $\operatorname{Re} z > 0$ and counterclockwise if $\operatorname{Re} z < 0$).

Proof. A straightforward computation shows that $F(\gamma(s)) = \exp(\frac{\pi}{2}sj)$. Hence γ is a geodesic. The remaining facts are easily verified. ■

Remarks. a) We recall that a Fermi coordinate system ϕ along a geodesic γ in a Riemannian manifold of dimension $n + 1$ is given by

$$\phi(t, t_1, \ldots, t_n) = \operatorname{Exp}_{\gamma(t)}\left(\sum_{i=1}^n t_i v_i(t)\right),$$

where Exp denotes the geodesic exponential map and $\{v_i\}$ is a parallel orthonormal frame along γ orthogonal to $\gamma'(t)$ at any t. Notice that since G is diffeomorphic to $S^1 \times \Delta$ via $uT_\alpha \mapsto (u, \alpha)$, if one looks just for a compactification of G as an open dense subset of the three-sphere, without extra properties, the simplest way is by using a slight modification of Fermi coordinates along the geodesic $s \mapsto e^{si}$ in S^3: $\overline{F}(uT_\alpha) = \operatorname{Exp}_u\left(\frac{\pi}{2}\alpha j\right)$, where $u \in S^1 \subset S^3$. The maps $\overline{F}$ and F do not coincide on G, since the mapping $s \mapsto m_{e^{is}}$ is not a one-parameter subgroup of transvections translating that geodesic (their differentials do not realize the parallel transport along it).

b) The situations of particles concentrating in a point or a point spreading instantaneously onto the whole space, is present in the literature in a *different* context, the study of volume preserving flows by geometric means, with the notions of polymorphisms [8] and generalized flows [3]. An overview of the subject can be found in [1].

For the sake of connectedness of mathematics we cite [4, 9]. Finally, we comment on the compactifications known to us of classical groups whose identity component is isomorphic to G or its double covering. The classical one is obtained as follows: Let $Sl\,(2,\mathbb{C}) = SU\,(2)\,AN$ be an Iwasawa decomposition. Since $SU\,(1,1)$ intersects AN only at the identity, its projection P to $SU\,(2) \cong S^3$ is an embedding, which is given explicitly by

$$P\begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix} = \frac{u+vj}{|u+vj|}, \qquad (u,v \in \mathbb{C},\ |u|^2 - |v|^2 = 1).$$

The image of P is the interior of the solid torus $\{u+vj \in S^3 \mid |v| \leq |u|\}$. If one wants $SU\,(1,1)$ to be dense in its compactification, one can consider for instance $p \circ P$ instead of P, where $p : S^3 \to S^3/\{1,j\}$ is the canonical projection. In this case, the frontier of the image of $SU\,(1,1)$ is a torus.

On the other hand, recently, H. He, based on suggestions of D. Vogan, obtained a general method to compactify the classical simple Lie groups [5, 6] (see also [2, 7]). The groups $O\,(1,2)$ and $Sl\,(2,\mathbb{R}) \cong Sp\,(2,\mathbb{R})$ are embedded as open dense subsets of $O\,(3)$ and of a manifold double covered by $S^2 \times S^1$, respectively. In both cases the frontier is a surface.

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