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Integral identities and constructions of approximations to zeta-values

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Résumé. Nous présentons une construction générale de combinaisons linéaires à coefficients rationnels en les valeurs de la fonction zêta de Riemann aux entiers. Ces formes linéaires sont exprimées en termes d'intégrales complexes, dites de Barnes, ce qui permet de les estimer. Nous montrons quelques identités reliant ces intégrales à des intgrales multiples sur le cube unité réel.

Abstract. Some general construction of linear forms with rational coefficients in values of Riemann zeta-function at integer points is presented. These linear forms are expressed in terms of complex integrals of Barnes type that allows to estimate them. Some identity connecting these integrals and multiple integrals on the real unit cube is proved.

1. Introduction

Apery's proof of irrationality of $\zeta(3)$ uses an elementary and rather complicated construction of rational approximations $u_n/v_n \in \mathbb{Q}$ to this number based on a recurrence relation. In [1] the following integral interpretation of these sequences u_n, v_n was proposed

$$(1)\quad \int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} dxdydz = 2(v_n\zeta(3)-u_n).$$

Another presentation for the same sequences can be found in [4]

$$(2)\qquad \frac{1}{2\pi i}\int_C \frac{\Gamma^4(-s)\Gamma^2(n+1+s)}{\Gamma^2(n+1-s)} ds = 2(v_n\zeta(3)-u_n).$$

Here $\Gamma(s)$ is Euler gamma-function and the contour C is the vertical straight line that begins at $-1/2-i\infty$ and ends at $-1/2+i\infty$.

In particular the aim of this article is to prove the coincidence of the integrals (1) and (2). Besides we prove a more general integral identity

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(Theorem 2) connected to the construction of functional linear forms in polylogarithmic functions

$$L_k(z) = \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu^k}, \qquad k \geq 1.$$

In this way since we have the equality $L_k(1) = \zeta(k)$ one can construct small linear forms in zeta-values at integer points, which are more general than right-hand sides of (1) and (2).

2. General construction of linear forms in polylogarithms.

Let $a_j \geq 1, b_j \geq 1, j = 1, \dots, m$, be integers. Define

$$R(s) = \gamma \prod_{j=1}^{m} \frac{\Gamma(s + a_j)}{\Gamma(s + b_j)}, \tag{3}$$

where γ is a rational number that will be defined later.

Denote

$$S_1 = \{j \mid \quad a_j \geq b_j\}, \qquad S_2 = \{j \mid \quad a_j < b_j\},$$

$$R_j(s) = \frac{(b_j + s)(b_j + 1 + s) \cdots (a_j - 1 + s)}{(a_j - b_j)!}, \quad \text{if} \quad j \in S_1,$$

and

$$R_j(s) = \frac{(b_j - a_j - 1)!}{(a_j + s)(a_j + 1 + s) \cdots (b_j - 1 + s)}, \quad \text{if} \quad j \in S_2.$$

Then

$$R(s) = \gamma \prod_{j=1}^{m} \frac{s(s+1) \cdots (s + a_j - 1)}{s(s+1) \cdots (s + b_j - 1)} = \prod_{j=1}^{m} R_j(s) \in \mathbb{Q}(s). \tag{4}$$

The last equality defines the constant $\gamma = \gamma(\overline{a}, \overline{b}) > 0$.

In what follows an important role belongs to the function

$$F(z) = \sum_{\nu=0}^{\infty} R(\nu) z^\nu, \qquad z \in \mathbb{C}. \tag{5}$$

The parameter (5)

$$\delta = b_1 + \cdots + b_m - a_1 - \cdots - a_m \tag{6}$$

is useful for a description of the convergence domain of (5). We will assume that $\delta \geq 1$. If $\delta \geq 2$, then the series (5) converges in the circle $|z| \leq 1$. In the case $\delta = 1$, this series diverges at the point $z = 1$.

In the sequel the notation Δ_j will be used for special segments of the real line. Define

$$\Delta_j = [b_j, a_j - 1], \text{ if } j \in S_1, \qquad \text{and} \qquad \Delta_j = [a_j, b_j - 1], \text{ if } j \in S_2.$$

For the length of Δ_j we will use notation $|\Delta_j|$.

For any integer ℓ define

$$\mathcal{M}(\ell) = \{j \in S_2 \,|\quad \ell \in \Delta_j\}$$

and denote

$$d(\ell) = \operatorname{Card} \mathcal{M}(\ell).$$

The rational function $R(s)$ can be presented as a sum of simple fractions

$$R(s) = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} \frac{B_{\ell,k}}{(s+\ell)^k}, \tag{7}$$

where $\mathcal{P}$ is the union of sets $\Delta_j, j \in S_2$. Note that the equality $d(\ell) = 0$ is possible. For coefficients $B_{\ell,k}$ we have the expression

$$B_{\ell,k} = \frac{1}{(d-k)!} \left(\frac{d}{ds}\right)^{d-k} \big(R(s)(s+\ell)^d\big)_{s=-\ell} \in \mathbb{Q}, \tag{8}$$

where $d = d(\ell)$. Farther denote $q = \max_{\ell \in \mathcal{P}} d(\ell)$.

Due to (7) one can find the following equalities

$$\begin{aligned} F(z) &= \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell,k} \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\nu+\ell)^k} = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell,k} z^{-\ell} \sum_{\nu=\ell}^{\infty} \frac{z^\nu}{\nu^k} \\ &= \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} B_{\ell,k} z^{-\ell} \left(L_k(z) - \sum_{\nu=1}^{\ell-1} \frac{z^\nu}{\nu^k} \right). \end{aligned}$$

This confirms that the sum (5) is a linear form in 1 and polylogarithms with polynomial in $1/z$ coefficients. It is clear that analogous result can be proved if we put as coefficients of the series any derivative of $R(s)$ and shift the lower limit of summation on any admissible integer number. The following Proposition defines our general construction.

Proposition 1. *Let a, μ be integers satisfying inequalities $a \le a_j$, $j \in S_2$, and $\mu \ge 1$. Then for any z from the convergence domain of the series*

$$G_\mu(z) = \frac{(-1)^{\mu-1}}{(\mu-1)!} \sum_{\nu=1}^{\infty} R^{(\mu-1)}(\nu - a) z^\nu \tag{9}$$

the following identity holds

$$G_\mu(z) = A_0(z^{-1}) + \sum_{k=1}^{q} A_k(z^{-1}) L_{k+\mu-1}(z).$$

Here

$$A_k(x) = \binom{k+\mu-2}{\mu-1} \sum_{\substack{\ell \in \mathcal{P} \\ d(\ell) \ge k}} B_{\ell,k} x^{\ell-a}, \qquad k = 1, \dots, q, \tag{10}$$

$$A_0(x) = -\sum_{\ell\in\mathcal{P}}\sum_{k=1}^{d(\ell)}\sum_{\nu=1}^{\ell-a}\binom{k+\mu-2}{\mu-1}B_{\ell,k}\nu^{1-k-\mu}x^{\ell-a-\nu}. \tag{11}$$

Proof. By (7) one can find

$$\frac{(-1)^{\mu-1}}{(\mu-1)!}R^{(\mu-1)}(s-a) = \sum_{k=1}^{q}\binom{k+\mu-2}{\mu-1}\sum_{\substack{\ell\in\mathcal{P}\\ d(\ell\geq k}}\frac{B_{\ell,k}}{(s-a+\ell)^{k+\mu-1}},$$

and

$$G_\mu(z) = \sum_{k=1}^{q}\binom{k+\mu-2}{\mu-1}\sum_{\substack{\ell\in\mathcal{P}\\ d(\ell\geq k}}B_{\ell,k}\sum_{\nu=1}^{\infty}\frac{z^\nu}{(\nu-a+\ell)^{k+\mu-1}} =$$

$$\sum_{k=1}^{q}\binom{k+\mu-2}{\mu-1}\sum_{\substack{\ell\in\mathcal{P}\\ d(\ell)\geq k}}B_{\ell,k}z^{a-\ell}\left(L_{k+\mu-1}(z) - \sum_{\nu=1}^{\ell-a}\frac{z^\nu}{\nu^{k+\mu-1}}\right).$$

This proves Proposition 1. □

Note that in the case $\delta\geq 2$ we have $A_1(1)=0$, since

$$A_1(1) = \sum_{\ell\in\mathcal{P}}B_{\ell,1} = \sum_{\ell\in\mathcal{P}}\mathrm{Res}_{s=-\ell}\,R(s) = -\,\mathrm{Res}_{s=\infty}\,R(s) = 0.$$

Another reason for this equality is the divergence of $L_1(z)$ and the convergence of $L_k(z)$, $k\geq 2$, at the point $z=1$.

Consider some more interesting partial choices of parameters a_i, b_i. The following Proposition describes a construction of simultaneous Pade approximations for polylogarithms at neighbourhood of infinity.

Proposition 2. *Let r be integer, $1\leq r\leq m$, and parameters a_j, b_j satisfy following conditions*

$$1\leq b_1\leq\ldots\leq b_r < a_1 = \ldots = a_m < b_m \leq\ldots\leq b_{r+1}. \tag{12}$$

For any μ, $1\leq\mu\leq r$, define a function $H_\mu(z)$ by the equality

$$H_\mu(z) = \frac{(-1)^{\mu-1}}{(\mu-1)!}\sum_{\nu=1}^{\infty}R^{(\mu-1)}(\nu-a_1)z^{-\nu}.$$

These functions have the representation

$$H_\mu(z) = \sum_{k=1}^{m-r}P_{r+k}(z)\binom{k+\mu-2}{\mu-1}L_{k+\mu-1}(z^{-1}) + P_\mu(z), \tag{13}$$

where

$$P_{r+k}(z) = \sum_{\ell\in\Delta_{r+k}}B_{\ell,k}z^{\ell-a_{r+k}}, \qquad 1\leq k\leq m-r, \tag{14}$$

$$(15)\quad P_\mu(z) = -\sum_{k=1}^{m-r}\sum_{\ell\in\Delta_{r+k}}\sum_{\nu=1}^{\ell-a_{r+k}}\binom{k+\mu-2}{\mu-1}B_{\ell,k}\frac{z^{\ell-a_{r+k}-\nu}}{\nu^{k+\mu-1}},\ \ 1\le\mu\le r.$$

In particular

$$(16)\qquad \deg P_k(z)\le|\Delta_k|,\qquad k=r+1,\dots,m,$$

and

$$(17)\qquad \operatorname{ord}_\infty H_\mu(z)\ge|\Delta_\mu|+2,\qquad \mu=1,\dots,r.$$

Note that polynomials $P_{r+k}(z)$ do not depend on the index μ. In the case $r=1$ and $\delta=1$ the above construction gives the Pade approximation of the first kind and for $r=m-1$ and $\delta=1$ it gives the approximations of the second kind.

Proof. In the conditions we have

$$\{\ell\in\mathcal{P}\,|,\ d(\ell)\ge k\,\}=\Delta_{r+k}=[a_{r+k},b_{r+k}-1].$$

This is a reason why the equalities (13)-(15) follow from (10) and (11).

The inequalities (16) follow from (14).

The inequalities (17) are valid, since due to (12) we have $R^{(\mu-1)}(\nu)=0$ for $\nu=1-a_1,\dots,-b_\mu$. □

Proposition 3. *Assume that the parameters a_j, b_j satisfy the conditions*

$$(18)\qquad a_j+b_j=c+1,\qquad j=1,\dots,m,$$

for some c. Then for polynomials $A_k(x)$, defined in Proposition 1 the following equalities hold

$$A_k(1)=0,\qquad k\equiv\delta+1\pmod 2.$$

The condition (18) in this context was proposed by K. Ball and used in first by T. Rivoal [6].

Proof. Since all parameters are integers and due to identity $\Gamma(z)\cdot\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$, we find

$$R(-c-s)=\gamma\prod_{j=1}^{m}\frac{\Gamma(-s-c+a_j)}{\Gamma(-s-c+b_j)}=\gamma(-1)^\delta\prod_{j=1}^{m}\frac{\Gamma(s+1+c-b_j)}{\Gamma(s+1+c-a_j)}.$$

Therefore the function $R(s)$ has the property

$$R(-c-s)=(-1)^\delta R(s).$$

The representation (7) implies

$$(19)\qquad R(s)=\sum_{\ell\in\mathcal{P}}\sum_{k=1}^{d(\ell)}\frac{(-1)^\delta B_{\ell,k}}{(-s-c+\ell)^k}.$$

Since (18) we derive that for $\ell \in \mathcal{P}$ the inclusion $c - \ell \in \mathcal{P}$ and equality $d(c-\ell) = d(\ell)$ hold. Moreover when ℓ runs through the set $\mathcal{P}$, also the number $c - \ell$ runs through this set. Hence (19) implies that

$$R(s) = \sum_{\ell \in \mathcal{P}} \sum_{k=1}^{d(\ell)} \frac{(-1)^{\delta+k} B_{c-\ell,k}}{(s+\ell)^k}.$$

Compared together the last equality and (7), we find

$$B_{\ell,k} = (-1)^{\delta+k} B_{c-\ell,k}.$$

Hence

$$\sum_{\substack{\ell \in \mathcal{P} \\ d(\ell) \geq k}} B_{\ell,k} = (-1)^{\delta+k} \sum_{\substack{\ell \in \mathcal{P} \\ d(\ell) \geq k}} B_{c-\ell,k} = (-1)^{\delta+k} \sum_{\substack{\ell \in \mathcal{P} \\ d(\ell) \geq k}} B_{\ell,k}.$$

Then $A_k(1) = (-1)^{\delta+k} A_k(1)$, and this proves Proposition 3. □

In particular the last Proposition implies that in case under consideration with $\delta \geq 2$ and even $\mu + \delta$ the number $G(1)$ is a linear combination of 1 and values of Riemann zeta-function at odd points with rational coefficients. But for odd $\mu + \delta$ the number $G(1)$ is a polynomial in π^2 with rational coefficients.

3. Integral representations.

For applications of above construction of linear forms one need upper bounds for the absolute value of these forms. Here we will find analytic representations for functions $G_\mu(z)$, see (9), which allow in some cases to find estimates and even asymptotics for their values.

3.1. Hypergeometric functions. The generalized hypergeometric function with parameters $a_1, \ldots, a_m$, $b_2, \ldots, b_m \in \mathbb{C}$ is given by the series

$$(20) \qquad {}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_2, \ldots, b_m \end{matrix} \middle| z \right) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu \ldots (a_m)_\nu}{(b_2)_\nu \ldots (b_m)_\nu} \cdot \frac{z^\nu}{\nu!},$$

where the symbol $(a)_0 = 1$, $(a)_\nu = a(a+1)\cdots(a+\nu-1)$, $\nu \geq 1$. Here one assumes that a_j, b_j are distinct from negative integers. The series (20) absolutely converges for any $|z| < 1$ and for $|z| = 1$ if an additional condition

$$(21) \qquad \delta = 1 + \Re(b_2 + \cdots + b_m - a_1 - \cdots - a_m) > 1,$$

is satisfied, see [3].

In this subsection we consider a partial case of the construction (9), corresponding to the choice $\mu = 1$, $a = a_1$.

Lemma 1. *Let $b_1 = 1$ $a_1, \dots, a_m$, $b_2, \dots, b_m$ be positive integers with $a = a_1 \le a_j$, $j \in S_2$, and $R(s)$, $G_1(z)$ be functions defined by (3) and (9). Then the following identity holds*

$$G_1(z) = z^{a_1}\gamma \prod_{j=1}^{m} \frac{\Gamma(a_j)}{\Gamma(b_j)} \cdot {}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \middle| z \right). \tag{22}$$

Proof. Since $R(\varkappa) = 0$, $1 - a_1 \le \varkappa \le -1$, we derive

$$G_1(z) = \sum_{\nu=1}^{\infty} R(\nu - a_1) z^{\nu} = \sum_{\varkappa=1-a_1}^{\infty} R(\varkappa) z^{\varkappa + a_1} =$$

$$\sum_{\varkappa=0}^{\infty} R(\varkappa) z^{\varkappa+a_1} = z^{a_1}\gamma \sum_{\varkappa=0}^{\infty} \prod_{j=1}^{m} \frac{\Gamma(\varkappa + a_j)}{\Gamma(\varkappa + b_j)} z^{\varkappa}.$$

Due to $\Gamma(\nu + a) = \Gamma(a) \cdot (a)_\nu$, $\nu \ge 0$, the above equalities prove the identity. □

In particular for

$$\mu = 1, \quad a = a_1 = \dots = a_m = 1, \quad b_1 = 1,\, b_2 = \dots = b_m = 2,$$

Lemma 1 implies

$$L_{m-1}(z) = z \cdot {}_mF_{m-1}\left(\begin{matrix} 1, 1, \dots, 1 \\ 2, \dots, 2 \end{matrix} \middle| z \right).$$

The generalized hypergeometric function has several integral representations, see [3], [7], and in this way it has analytic continuation in the whole complex plain with some cuttings. Although some of the formulae in this subsection are classical, we have included their proofs for the convenience of the reader. In two following lemmas we assume that the parameters a_j, b_j are complex numbers.

The first one is a multidimentional generalization of Euler representation for Gaussian hypergeometric function.

Lemma 2. *Let $b_1 = 1$ and $a_1, \dots, a_m$, $b_2, \dots,$ b_m be complex numbers satisfying*

$$\Re b_j > \Re a_j > 0, \qquad j = 2, \dots, m.$$

The integral $J_1(z)$ is defined by

$$J_1(z) = \int_0^1 \cdots \int_0^1 \frac{\prod_{j=2}^{m} x_j^{a_j - 1}(1 - x_j)^{b_j - a_j - 1}}{(1 - z x_2 \cdots x_m)^{a_1}} dx_2 \cdots dx_m.$$

In these conditions the following assertions hold.

1. The integral $J_1(z)$ absolutely converges for any $z \in \mathbb{C}$, $|\arg(1 - z)| < \pi$,.

2. For any z, $|z|<1$, we have

$$\text{(23)}\qquad {}_mF_{m-1}\left(\begin{matrix} a_1,a_2,\ldots,a_m \\ b_2,\ldots,b_m\end{matrix}\,\middle|\, z\right)=\prod_{j=2}^{m}\frac{\Gamma(b_j)}{\Gamma(a_j)\Gamma(b_j-a_j)}J_1(z).$$

3. If $\delta>1$, see (21), we have (23) for any complex z, $|z|\le 1$.

Proof. 1. The first assertion is trivial, since in the conditions the function $1-zx_2\cdots x_m$ is distinct from 0 on the cube $[0;1]^{m-1}$.

2. To prove the second one, see [7], subsection 4.1, we apply the identity

$$(1-x)^{-a}=\sum_{\nu=0}^{\infty}\frac{(a)_\nu}{\nu\,!}x^\nu,$$

to the function $(1-zx_2\cdots x_m)^{-a_1}$. Since $|z|<1$, the corresponding series uniformelly converges on the cube $[0;1]^{m-1}$. Hence

$$J_1(z)=\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu}{\nu\,!}z^\nu\int_0^1\cdots\int_0^1\prod_{j=2}^{m}x_j^{a_j+\nu-1}(1-x_j)^{b_j-a_j-1}dx_2\cdots dx_m=$$
$$\sum_{\nu=0}^{\infty}\frac{(a_1)_\nu}{\nu\,!}z^\nu\prod_{j=2}^{m}\frac{\Gamma(\nu+a_j)\Gamma(b_j-a_j)}{\Gamma(\nu+b_j)},$$

and this proves the identity.

3. Primarily let us prove for any real a, $0\le a<m-1$, the following identity

$$\text{(24)}\qquad \int_0^1\cdots\int_0^1\frac{dx_2\cdots dx_m}{(1-x_2\cdots x_m)^a}={}_mF_{m-1}\left(\begin{matrix}1,1,\ldots,1,a\\ 2,\ldots,2,2\end{matrix}\,\middle|\,1\right),$$

where both the series and the integral converge.

Convergence of the series is the consequence of $2(m-1)-(m-1)-a=m-a-1>0$, see [7], subsection 2.2.

Let us prove that the integral in (24) converges. Let λ be real number such that $0<\lambda<1$. Then we derive

$$\int_0^\lambda\int_0^1\cdots\int_0^1\frac{dx_2\cdots dx_m}{(1-x_2\cdots x_m)^a}=\lambda\int_0^1\cdots\int_0^1\frac{dt_2\cdots dt_m}{(1-\lambda t_2\cdots t_m)^a}=$$
$$\lambda\cdot{}_mF_{m-1}\left(\begin{matrix}1,1,\ldots,1,a\\ 2,\ldots,2,2\end{matrix}\,\middle|\,\lambda\right).$$

The first equality is obtained by the transformation of variables $x_2=\lambda t_2$, $x_j=t_j$, $j\ge 3$, and the second one is a consequence of (23).

Since the convergence of the series (24) one can compute the limit for $\lambda \to 1$. This proves that the integral in (24) converges and the equality (24) itself.

Since $1 - x_j \le 1 - x_2 \cdots x_m$ on the cube $[0,1]^{m-1}$, we obtain

$$\left| \frac{\prod_{j=2}^m x_j^{a_j - 1}(1-x_j)^{b_j - a_j - 1}}{(1 - x_2 \cdots x_m)^{a_1}} \right| \le (1 - x_2 \ldots x_m)^{\Re\left(\sum_{j=2}^m (b_j - a_j - 1) - a_1\right)} = (1 - x_2 \ldots x_m)^{\delta - m}.$$

This proves the convergence of the integral $J_1(1)$.

For any $z, |z| \le 1$, we have $|1 - zx_2 \cdots x_m| \ge 1 - x_2 \cdots x_m$. Therefore the integral $J_1(z)$ uniformely converges on the set $|z| \le 1$, and is continuous function on this set. The sum of (20) is continuous function too. Hence the equality (23) is valid for every z with $|z| \le 1$. □

Lemma 2 demonstrates that the function $G_1(z)$ from Lemma 1 can be expressed by Euler's multiple integral only in the case when the set S_1 consists of one element. Following representation of generalized hypergeometric function as a Mellin-Barnes type integral is valid in less restrictive conditions on parameters, see [3], subsection 5.3.1.

Lemma 3. *Let $b_1 = 1$ and $a_1, \ldots, a_m, b_2, \ldots, b_m$ be complex numbers, $a_j \ne 0, -1, -2, \ldots,$ and let the integral $I(z)$ be defined by the equality*

$$I(z) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(s + a_j)}{\prod_{j=2}^m \Gamma(s + b_j)} \cdot \Gamma(-s) \cdot (-z)^s ds, \tag{25}$$

where the path of integration L is coming from $-i\infty$ to $i\infty$ and separates the poles of $\Gamma(s + a_j)$, $1 \le j \le m$ from the poles of $\Gamma(-s)$. Besides

$$(-z)^s = \exp(s \cdot (\log|z| + i \arg(-z))).$$

Under these conditions the following assertions hold.

1. For any $z \in \mathbb{C}$ such that $|\arg(-z)| < \pi$, $z \ne 0$, the integral $I(z)$ absolutely converges.

2. Suppose z is a positive real number $\arg(-z) = \pm\pi$ and $\delta > 1$, see (21); then the integral $I(z)$ converges absolutely.

3. For any z, $|\arg(-z)| \le \pi$ such that both the integral $I(z)$ and the series (20) converge the following equality holds

$${}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_2, \ldots, b_m \end{matrix} \middle| z \right) = \prod_{j=1}^m \frac{\Gamma(b_j)}{\Gamma(a_j)} \cdot I(z).$$

Proof. By $\Psi(\zeta)$ denote the function under the integral (25). Then we obtain

$$\Psi(\zeta) = \prod_{j=1}^m \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + b_j)} \cdot \frac{-\pi}{\sin(\pi\zeta)} (-z)^\zeta.$$

Due to

$$\frac{\Gamma(\zeta+a)}{\Gamma(\zeta+b)} = \zeta^{a-b}\cdot(1+O(|\zeta|^{-1})), \qquad |\arg\zeta| < \pi-\varepsilon, \tag{26}$$

and

$$\left|\frac{1}{\sin\zeta}\right| \le 3e^{-|t|} \qquad \zeta=\sigma+it, \quad |t|\ge 1, \tag{27}$$

see [8], we find

$$\left|\prod_{j=1}^{m}\frac{\Gamma(\zeta+a_j)}{\Gamma(\zeta+b_j)}\cdot\frac{-\pi}{\sin(\pi\zeta)}\right| \le c_1|\zeta|^{-\delta}e^{-\pi|t|},$$

where c_i here and later are positive constants depending only on z, a_j, b_j and the path L. Since

$$\left|(-z)^{\zeta}\right| = e^{\sigma\log|z|-t\arg(-z)} \le c_2 e^{|t\arg(-z)|},$$

then

$$|\Psi(\zeta)| \le c_3|\zeta|^{-\delta}e^{-|t|(\pi-|\arg(-z)|)}. \tag{28}$$

This inequality implies the absolute convergence of the integral from (32) for $|\arg(-z)|<\pi$ and for $|\arg(-z)|=\pi$, if we additionaly postulate $\delta>1$.

To prove the last assertion we suppose $|z|<1$ and $|\arg(-z)|<\pi$. Due to uniqueness theorem the identity will be true for other points of converge at $|z|\le 1$.

Denote $T=N+1/2$, where N is sufficiently large integer number. Since the inequality (28) is satisfied on the contour C composed of segments connecting the points iT, $T+iT$, $T-iT$, $-iT$, the integral $\int_C\Psi(s)ds$ tends to zero when $N\to+\infty$. Then the integral $I(z)$ can be evaluated as the sum of the residues of $\Psi(\zeta)$ at points $\zeta=0,1,2,\dots$ taken with the sign minus

$$I(z)=\sum_{n=0}^{\infty}\prod_{j=1}^{m}\frac{\Gamma(a_j+n)}{\Gamma(b_j+n)}z^n.$$

Since $\Gamma(a+n)=\Gamma(a)\cdot(a)_n$ the last identity proves the assertion. □

Corollary 1. *Let $b_1=1$ and $a_1,\dots,a_m$, $b_2,\dots,$ b_m be complex numbers satisfying*

$$\Re b_j > \Re a_j > 0, \qquad j=2,\dots,m.$$

Then for any $z \in \mathbb{C}$ under condition $|\arg(-z)| < \pi$ the following identity holds

$$\int_0^1 \cdots \int_0^1 \frac{\prod_{j=2}^m x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{(1-zx_2\cdots x_m)^{a_1}} dx_2 \cdots dx_m =$$

$$\frac{\prod_{j=2}^m \Gamma(b_j-a_j)}{\Gamma(a_1)} \cdot \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(s+a_j)}{\prod_{j=2}^m \Gamma(s+b_j)} \cdot \Gamma(-s)\cdot(-z)^s ds,$$

where the path of integration L is the same as in Lemma 3.

Suppose that $\delta > 1$ and z is a real number $0 < z \leq 1$, $\arg(-z) = \pm\pi$; then the above identity is satisfied.

Proof. For $|z| < 1$ the identity is a consequence of Lemmas 2 and 3. For remaining z it is valid due to uniqueness of the analytic continuation. □

3.2. Hypergeometric integrals. Here we will find integral representations for sums $G_r(z)$ defined in Proposition 1. These representations are useful in applications for computation of asimptotics for constructed linear forms.

Let $r \geq 2$ be integer and u be complex number. Denote

$$I_r(u) = \frac{1}{2\pi i}\int_L R(s)\left(\frac{\pi}{\sin \pi s}\right)^r e^{\pi i u s} ds, \tag{29}$$

where L is a path in complex plane the same as in Lemma 3. It is easy to check with the inequalities like (28), that the integral (29) converges for $|\Re u| < r$. The following assertion express the function $G_r(z)$ in terms of integrals (29) and is analogous in this respect to Lemma 3.

Theorem 1. *In conditions of Proposition 1 there exist constants $c_{\mu,\lambda}$ depending only on r, such that for any $z = e^{\pi i u}$, $|\Re u| < 2$, the following inequalities hold*

$$G_r(z) = \sum_{\mu=1}^r \sum_{\substack{|\lambda|<\mu \\ \lambda\equiv\mu \pmod 2}} c_{\mu,\lambda} I_\mu(u+\lambda)(\pi i u)^{r-\mu}.$$

All the integralls are evaluated on stright line $\Re s = 1/2 - a$ comming bottom-up.

Relations of this kind were used at first by L. A. Gutnik, [2] in opposite direction, and T. Hessami-Pilerhood, [5].

The proof of following Lemma gives the way to compute all coefficients $c_{\mu,\lambda}$.

Lemma 4. *For any integer $r \geq 2$ there exist rational numbers d_λ such that the following asymptotic equality holds*

$$(30) \qquad (\sin x)^{-r} \sum_{\substack{|\lambda|<r \\ \lambda \equiv r \pmod 2}} d_\lambda e^{i\lambda x} = \frac{1}{x^r} + O(1),$$

as $x \to 0$.

Proof. We prove this assertion by induction on r. For $r = 2$ it is true with $d_0 = 1$. Assume that this assertion is true for some $r \geq 1$.

Differentiate (30) and apply formulas

$$e^{i\lambda x} \cos x = \frac{1}{2}\left(e^{i(\lambda+1)x} + e^{-i(\lambda-1)x}\right),$$
$$e^{i\lambda x} \sin x = \frac{1}{2i}\left(e^{i(\lambda+1)x} - e^{-i(\lambda-1)x}\right),$$

we derive

$$(\sin x)^{-r-1} \sum_{\substack{|\lambda|<r \\ \lambda \equiv r \pmod 2}} \frac{d_\lambda}{2} \cdot \left((\lambda - r)e^{i(\lambda+1)x} - (\lambda + r)e^{-i(\lambda-1)x}\right) =$$
$$- rx^{-r-1} + O(1).$$

Last equality proves the assertion needed. $\square$

Another proof of Lemma 4 one can find in [9, Lemma 2.2.].

Proof of Theorem 1. With the coefficients d_λ introduced in Lemma 4 we find

$$(31) \qquad \sum_{\substack{|\lambda|<r \\ \lambda \equiv r \pmod 2}} d_\lambda I_r(u + \lambda) = \frac{1}{2\pi i} \int_L R(s)U(s)e^{\pi i u s} ds,$$

where

$$U(s) = \left(\frac{\pi}{\sin \pi s}\right)^r \sum_{\substack{|\lambda|<r \\ \lambda \equiv r \pmod 2}} d_\lambda e^{\pi i \lambda s}.$$

Lemma 4 implies that the function $U(s)$ has 1 as period, besides in a neighbourhood af every integer point ν the assymptotic equality holds

$$U(s) = \frac{1}{(s - \nu)^r} + O(1).$$

The integral in right-hand side of (31) equals to the sum of the residues at integer points $\nu > 1/2 - a$. Therefore

$$\sum_{\substack{|\lambda|<r \\ \lambda\equiv r \pmod 2}} d_\lambda I_r(u+\lambda) = \sum_{\nu=1-a}^{\infty} \mathrm{Res}_{s=\nu}\left(R(s)e^{\pi ius}\frac{1}{(s-\nu)^r}\right) =$$

$$\sum_{\nu=1-a}^{\infty}\sum_{\mu=1}^{r}\frac{1}{(\mu-1)!}R^{(\mu-1)}(\nu)e^{\pi i\nu u}\frac{(\pi iu)^{r-\mu}}{(r-\mu)!} = \sum_{\mu=1}^{r} G_\mu(e^{\pi iu})\frac{(\pi iu)^{r-\mu}}{(r-\mu)!}.$$

The last equality implies

$$G_r(e^{\pi iu}) = \sum_{\substack{|\lambda|<r \\ \lambda\equiv r \pmod 2}} d_\lambda I_r(u+\lambda) - \sum_{\mu=1}^{r-1} G_\mu(e^{\pi iu})\frac{(\pi iu)^{r-\mu}}{(r-\mu)!}.$$

This allows to enforce the inductive argument on r. □

The following Theorem connects multiple integrals of special kind and complex integrals (29).

Theorem 2. *Let m, r, $1 \le r < m$, be integers, c be negative real number and $a_1, \dots a_m$, $b_2, \dots, b_m$ be complex numbers satisfying*

$$\Re b_k > \Re a_k, \qquad k = 2, \dots, m,$$

$$-\Re a_k < c < \Re(b_j - a_j - a_1), \qquad k = 1, \dots, m, \quad j = 2, \dots, r.$$

Then for any complex number z, $|\arg z| < \pi$ the following identity holds

$$\int\limits_{[0,1]^{m-1}} \frac{\prod_{j=2}^m x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}dx_2\cdots dx_m}{((1-x_2)\cdots(1-x_r)+zx_2\cdots x_m)^{a_1}} = \frac{\prod_{k=r+1}^m \Gamma(b_k-a_k)}{\Gamma(a_1)\prod_{k=2}^r \Gamma(b_k-a_1)} \tag{32}$$

$$\times \frac{1}{2\pi i}\int\limits_L \frac{\prod_{j=1}^m \Gamma(\zeta+a_j)\prod_{j=2}^r \Gamma(b_j-a_j-a_1-\zeta)}{\prod_{j=r+1}^m \Gamma(\zeta+b_j)}\Gamma(-\zeta)z^\zeta d\zeta,$$

where both integrals converge. Here

$$z^\zeta = \exp(\zeta\cdot(\log|z| + i\arg z)),$$

and the path of integration L is the stright line $\Re\zeta = c$, coming from $-i\infty$ to $i\infty$.

Proof. Obviously, the integral on the left-hand side of (32) converges.

Let us prove the absolute convergence of the integral in right-hand side for $|\arg z| < \pi r$.

Denote the function under the right integral of (32) as $\Phi(\zeta)$. Then we have

$$\Phi(\zeta) = \prod_{j=1}^m \frac{\Gamma(\zeta+a_j)}{\Gamma(\zeta+\xi_j)}\cdot\prod_{j=1}^r \frac{\pi}{\sin(\pi(\zeta+\xi_j))}z^\zeta,$$

where

$$\xi_1 = 1, \quad \xi_j = 1 + a_1 + a_j - b_j, \quad j = 2, \dots, r, \quad \xi_j = b_j, \quad j = r+1, \dots, m.$$

Using (26), (27) we obtain

$$\left| \prod_{j=1}^{m} \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + \xi_j)} \cdot \prod_{j=1}^{r} \frac{\pi}{\sin(\pi(\zeta + \xi_j))} \right| \leq c_4 |\zeta|^{c_5} e^{-\pi r |t|}.$$

Since on the path of integration

$$\left| z^{\zeta} \right| = e^{c \log|z| - t \arg z} \leq c_6 e^{|t \arg z|},$$

we get

$$|\Phi(\zeta)| \leq c_7 |t|^{c_8} e^{-|t|(\pi r - |\arg z|)}. \tag{33}$$

This inequality implies absolute convergence of the integral in right-hand side of (32) for $|\arg z| < r\pi$.

According to Lemma 2 the left-hand side of (32) can be expressed in the form

$$I = \int\limits_{[0,1]^{r-1}} \prod_{j=2}^{r} x_j^{a_j - 1} (1 - x_j)^{b_j - a_j - a_1 - 1} \cdot \Bigg(\int\limits_{[0,1]^{m-r-1}} \prod_{j=r+1}^{m} x_j^{a_j - 1} \tag{34}$$

$$\times (1 - x_j)^{b_j - a_j - 1} \left(1 + \frac{z x_2 \cdots x_m}{(1 - x_2) \cdots (1 - x_r)} \right)^{-a_1} dx_{r+1} \cdots dx_m \Bigg) dx_2 \dots dx_r.$$

Apply to the inner integral of the right-hand side of (34) Corollary 1 with the path L given in conditions of the Proposition. Then we derive

$$I = \frac{\prod_{k=r+1}^{m} \Gamma(b_k - a_k)}{\Gamma(a_1)} \int\limits_{[0,1]^{r-1}} \prod_{j=2}^{r} x_j^{a_j - 1} (1 - x_j)^{b_j - a_j - a_1 - 1}$$

$$\times \left(\frac{1}{2\pi i} \int\limits_{L} \prod_{j=r+1}^{m} \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + b_j)} \Gamma(\zeta + a_1) \Gamma(-\zeta) \left(\frac{z x_2 \cdots x_r}{(1 - x_2) \cdots (1 - x_r)} \right)^{\zeta} d\zeta \right) dx_2 \cdots dx_r,$$

and

$$I = \frac{\prod_{k=r+1}^{m} \Gamma(b_k - a_k)}{\Gamma(a_1)} \cdot \int\limits_{[0,1]^{r-1}} \Bigg(\frac{1}{2\pi i} \int\limits_{L} \prod_{j=2}^{r} x_j^{a_j + \zeta - 1} \tag{35}$$

$$\times (1 - x_j)^{b_j - a_j - a_1 - 1 - \zeta} \prod_{j=r+1}^{m} \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + b_j)} \Gamma(\zeta + a_1) \Gamma(-\zeta) \cdot z^{\zeta} d\zeta \Bigg) dx_2 \cdots dx_r.$$

Now one can change the order of integration and to make the external integration on ζ:

$$I = \frac{\prod_{k=r+1}^{m}\Gamma(b_k - a_k)}{\Gamma(a_1)} \cdot \frac{1}{2\pi i}\int\limits_L \prod_{j=r+1}^{m} \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + b_j)}\Gamma(\zeta + a_1)\Gamma(-\zeta)\cdot z^{\zeta}\cdot$$

$$\Big(\int\limits_{[0,1]^{r-1}} \prod_{j=2}^{r} x_j^{a_j+\zeta-1}(1-x_j)^{b_j-a_j-a_1-1-\zeta}dx_2\cdots dx_r\Big)d\zeta.$$

On the path L the following equalities hold

$$\Re(\zeta + a_j) = \Re a_j + c > 0, \qquad \Re(b_j - a_j - a_1 - \zeta) = \Re(b_j - a_j - a_1) - c > 0.$$

Therefore

$$\int\limits_0^1 x_j^{a_j+\zeta-1}(1-x_j)^{b_j-a_j-a_1-1-\zeta}dx_j = \frac{\Gamma(a_j+\zeta)\Gamma(b_j-a_j-a_1-\zeta)}{\Gamma(b_j-a_1)},$$

and we obtain (32).

For any $\zeta \in L$ we have

$$\int\limits_{[0,1]^{r-1}} \Big|\prod_{j=2}^{r} x_j^{a_j+\zeta-1}(1-x_j)^{b_j-a_j-a_1-1-\zeta}\Big|dx_2\cdots dx_r =$$

$$\int\limits_{[0,1]^{r-1}} \prod_{j=2}^{r} x_j^{\Re a_j+c-1}(1-x_j)^{\Re(b_j-a_j-a_1)-1-c}dx_2\cdots dx_r.$$

This implies that the integral converges and does not depend on the variable ζ. In addition the integral

$$\frac{1}{2\pi i}\int\limits_L \prod_{j=r+1}^{m} \frac{\Gamma(\zeta + a_j)}{\Gamma(\zeta + b_j)}\Gamma(\zeta + a_1)\Gamma(-\zeta)\cdot z^{\zeta}d\zeta,$$

absolutely converges due to Lemma 3. This grounds the possibility to change the order of integration in (35). $\square$

Note that for integer parameters a_j, b_j the integral in right-hand side of (32) owing to the relation $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$ can be written in the form (29). And conversely the integral (29) can be expressed as multiple integral. Namely the integral $I_r(u)$ for $z = e^{\pi i u}$ concides up to the sign with

$$\frac{\Gamma(a_1)\prod_{k=2}^{r}\Gamma(1+a_k-b_k)}{\prod_{k=r+1}^{m}\Gamma(b_k-a_k)}\cdot$$
$$\int\limits_{[0,1]^{m-1}} \frac{\prod_{j=2}^{r} x_j^{a_j-1}(1-x_j)^{a_1-b_j}\prod_{j=r+1}^{m} x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{((1-x_2)\cdots(1-x_r)+zx_2\cdots x_m)^{a_1}}dx_2\cdots dx_m,$$

in the case of convergence.

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