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A study of the mean value of the error term in the mean square formula of the Riemann zeta-function in the critical strip $3/4 \leq \sigma < 1$

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Résumé. Pour σ dans la bandre critique $1/2 < \sigma < 1$, on note $E_\sigma(T)$ le terme d'erreur de la formule asymptotique de $\int_1^T |\zeta(\sigma+it)|^2\,dt$ (pour T grand). C'est un analogue du terme d'erreur classique $E(T)$ $(= E_{1/2}(T))$. L'étude de $E(T)$ a une longue histoire, mais celle de $E_\sigma(T)$ est assez récente. En particulier, lorsque $3/4 < \sigma < 1$, on connaît peu d'informations sur $E_\sigma(T)$. Pour en gagner, nous étudions la moyenne $\int_1^T E_\sigma(u)\,du$. Dans cet article, nous donnons une expression en série de type Atkinson et explorons quelques une des propriétés de la moyenne comme fonction en T.

Abstract. Let $E_\sigma(T)$ be the error term in the mean square formula of the Riemann zeta-function in the critical strip $1/2 < \sigma < 1$. It is an analogue of the classical error term $E(T)$. The research of $E(T)$ has a long history but the investigation of $E_\sigma(T)$ is quite new. In particular there is only a few information known about $E_\sigma(T)$ for $3/4 < \sigma < 1$. As an exploration, we study its mean value $\int_1^T E_\sigma(u)\,du$. In this paper, we give it an Atkinson-type series expansion and explore many of its properties as a function of T.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function, and let

$$E(T) = \int_0^T |\zeta(1/2+it)|^2\,dt - T(\log\frac{T}{2\pi} + 2\gamma - 1)$$

denote the error term in the mean-square formula for $\zeta(s)$ (on the critical line). The behaviour of $E(T)$ is interesting and many papers are devoted

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to study this function. Analogously, it is defined for $1/2 < \sigma < 1$,

$$E_\sigma(T) = \int_0^T |\zeta(\sigma + it)|^2\, dt - \left(\zeta(2\sigma)T + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}\right).$$

The behaviour of $E_\sigma(T)$ is very interesting too, and in fact, more delicate analysis is required to explore its properties such as the Atkinson-type series expansion and mean square formula, see ([17]-[20]). Excellent surveys are given in [11] and [18].

In the critical strip $1/2 < \sigma < 1$, our knowledge of $E_\sigma(T)$ is not 'uniform', for example, an asymptotic formula for the mean square is available for $1/2 < \sigma \leq 3/4$ but not for the other part. In fact, not much is known for the case $3/4 < \sigma < 1$, except perhaps some upper bound estimates and

$$T \ll \int_1^T E_\sigma(t)^2\, dt \ll T \qquad (3/4 < \sigma < 1). \tag{1.1}$$

(See [7], [20] and [14].) To furnish this part, we look at the mean value $\int_1^T E_\sigma(u)\, du$. The mean values of $E(T)$ and $E_\sigma(T)$ $(1/2 < \sigma < 3/4)$ are respectively studied in [2] and [6], each of which gives an Atkinson-type expansion. Correspondingly, we prove an analogous formula with a good error term in the case $3/4 \leq \sigma < 1$. Actually, the tight lower bound in (1.1) is shown in [14] based on this formula. The proof of the asymptotic formula relies on the argument of [2] and uses the tools available in [2] and [19]. But there is a difficulty which we need to get around. In [2], Hafner and Ivić used a result of Jutila [9] on transformation of Dirichlet Polynomials, which depends on the formula

$$\textstyle\sum'_{a\leq n\leq b} d(n)f(n) = \displaystyle\int_a^b (\log x + 2\gamma) f(x)\, dx + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\alpha(nx)\, dx,$$

where $\alpha(x) = 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x})$ is a combination of the Bessel functions K_0 and Y_0. It is not available in our case but this can be avoided by using an idea in [19].

In addition, we shall regard the mean value as a function of T and study its behaviour; more precisely, we consider

$$G_\sigma(T) = \int_1^T (E_\sigma(t) + 2\pi\zeta(2\sigma - 1))\, dt. \tag{1.2}$$

(The remark below Corollary 1 explains the inclusion of $2\pi\zeta(2\sigma-1)T$.) Unlike the case $1/2 \leq \sigma < 3/4$, the function $G_\sigma(T)$ is now more fluctuating. Nevertheless we can still explore many interesting properties, including some power moments, $\Omega_\pm$-results, gaps between sign-changes and limiting distribution functions, by using the tools in [19], [23], [4], [3], [1] and [13]. Particularly, we can determine the exact order of magnitude of the gaps of sign-changes (see Theorems 5 and 6). The limiting distribution function is not computed in the case $1/2 \le \sigma < 3/4$, perhaps because it is less interesting in the sense that the exact order of magnitude of $G_\sigma(t)$ $(1/2 \le \sigma < 3/4)$ is known; therefore, the limiting distribution is 'compactly supported'. Here, a limiting distribution $P(u)$ is said to be compactly supported if $P(u) = 0$ for all $u \le a$ and $P(u) = 1$ for all $u \ge b$, for some constants $a < b$. (Note that a distribution function is non-decreasing.) However, in our case the distribution never vanishes (i.e. never equal to 0 or 1), and we evaluate the rate of decay.

2. Statement of results

Throughout the paper, we assume $3/4 \le \sigma < 1$ to be fixed and use c, c' and c'' to denote some constants which may differ at each occurrence. The implied constants in $\ll$- or O-symbols and the unspecified positive constants c_i $(i = 1, 2, \ldots)$ may depend on σ.

Let $\sigma_a(n) = \sum_{d|n} d^a$ and $\operatorname{arsinh} x = \log(x + \sqrt{x^2+1})$. We define

$$\Sigma_1(t,X) = \sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \sum_{n\le X} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(t,n) \sin f(t,n),$$

$$\Sigma_2(t,X) = 2\left(\frac{t}{2\pi}\right)^{1/2-\sigma} \sum_{n\le B(t,\sqrt{X})} \frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}} \left(\log\frac{t}{2\pi n}\right)^{-2} \sin g(t,n),$$

where

$$e_2(t,n) = \left(1+\frac{\pi n}{2t}\right)^{-1/4} \left(\sqrt{\frac{2t}{\pi n}}\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2},$$

$$f(t,n) = 2t\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}} + (2\pi n t + \pi^2 n^2)^{1/2} - \frac{\pi}{4},$$

$$g(t,n) = t\log\frac{t}{2\pi n} - t + \frac{\pi}{4},$$

$$B(t,\sqrt{X}) = \frac{t}{2\pi} + \frac{X}{2} - \sqrt{X}\Big(\frac{t}{2\pi}+\frac{X}{4}\Big)^{1/2} = \Big(\Big(\frac{t}{2\pi}+\frac{X}{4}\Big)^{1/2} - \frac{\sqrt{X}}{2}\Big)^2.$$

Theorem 1. *Let $\sigma \in [3/4, 1)$, $T \ge 1$ and $N \asymp T$. We have*

$$\int_1^T E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + \Sigma_1(T,N) - \Sigma_2(T,N) + O(\log^2 T).$$

The next result follows with the trivial bounds on $\Sigma_1(T,N)$ and $\Sigma_2(T,N)$.

Corollary 1. *For all $T \ge 3$, we have*

$$\int_1^T E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + O(\sqrt{T}) \qquad (3/4 < \sigma < 1),$$

and

$$\int_1^T E_{3/4}(t)\,dt = -2\pi\zeta(1/2)T + O(\sqrt{T}\log T).$$

Remark. It suggests that $E_\sigma(t)$ is a superimposition of the constant $-2\pi\zeta(2\sigma-1)$ and an oscillatory function, say, $E^*_\sigma(t)$. (Indeed this viewpoint had appeared in [18].)

Define $G_\sigma(t) = \int_1^t E^*_\sigma(t)\,dt$, which is (1.2). Then,

$$G_\sigma(t) \ll t^{1/2} \quad (3/4 < \sigma < 1) \quad \text{and} \quad G_{3/4}(t) \ll t^{1/2}\log T.$$

Integrating termwisely with partial integrations, one gets

$$\int_T^{2T} G_\sigma(t)\,dt = o(T^{1+(5/4-\sigma)}) \qquad (3/4 \le \sigma < 1). \tag{2.1}$$

In addition, we have the following higher power moments.

Theorem 2. *Let $\sigma \in [3/4,1)$ and $T \ge 1$. We have*

(1) $\displaystyle\int_T^{2T} G_\sigma(t)^2\,dt = B(\sigma)\int_T^{2T} (t/(2\pi))^{5/2-2\sigma}\,dt + O(T^{3-2\sigma})$,

(2) $\displaystyle\int_T^{2T} G_\sigma(t)^3\,dt = -C(\sigma)\int_T^{2T} (t/(2\pi))^{15/4-3\sigma}\,dt + O(T^{(13-8\sigma)/3})$,

where $B(\sigma)$ and $C(\sigma)$ are defined by

$$B(\sigma) = \sum_{n=1}^\infty \sigma_{1-2\sigma}(n)^2 n^{2\sigma-7/2} = \zeta(7/2-2\sigma)\zeta(3/2+2\sigma)\zeta(5/2)^2\zeta(5)^{-1},$$

$$C(\sigma) = \frac{3}{2}\sum_{s=1}^\infty \frac{\mu(s)^2}{s^{21/4-3\sigma}} \sum_{a,b=1}^\infty \frac{\sigma_{1-2\sigma}(sa^2)}{a^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(sb^2)}{b^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(s(a+b)^2)}{(a+b)^{7/2-2\sigma}}.$$

(3) *for any real $k \in [0, A_0)$ and any odd integer $l \in [0, A_0)$ where $A_0 = (\sigma-3/4)^{-1}$,*

$$\int_1^T |G_\sigma(t)|^k\,dt \sim \alpha_k(\sigma)T^{1+k(5/4-\sigma)}$$

and

$$\int_1^T G_\sigma(t)^l\,dt \sim \beta_l(\sigma)T^{1+l(5/4-\sigma)}.$$

for some constants $\alpha_k(\sigma) > 0$ and $\beta_l(\sigma)$ depending on σ. (A_0 denotes ∞ when $\sigma = 3/4$.)

Remark. We have no information about the value of $\beta_l(\sigma)$, which may be positive, negative or even zero. (See [16] for possible peculiar properties of series of this type.)

It is expected that $G_\sigma(t)$ is oscillatory and its order of magnitude of $G_\sigma(t)$ is about $t^{5/4-\sigma}$. (2.1) shows a big cancellation between the positive and negative parts, but Theorem 2 (2) suggests that it skews towards negative. This phenomenon also appears in the case $1/2 \le \sigma < 3/4$. Now, we look at its distribution of values from the statistical viewpoint.

Theorem 3. *For $3/4 \le \sigma < 1$, the limiting distribution $D_\sigma(u)$ of the function $t^{\sigma-5/4}G_\sigma(t)$ exists, and is equal to the distribution of the random series $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ where*

$$a_n(t) = \sqrt{2}\frac{\mu(n)^2}{n^{7/4-\sigma}}\sum_{r=1}^{\infty}(-1)^{nr}\frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}}\sin(2\pi rt - \pi/4)$$

and t_n's are independent random variables uniformly distributed on $[0,1]$. Define $\mathrm{tail}(D_\sigma(u)) = 1 - D_\sigma(u)$ *for $u \ge 0$ and $D_\sigma(u)$ for $u < 0$. Then*

$$\begin{gathered}\exp(-c_1\exp(|u|)) \ll \mathrm{tail}(D_{3/4}(u)) \ll \exp(-c_2\exp(|u|)),\\ \exp(-c_3|u|^{4/(4\sigma-3)}) \ll \mathrm{tail}(D_\sigma(u)) \ll \exp(-c_4|u|^{4/(4\sigma-3)})\end{gathered} \tag{2.2}$$

for $3/4 < \sigma < 1$.

Remark. $D_\sigma(u)$ is non-symmetric and skews towards the negative side because of Theorem 2 (2). Again it is true for $1/2 \le \sigma < 3/4$. But in the case $1/2 \le \sigma < 3/4$, the closure of the set $\{u \in \mathbb{R} : 0 < D_\sigma(u) < 1\}$ is compact and it differs from our case.

To investigate the oscillatory nature, we consider the extreme values of $G_\sigma(t)$ and the frequency of occurrence of large values. These are revealed in the following three results.

Theorem 4. *We have*

$$G_{3/4}(T) = \Omega_-(\sqrt{T}\log\log T) \quad \text{and} \quad G_{3/4}(T) = \Omega_+(\sqrt{T}\log\log\log T).$$

For $3/4 < \sigma < 1$,

$$G_\sigma(T) = \Omega_-(T^{5/4-\sigma}(\log T)^{\sigma-3/4})$$

and

$$G_\sigma(T) = \Omega_+\Big(T^{5/4-\sigma}\exp\big(c_5\frac{(\log\log T)^{\sigma-3/4}}{(\log\log\log T)^{7/4-\sigma}}\big)\Big).$$

Theorem 5. *For $\sigma \in [3/4, 1)$ and for every sufficiently large T, there exist t_1, $t_2 \in [T, T + c_6\sqrt{T}]$ such that $G_\sigma(t_1) \ge c_7 t_1^{5/4-\sigma}$ and $G_\sigma(t_2) \le -c_7 t_2^{5/4-\sigma}$. In particular, $G_\sigma(t)$ has (at least) one sign change in every interval of the form $[T, T + c_8\sqrt{T}]$.*

Theorem 6. *Let $\sigma \in [3/4, 1)$ and $\delta > 0$ be a fixed small number. Then for all sufficiently large $T \ge T_0(\delta)$, there are two sets S^+ and S^- of disjoint intervals in $[T, 2T]$ such that*

1. *every interval in $S^\pm$ is of length $c_9\delta\sqrt{T}$,*
2. *the cardinality of $S^\pm \ge c_{10}\delta^{4(1-\sigma)}\sqrt{T}$,*
3. *$\pm G_\sigma(t) \ge (c_{11} - \delta^{5/2-2\sigma})t^{5/4-\sigma}$ for all $t \in I$ with $I \in S^\pm$ respectively.*

Remark. Theorems 5 and 6 determine the order of magnitude of the gaps between sign-changes.

3. Series representation

This section is to prove Theorem 1 and we need two lemmas, which come from [2, Lemma 3] and [19, Lemma 1] with [22] respectively.

Lemma 3.1. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \ne 1$, $0 < a < 1/2$, $a < T/(8\pi k)$, $b \ge T$, $k \ge 1$, $T \ge 1$,*

$$U(t) = \left(\frac{t}{2\pi k} + \frac{1}{4}\right)^{1/2}, V(t) = 2\operatorname{arsinh}\sqrt{\frac{\pi k}{2t}},$$

$$L_k(t) = (2ki\sqrt{\pi})^{-1} t^{1/2} V(t)^{-\gamma-1} U(t)^{-1/2} \left(U(t) - \frac{1}{2}\right)^{-\alpha} \left(U(t) + \frac{1}{2}\right)^{-\beta} \times \exp\left(itV(t) + 2\pi ikU(t) - \pi ik + \frac{\pi i}{4}\right),$$

and

$$J(T) = \int_T^{2T} \int_a^b y^{-\alpha}(1+y)^{-\beta} \left(\log\frac{1+y}{y}\right)^{-\gamma} \times \exp(it\log(1+1/y) + 2\pi iky)\,dy\,dt.$$

Then uniformly for $|\alpha - 1| \ge \epsilon$, $1 \le k \le T + 1$, we have

$$J(T) = L_k(2T) - L_k(T) + O(a^{1-\alpha}) + O(Tk^{-1}b^{\gamma-\alpha-\beta}) + O((T/k)^{(\gamma+1-\alpha-\beta)/2}T^{-1/4}k^{-5/4}).$$

In the case $-k$ in place of k, the result holds without $L_k(2T) - L_k(T)$ for the corresponding integral.

Lemma 3.2. *Let*

$$\Delta_{1-2\sigma}(t) = {\textstyle\sum'_{n\le t}}\sigma_{1-2\sigma}(n) - \left(\zeta(2\sigma)t + \frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma} - \frac{1}{2}\zeta(2\sigma-1)\right)$$

where the sum $\sum'_{n\le t}$ *counts half of the last term only when* t *is an integer. Define* $\tilde{\Delta}_{1-2\sigma}(\xi) = \int_0^\xi \Delta_{1-2\sigma}(t)\,dt - \zeta(2\sigma-2)/12$. *Assuming* $3/4 \le \sigma < 1$, *we have for* $\xi \ge 1$,

$$\begin{aligned}\tilde{\Delta}_{1-2\sigma}(\xi) = C_1\xi^{5/4-\sigma}&\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\cos(4\pi\sqrt{n\xi}+\pi/4)\\ &+ C_2\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-9/4}\cos(4\pi\sqrt{n\xi}-\pi/4)\\ &+ O(\xi^{1/4-\sigma})\end{aligned}$$

where the two infinite series on the right-hand side are uniformly convergent on any finite closed subinterval in $(0,\infty)$, *and the values of the constants are* $C_1 = -1/(2\sqrt{2}\pi^2)$, $C_2 = (5-4\sigma)(7-4\sigma)/(64\sqrt{2}\pi^3)$. *In addition, we have for* $3/4 \le \sigma < 1$,

$$\Delta_{1-2\sigma}(v) \ll v^{1-\sigma}, \qquad \int_1^x \Delta_{1-2\sigma}(v)^2\,dv \ll x\log x,$$

$$\tilde{\Delta}_{1-2\sigma}(\xi) \ll \xi^r\log\xi, \qquad \int_1^x \tilde{\Delta}_{1-2\sigma}(v)^2\,dv \ll x^{7/2-2\sigma}$$

where $0 < r = -(4\sigma^2-7\sigma+2)/(4\sigma-1) \le 1/2$.

Proof of Theorem 1. From [17, (3.4)] and [20, (3.1)], we have

$$\begin{aligned}\int_{-t}^{t}|\zeta(\sigma+iu)|^2\,du = 2\zeta(2\sigma)t &+ 2\zeta(2\sigma-1)\Gamma(2\sigma-1)\frac{\sin(\pi\sigma)}{1-\sigma}t^{2-2\sigma}\\ &- 2i\int_{\sigma-it}^{\sigma+it}g(u,2\sigma-u)\,du + O(\min(1,|t|^{-2\sigma})).\end{aligned}$$

(Note that the value of c_3 in [20, (3.1)] is zero.) Hence, we have

$$E_\sigma(t) = -i\int_{\sigma-it}^{\sigma+it}g(u,2\sigma-u)\,du + O(\min(1,t^{-2\sigma})).$$

Define

$$h(u,\xi) = 2\int_0^\infty y^{-u}(1+y)^{u-2\sigma}\cos(2\pi\xi y)\,dy. \tag{3.1}$$

Assume $AT \le X \le T$ and X is not an integer where $0 < A < 1$ is a constant. Then, following [19, p.364-365], we define

$$
\begin{aligned}
\mathrm{G}_1(t) &= \sum_{n \le X} \sigma_{1-2\sigma}(n) \int_{\sigma-it}^{\sigma+it} h(u,n)\,du,\\
\mathrm{G}_2(t) &= \Delta_{1-2\sigma}(X) \int_{\sigma-it}^{\sigma+it} h(u,X)\,du,\\
\mathrm{G}_3(t) &= \int_{\sigma-it}^{\sigma+it} \int_X^\infty (\zeta(2\sigma)+\zeta(2-2\sigma)\xi^{1-2\sigma}) h(u,\xi)\,d\xi\,du,\\
\mathrm{G}_4^*(t) &= \tilde{\Delta}_{1-2\sigma}(X) \int_{\sigma-it}^{\sigma+it} \frac{\partial h}{\partial \xi}(u,X)\,du,\\
\mathrm{G}_4^{**}(t) &= \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi) \int_{\sigma-it}^{\sigma+it} \frac{\partial^2 h}{\partial \xi^2}(u,\xi)\,du\,d\xi.
\end{aligned} \tag{3.2}
$$

Then, we have

$$
\begin{aligned}
\int_T^{2T} E_\sigma(t)\,dt = -i \int_T^{2T} \mathrm{G}_1(t)\,dt + i \int_T^{2T} \mathrm{G}_2(t)\,dt - i \int_T^{2T} \mathrm{G}_3(t)\,dt\\
- i \int_T^{2T} \mathrm{G}_4^*(t)\,dt - i \int_T^{2T} \mathrm{G}_4^{**}(t)\,dt + O(1)
\end{aligned} \tag{3.3}
$$

1) Evaluation of $\int_T^{2T} \mathrm{G}_1(t)\,dt$. By Lemma 3.1 with $\gamma = 1$, $\alpha = \beta = \sigma$, we have from (3.1),

$$
\begin{aligned}
\int_T^{2T} \int_{\sigma-it}^{\sigma+it} h(u,n)\,du\,dt &= 4i \int_T^{2T} \int_0^\infty (y(1+y))^{-\sigma} (\log(1+1/y))^{-1}\\
&\qquad \times \sin(t\log((1+y)/y)) \cos(2\pi n y)\,dy\,dt\\
&= 2i \;\mathrm{Im} \int_T^{2T} \int_0^\infty (y(1+y))^{-\sigma} (\log(1+1/y))^{-1}\\
&\qquad \times \Big\{ \exp(i(t\log((1+y)/y) + 2\pi n y))\\
&\qquad + \exp(i(t\log((1+y)/y) - 2\pi n y)) \Big\}\,dy\,dt\\
&= 2i \;\mathrm{Im}\; (L_n(2T) - L_n(T)) + O(T^{3/4-\sigma} n^{\sigma-9/4})
\end{aligned}
$$

Noting that

$$
L_n(t) = (i\sqrt{2})^{-1} (t/(2\pi))^{5/4-\sigma} (-1)^n n^{\sigma-7/4} e_2(t,n) \exp(i(f(t,n)+\pi/2)),
$$

we get with (3.2) that

$$\int_T^{2T} \mathrm{G}_1(t)\,dt = \sqrt{2}i\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \sum_{n\le X}(-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(t,n)\sin f(t,n)\Bigg|_T^{2T}$$

$$+\,O(T^{3/4-\sigma}). \tag{3.4}$$

2) Evaluation of $\int_T^{2T} \mathrm{G}_2(t)\,dt$. The treatment is similar to G_1. From (3.2) and Lemma 3.1,

$$\int_T^{2T}\int_{\sigma-it}^{\sigma+it} h(u,X)\,du\,dt = 2i\ \mathrm{Im}(L_X(2T)-L_X(T)) + O(T^{3/4-\sigma}X^{\sigma-9/4}).$$

Since $L_X(t) \ll t^{5/4-\sigma}X^{\sigma-7/4} \ll T^{-1/2}$ for $t=T$ or $2T$, we have

$$\int_T^{2T} \mathrm{G}_2(t)\,dt \ll \Delta_{1-2\sigma}(X)T^{-1/2} \ll T^{1/2-\sigma}. \tag{3.5}$$

3) Evaluation of $\int_T^{2T} \mathrm{G}_3(t)\,dt$. Using [17, (4.6)], we have

$$\begin{aligned}\mathrm{G}_3(t) = -2i\pi^{-1}(\zeta(2\sigma)&\\ +\zeta(2-2\sigma)X^{1-2\sigma})&\int_0^\infty y^{-\sigma-1}(1+y)^{-\sigma}(\log(1+1/y))^{-1}\\ &\times \sin(2\pi Xy)\sin(t\log(1+1/y))\,dy\\ +(1-2\sigma)\pi^{-1}\zeta(2-2\sigma)X^{1-2\sigma}&\int_0^\infty y^{-1}(1+y)^{1-2\sigma}\sin(2\pi Xy)\\ &\times\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(\frac{1+y}{y}\right)^u du\,dy.\end{aligned}$$

Direct computation shows that for $y>0$,

$$\begin{aligned}\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}(1+1/y)^u\,du &= 2\pi i\left(\frac{1+y}{y}\right)^{2\sigma-1}\\ &+\Big(\int_{-\infty+it}^{\sigma+it}+\int_{\sigma-it}^{-\infty-it}\Big)(1+1/y)^u(u+1-2\sigma)^{-1}\,du.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_T^{2T}\mathrm{G}_3(t)\,dt &= 2i(1-2\sigma)\zeta(2-2\sigma)TX^{1-2\sigma}I_1\\ &-2i\pi^{-1}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})I_2\\ &+\pi^{-1}(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}I_3\end{aligned} \tag{3.6}$$

where

$$I_1 = \int_0^\infty y^{-2\sigma} \sin(2\pi Xy)\, dy$$

$$I_2 = \int_T^{2T} \int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}(\log(1+1/y))^{-1} \times \sin(2\pi Xy)\sin(t\log(1+1/y))\, dy\, dt,$$

$$I_3 = \int_T^{2T} \int_0^\infty y^{-1}(1+y)^{1-2\sigma} \sin(2\pi Xy) \times \left(\int_{-\infty+it}^{\sigma+it} + \int_{\sigma-it}^{-\infty-it}\right)(1+1/y)^u (u+1-2\sigma)^{-1}\, du\, dy\, dt.$$

Then, $I_1 = 2^{2\sigma-2}\pi^{2\sigma}X^{2\sigma-1}/(\Gamma(2\sigma)\sin(\pi\sigma))$ which is the main contribution. Interchanging the integrals, we have

$$I_2 = -\int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}\big(\log(1+1/y)\big)^{-2} \times \sin(2\pi Xy)\cos(t\log(1+1/y))\Big|_{t=T}^{t=2T}\, dy.$$

We split the integral into two parts $\int_0^c + \int_c^\infty$ for some large constant $c > 0$. Expressing the product $\sin(\cdots)\cos(\cdots)$ as a combination of $\exp(i(t\log(1+1/y) \pm 2\pi Xy))$, since $(d/dy)(t\log(1+1/y) \pm 2\pi Xy) = \pm 2\pi X - t/(y(1+y)) \gg X$ for $y \geq c$ (recall $t = T$ or $2T$), the integral $\int_c^\infty$ is $\ll X^{-1}$ by the first derivative test. Applying the mean value theorem for integrals, we have

$$\int_0^c \ll \left| \int_{c'}^{c''} y^{-1-\sigma}(1+y)^{-1}\sin(2\pi Xy)\cos(t\log(1+1/y))\, dy \right|.$$

Integration by parts yields that the last integral $\int_{c'}^{c''}$ equals

$$t^{-1}\Big(y^{-\sigma}\sin(2\pi Xy)\sin(t\log(1+1/y))\Big|_{c'}^{c''} - \int_{c'}^{c''} O(y^{-\sigma-1}|\sin(2\pi Xy)| + y^{-\sigma}X)\, dy\Big) \ll 1. \tag{3.7}$$

Hence $I_2 \ll 1$. For I_3, the extra integration over t is in fact not necessary to yield our bound. Thus, we write $I_3 = \int_T^{2T}(I_{31}+I_{32})\, dt$, separated according to the integrals over u. I_{31} and I_{32} are treated in the same way, so we work out I_{31} only. Using integration by parts over u,

$$I_{31} = \int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y)) \times \left\{ \frac{(1+1/y)^\alpha}{\alpha+1-2\sigma+it}\bigg|_{\alpha=-\infty}^{\alpha=\sigma} + \int_{-\infty}^{\sigma}(1+1/y)^\alpha \frac{d\alpha}{(\alpha+1-2\sigma+it)^2} \right\} dy.$$

Then we consider

$$\int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))(1+1/y)^\alpha dy.$$

Again, we split the integral into $\int_0^c + \int_c^\infty$. Then $\int_c^\infty \ll X^{-1}$. If $\alpha \le -2$, then $\int_0^c \ll 1$ trivially; otherwise, we have (see (3.7))

$$\int_0^c \ll \left| \int_{c'}^{c''} y^{-1-\alpha}(1+y)^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))\,dy \right| \ll 1.$$

Therefore, $I_{31} \ll T^{-1}$ and so $I_3 \ll 1$. Putting these estimates into (3.6), we get

$$\int_T^{2T} \mathrm{G}_3(t)\,dt = i2^{2\sigma-1}\pi^{2\sigma}\frac{(1-2\sigma)\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\pi\sigma)}T + O(1)$$

$$= -2\pi i\zeta(2\sigma-1)T + O(1). \tag{3.8}$$

4) Evaluation of $\int_T^{2T} \mathrm{G}_4^*(t)\,dt$. From [19, Section 4], we obtain

$$\int_T^{2T} \mathrm{G}_4^*(t)\,dt = 4i\tilde{\Delta}_{1-2\sigma}(X)((2\sigma-1)I_1 + I_2 - \sigma I_3 - I_4)$$

where by Lemma 3.1, (recall $L_X(t) \ll T^{-1/2} \ll X^{-1/2}$ for $t = T$ or $2T$)

$$\begin{aligned} I_1 &= X^{2\sigma-2}\int_T^{2T}\int_0^\infty \frac{\cos(2\pi y)\sin(t\log(1+X/y))}{y^\sigma(X+y)^\sigma\log(1+X/y)}\,dy\,dt \\ &= X^{-1}\int_T^{2T}\int_0^\infty \frac{\cos(2\pi Xy)\sin(t\log(1+1/y))}{y^\sigma(1+y)^\sigma\log(1+1/y)}\,dy\,dt \ll X^{-3/2} \\ I_2 &= X^{2\sigma-1}\int_T^{2T} t\int_0^\infty \frac{\cos(2\pi y)\cos(t\log(1+X/y))}{y^\sigma(X+y)^{\sigma+1}\log(1+X/y)}\,dy\,dt \\ &\ll X^{-1}T \sup_{T\le T_1\le T_2\le 2T}\left|\int_{T_1}^{T_2}\int_0^\infty \frac{\cos(2\pi Xy)\cos(t\log(1+1/y))}{y^\sigma(1+y)^{\sigma+1}\log(1+1/y)}\,dy\,dt\right| \\ &\ll X^{-1/2} \end{aligned}$$

and similarly I_3, $I_4 \ll X^{-3/2}$. With Lemma 3.2,

$$\int_T^{2T} \mathrm{G}_4^*(t)\,dt \ll T^{r-1/2}\log T \ll \log T. \tag{3.9}$$

5) Evaluation of $\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt$. [19, (3.6) and Section 5] gives

$$\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt = -4iI_1 + 4iI_2 + 4iI_3. \tag{3.10}$$

I_1, I_2 and I_3 are defined as follows: write

$$w(\xi, y) = \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2}y^{-\sigma}(1+y)^{-\sigma-2}(\log(1+1/y))^{-1}\cos(2\pi\xi y), \tag{3.11}$$

then

$$I_1 = \int_X^\infty \int_T^{2T} t^2 \int_0^\infty w(\xi, y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi$$
$$I_2 = \int_X^\infty \int_T^{2T} t \int_0^\infty w(\xi, y)H_1(y)\cos(t\log(1+1/y))\,dy\,dt\,d\xi$$
$$I_3 = \int_X^\infty \int_T^{2T} \int_0^\infty w(\xi, y)H_0(y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi$$

where $H_0(y)$ and $H_1(y)$ are linear combinations of $y^\mu(\log(1+1/y))^{-\nu}$ with $\mu+\nu \le 2$ and $\mu+\nu \le 1$ respectively. (Remark: It is stated in [19] $\mu+\nu \le 2$ only for both $H_0(y)$ and $H_1(y)$.)

When $\xi \ge X \asymp T \asymp t$ and $\mu+\nu \le 2$, we have

$$\int_T^{2T} \int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy\,dt \ll 1, \tag{3.12}$$

$$\int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy \ll T^{-1/2}. \tag{3.13}$$

The estimate (3.13) can be seen from [19, p.368]. To see (3.12), we split the inner integral into $\int_0^c + \int_c^\infty$. First derivative test gives $\int_c^\infty \ll \xi^{-1}$. For $\int_0^c$, we integrate over t first and plainly $\int_0^c \int_T^{2T} \ll 1$.

Using (3.12) and Lemma 3.2, we have $I_3 \ll \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2}\,d\xi \ll T^{1/4-\sigma}$. Applying integration by parts to the t-integral, we find that $I_2 \ll T^{3/4-\sigma}$ with (3.12) and (3.13). (Here we have used $\mu+\nu \le 1$ for $H_1(y)$.) Since

$$\begin{aligned}\int_T^{2T} t^2\sin(t\log(1+1/y))\,dt &= -t^2(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\Big|_T^{2T} \\ &\quad + 2t(\log(1+1/y))^{-2}\sin(t\log(1+1/y))\Big|_T^{2T} \\ &\quad - 2(\log(1+1/y))^{-2}\int_T^{2T}\sin(t\log(1+1/y))\,dt,\end{aligned}$$

the last two terms contribute $T^{3/4-\sigma}$ and $T^{1/4-\sigma}$ in I_1 respectively by using (3.13) and (3.12). Substituting into (3.10), we get with [17, Lemma 3] (or

[5, Lemma 15.1]) and (3.11)

$$\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt$$

$$= 4it^2 \int_X^\infty \int_0^\infty w(\xi,y)(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\,dy\,d\xi\Big|_{t=T}^{t=2T}$$

$$+ O(T^{3/4-\sigma})$$

(3.14)

$$= i\pi^{-1/2}t^{5/2}\int_X^\infty \frac{\tilde{\Delta}_{1-2\sigma}(\xi)\cos(tV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3V^2U^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi\Big|_T^{2T}$$

$$+ O(T^{3/4-\sigma})$$

where U and V are defined as in Lemma 3.1 with k replaced by ξ. Applying the argument in [19, Section 6] to (3.14), we get

$$\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt$$

$$= -2i\left(\frac{t}{2\pi}\right)^{1/2-\sigma}\sum_{n\le B(t,\sqrt{X})}\frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}}\left(\log\frac{t}{2\pi n}\right)^{-2}\sin g(t,n)\Big|_{t=T}^{t=2T}$$

(3.15)

$$+ O(\log T).$$

(Remark: The σ in [19, Lemma 4] should be omitted, as mentioned in [18].)

Inserting (3.4), (3.5), (3.8), (3.9), (3.15) into (3.3), we obtain

$$\int_T^{2T} E_\sigma(t)\,dt$$

$$= -2\pi\zeta(2\sigma-1)T + \Sigma_1(t,X)\Big|_T^{2T} - \Sigma_2(t,X)\Big|_T^{2T} + O(\log T). \tag{3.16}$$

6) Transformation of Dirichlet Polynomial. Let $X_1,\ X_2 \asymp T$ (both are not integers) and denote $B_1 = B(T,\sqrt{X_1})$ and $B_2 = B(T,\sqrt{X_2})$. Assume $X_1 < X_2$. Write

$$F(x) = x^{\sigma-1}\left(\log\frac{T}{2\pi x}\right)^{-2}\exp(i(T\log\frac{T}{2\pi x}+2\pi x - T + \frac{\pi}{4})),$$

then we have

$$\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})}\sigma_{1-2\sigma}(n)n^{\sigma-1}(\log(T/(2\pi n)))^{-2}\sin g(T,n)$$

$$= \mathrm{Im}\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n). \tag{3.17}$$

Stieltjes integration gives

$$\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)$$
$$= \int_{B_2}^{B_1} F(t)(\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})\,dt + \Delta_{1-2\sigma}(t)F(t)\Big|_{B_2}^{B_1}$$
$$- \int_{B_2}^{B_1} \Delta_{1-2\sigma}(t)F'(t)\,dt$$
$$= I_1 + I_2 - I_3,\ \text{say.} \tag{3.18}$$

Now, since $(d/dt)(g(T,t)+2\pi t) = 2\pi - T/t < -c$ when $B_2 < t < B_1$, we have

$$I_1 = \int_{B_2}^{B_1} (\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})$$
$$\times\, t^{\sigma-1}(\log(T/(2\pi t)))^{-2}\exp(i(g(T,t)+2\pi t))\,dt$$
$$\ll T^{\sigma-1}.$$

By Lemma 3.2, $I_2 \ll 1$. Direct computation gives

$$F'(t) = i(2\pi-\frac{T}{t})t^{\sigma-1}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(T\log\frac{T}{2\pi t}+2\pi t-T+\frac{\pi}{4}))+O(t^{\sigma-2})$$

where $B_2 \le t \le B_1$. As $\int_{B_2}^{B_1}|\Delta_{1-2\sigma}(t)|t^{\sigma-2}\,dt \ll T^{\sigma-1}\sqrt{\log T}$, we have by (3.18) that

$$\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)$$
$$= -i\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\int_{B_2}^{B_1}\Delta_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}$$
$$\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t - T\log t))\,dt + O(1). \tag{3.19}$$

The integral $\int_{B_1}^{B_2}$ in (3.19) is, after by parts,

$$\tilde{\Delta}_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\Bigg|_{B_1}^{B_2}$$
$$-\int_{B_2}^{B_1}\tilde{\Delta}_{1-2\sigma}(t)\frac{d}{dt}\left\{(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\right\}dt$$

The first term is $\ll T^{\sigma-1/2}\log T$ by Lemma 3.2. Besides, computing directly shows that for $B_2 \le t \le B_1$,

$$\frac{d}{dt}\{\cdots\} = i(2\pi t - T)^2 t^{\sigma-3}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t - T\log t)) + O(t^{\sigma-2}).$$

Treating the O-term with Lemma 3.2, (3.19) becomes

$$\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)$$
$$= -\exp(i(T\log\frac{T}{2\pi} - T + \frac{\pi}{4}))\int_{B_2}^{B_1}\tilde{\Delta}_{1-2\sigma}(t)(2\pi t - T)^2 t^{\sigma-3}$$
$$\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t - T\log t))\,dt + O(T^{\sigma-1/2}\log T). \tag{3.20}$$

Inserting the Voronoi-type series of $\tilde{\Delta}_{1-2\sigma}(t)$ (see Lemma 3.2) into (3.20), we get

$$\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)$$
$$= -\exp(i(T\log\frac{T}{2\pi} - T + \frac{\pi}{4}))$$
$$\times\left\{C_1\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}J_1(n) + C_2\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{9/4-\sigma}}J_2(n)\right\}$$
$$+ O(T^{\sigma-1/2}\log T) \tag{3.21}$$

where

$$J_1(n) = \int_{B_2}^{B_1}(2\pi t - T)^2 t^{-7/4}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(2\pi t - T\log t))$$
$$\times\cos(4\pi\sqrt{nt} + \frac{\pi}{4})\,dt$$
$$J_2(n) = \int_{B_2}^{B_1}(2\pi t - T)^2 t^{-9/4}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(2\pi t - T\log t))$$
$$\times\cos(4\pi\sqrt{nt} - \frac{\pi}{4})\,dt.$$

Applying the first derivative test or bounding trivially, we have $J_2(n) \ll T^{-1/4}$ for $n \le cT$, $J_2(n) \ll T^{3/4}$ for $cT < n < c'T$ and $\ll T^{1/4}n^{-1/2}$ for

$n \geq c'T$. Thus, the second sum in (3.21) is

$$\ll \left(T^{-1/4} \sum_{n \leq cT} + T^{3/4} \sum_{cT < n < c'T} \right) \sigma_{1-2\sigma}(n) n^{\sigma - 9/4} + T^{1/4} \sum_{n \geq c'T} \sigma_{1-2\sigma}(n) n^{\sigma - 11/4}$$

$$\ll T^{\sigma - 1/2}. \tag{3.22}$$

After a change of variable $t = x^2$,

$$J_1(n) = \int_{\sqrt{B_2}}^{\sqrt{B_1}} (2\pi x^2 - T)^2 x^{-5/2} \left(\log \frac{T}{2\pi x^2} \right)^{-2} \times \Big\{ \exp\big(i(2\pi x^2 - 2T \log x + 4\pi\sqrt{n}x + \frac{\pi}{4})\big) + \exp\big(i(2\pi x^2 - 2T \log x - 4\pi\sqrt{n}x - \frac{\pi}{4})\big) \Big\} \, dx.$$

Then we use [5, Theorem 2.2], with $f(x) = x^2 - \pi^{-1}T \log x$, $\Phi(x) = x^{3/2}$, $F(x) = T$, $\mu(x) = x/2$ and $k = \pm 2\sqrt{n}$. Thus,

$$J_1(n) = \delta_n 2\pi^2 \left(\frac{T}{2\pi} \right)^{3/4} e_2(T,n) \exp(i(f(T,n) - T \log \frac{T}{2\pi} + T - \pi n + \frac{3\pi}{4})) + O\big(\delta_n T^{-1/4}\big) + O\big(T^{3/4} \exp(-c\sqrt{nT} - cT)\big) + O\big(T^{3/4} \min(1, |\sqrt{X_1} \pm \sqrt{n}|^{-1})\big) + O\big(T^{3/4} \min(1, |\sqrt{X_2} \pm \sqrt{n}|^{-1})\big)$$

where $\delta_n = 1$ if $B_2 < x_0 < B_1$ and $k > 0$, or $\delta_n = 0$ otherwise. ($x_0 = \sqrt{T/(2\pi) + n/4} - \sqrt{n}/2$ is the saddle point.) Note that $B_2 < x_0 < B_1$ is equivalent to $X_1 < n < X_2$. Thus, for the first term in (3.21), we have

$$-C_1 \exp\big(i(T \log \frac{T}{2\pi} - T + \frac{\pi}{4})\big) \sum_{n=1}^{\infty} \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} J_1(n) = \frac{1}{\sqrt{2}} \left(\frac{T}{2\pi} \right)^{3/4} \sum_{X_1 < n < X_2} \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(T,n) \exp(i(f(T,n) - \pi n + \pi)) + O(T^{\sigma - 1/2} \log T)$$

Together with (3.22), (3.21) and (3.17), we obtain

$$\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})} \sigma_{1-2\sigma}(n)n^{\sigma-1}(\log T/(2\pi n))^{-2}\sin g(T,n)$$
$$= -\frac{1}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{3/4}\sum_{X_1<n<X_2}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(T,n)\sin f(T,n)$$
$$+ O(T^{\sigma-1/2}\log T). \tag{3.23}$$

We can complete our proof now. Taking $X = [T] - 1/2$ in (3.16), we have $\Sigma_i(t,X) - \Sigma_i(t,T) \ll \log T$ for $i = 1,2$ and $t = T, 2T$; hence

$$\int_T^{2T} E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + \Sigma_1(t,t)\big|_T^{2T} - \Sigma_2(t,t)\big|_T^{2T}$$
$$- \Big(\big(\Sigma_1(2T,2T) - \Sigma(2T,T)\big) + \big(\Sigma_2(2T,T) - \Sigma_2(2T,2T)\big)\Big)$$
$$+ O(\log T).$$

Choosing X_1 and X_2 in (3.23) to be half-integers closest to T and $2T$ respectively, then $(\Sigma_1(2T,2T) - \Sigma(2T,T)) + (\Sigma_2(2T,T) - \Sigma_2(2T,2T)) \ll \log T$. Hence,

$$\int_1^T E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + \Sigma_1(T,T) - \Sigma_2(T,T) + O(\log^2 T).$$

The extra $\log T$ in the O-term comes from the number of dyadic intervals. Suppose $N \asymp T$. We apply (3.23) again with $X_1 = [N] + 1/2$ and $X_2 = [T] + 1/2$ to yield our theorem.

4. The second and third power moments

The proof of the second moment is quite standard, see [19], [21] or [5] for example. Part (1) of Theorem 2 follows from that for $N \asymp T$,

$$\int_T^{2T}\Sigma_2(t,N)^2\,dt \ll T, \qquad \int_T^{2T}\Sigma_1(t,N)\Sigma_2(t,N)\,dt \ll T\log T,$$
$$\int_T^{2T}\Sigma_1(t,N)^2\,dt = B(\sigma)\int_T^{2T}\left(\frac{t}{2\pi}\right)^{5/2-2\sigma}dt + O(T^{3-2\sigma}).$$

Moreover, one can show

Lemma 4.1. *Define $\Sigma_{M,M'}(t) = \Sigma_{1,M'}(t) - \Sigma_{1,M}(t)$ for $1 \le M \le M' \ll T$. Then, we have*

$$\int_T^{2T}\Sigma_{M,M'}(t)^2\,dt \ll T^{7/2-2\sigma}M^{2\sigma-5/2}.$$

The next result is of its own interest and will be used in the proof of the third moment.

Proposition 4.1. *Let $0 \le A < (\sigma - 3/4)^{-1}$. Then, we have*

$$\int_T^{2T} |G_\sigma(t)|^A \, dt \ll T^{1+A(5/4-\sigma)}.$$

Proof. The case $0 \le A \le 2$ is proved by Hölder's inequality and part (1) of Theorem 2. Consider the situation $2 < A < (\sigma - 3/4)^{-1}$. Then, for $T \le t \le 2T$ and $N \asymp T$, we have $\Sigma_2(t, N) \ll T^{1/2}$ and hence $\int_T^{2T} |\Sigma_2(t,N)|^A \, dt \ll T^{A/2}$. We take $N = 2^R - 1 \asymp T$ and write $M = 2^r$. Then $\Sigma_1(t, N) \le \sum_{r \le R} |\Sigma_{M,2M}(t)|$. By Hölder's inequality, we have

$$|\Sigma_1(t,N)|^A \ll \left(\sum_{r \le R} \alpha_r^A |\Sigma_{M,2M}(t)|^A \right) \left(\sum_{r \le R} \alpha_r^{-A/(A-1)} \right)^{A-1}.$$

Taking $\alpha_r = M^{(1-A(\sigma-3/4))/(2A)}$ with the trivial bound $\Sigma_{M,2M}(t) \ll T^{5/4-\sigma} M^{\sigma-3/4}$, we have

$$\begin{aligned} &\int_T^{2T} |\Sigma_1(t,N)|^A \, dt \\ &\qquad \ll_A T^{(5/4-\sigma)(A-2)} \sum_{r \le R} \alpha_r^A M^{(\sigma-3/4)(A-2)} \int_T^{2T} \Sigma_{M,2M}(t)^2 \, dt \\ &\qquad \ll_A T^{1+A(5/4-\sigma)} \end{aligned} \tag{4.1}$$

by Lemma 4.1.

Proof of Theorem 2 (2). We have, with $M = [\delta T^{1/3}]$ for some small constant $\delta > 0$,

$$\begin{aligned} &\int_T^{2T} G_\sigma(t)^3 \, dt \\ &\qquad = \int_T^{2T} \Sigma_{1,M}(t)^3 \, dt + O\Big(\int_T^{2T} |\Sigma_{M,T}| (G_\sigma(t)^2 + \Sigma_{1,M}^2(t)) \, dt\Big). \end{aligned} \tag{4.2}$$

Proposition 4.1 and (4.1) yields that the O-term is $O(T^{(13-8\sigma)/3})$. The integral on the right-sided of (4.2) is treated by the argument in [23]. Then the result follows.

5. Limiting distribution functions

We first quote some results from [1, Theorem 4.1] and [3, Theorem 6].

Let F be a real-valued function defined on $[1, \infty)$, and let $a_1(t), a_2(t), \ldots$ be real-valued, continuous and of period 1 such that $\int_0^1 a_n(t) \, dt = 0$ and $\sum_{n=1}^\infty \int_0^1 a_n(t)^2 \, dt < \infty$. Suppose that there are positive constants γ_1,

$\gamma_2, \ldots$ which are linearly independent over $\mathbf{Q}$, such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \min(1, |F(t) - \sum_{n\le N} a_n(\gamma_n t)|)\, dt = 0.$$

Fact I. For every continuous bounded function g on $\mathbb{R}$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F(t))\, dt = \int_{-\infty}^{\infty} g(x)\nu(dx),$$

where $\nu(dx)$ is the distribution of the random series $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ and t_n are independent random variables uniformly distributed on $[0,1]$. Equivalently, the distribution function of F, $P_T(u) = T^{-1}\mu\{t \in [1,T] : F(t) \le u\}$, converges weakly to a function $P(u)$, called the limiting distribution, as $T \to \infty$.

Fact II. If $\int_1^T |F(t)|^A\, dt \ll T$, then for any real $k \in [0, A)$ and integral $l \in [0, A)$, the following limits exist:

$$\lim_{T\to\infty} T^{-1} \int_1^T |F(t)|^k\, dt \text{ and } \lim_{T\to\infty} T^{-1} \int_1^T F(t)^l\, dt.$$

Now, let us take $F(t) = t^{2\sigma-5/2} G_\sigma(2\pi t^2)$, $\gamma_n = 2\sqrt{n}$ and

$$a_n(t) = \sqrt{2} \frac{\mu(n)^2}{n^{7/4-\sigma}} \sum_{r=1}^{\infty} (-1)^{nr} \frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}} \sin(2\pi r t - \pi/4). \tag{5.1}$$

Following the computation in [3, p.402] with Lemma 4.1, we get

$$\int_T^{2T} (F(t) - \sum_{n\le N} a_n(2\sqrt{n}t))^2\, dt \ll T N^{2\sigma-5/2} \qquad (N \le \sqrt{T}).$$

Then Theorem 2 (c) and the first part of Theorem 3 are immediate consequence of Facts I and II with Proposition 4.1.

We proceed to prove the lower bounds in (2.2) with the idea in [1, Section V].

Lemma 5.1. *Let n be squarefree. Define*

$$A_n = \{t \in [0,1] : a_n(t) > B^{-1}\sigma_{1-2\sigma}(n) n^{\sigma-7/4}\}$$

where $B = 4A(\sum_{r=1}^{\infty} r^{4\sigma-7})^{-1}$ and $A = \sqrt{2}\sum_{r=1}^{\infty} \sigma_{1-2\sigma}(r^2) r^{2\sigma-7/2}$. Then, we have $\mu(A_n) \ge 1/(AB)$ where μ is the Lebesgue measure.

The proof makes use of the fact that $\int_0^1 a_n^+(t)\, dt = \int_0^1 a_n^-(t)\, dt$ where $a_n^{\pm}(t) = \max(0, \pm a_n(t))$, and

$$\int_0^1 a_n^+(t)^2\, dt + \int_0^1 a_n^-(t)^2\, dt = \frac{1}{n^{7/2-2\sigma}} \sum_{r=1}^{\infty} \frac{\sigma_{1-2\sigma}(nr^2)^2}{r^{7-4\sigma}}.$$

The readers are referred to [1] for details.

Proof of lower bounds in Theorem 3. By Markov's inequality, we have

$$\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}) \ge 1 - \frac{1}{4K}\sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt \ge \frac{3}{4}$$

where $\Pr(\#)$ denotes the probability of the event # and

$$K = \sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt < +\infty.$$

Consider the set

$$E_n = \Big\{(t_1, t_2, \ldots) : t_m \in A_m \text{ for } 1 \le m \le n \text{ and } |\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}\Big\}$$

where $A_m = [0,1]$ if m is not squarefree. Then,

$$\Pr(E_n) = \prod_{m=1}^{n}\Pr(A_m)\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}) \ge \frac{3}{4(AB)^n}$$

due to $\Pr(A_m) = \mu(A_m)$ and Lemma 5.1. When $(t_1, t_2, \ldots) \in E_n$, we have

$$\sum_{m=1}^{\infty} a_m(t_m) \ge \frac{1}{B}\sum_{\substack{m\le n\\ m \text{ squarefree}}}\frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} - 2\sqrt{K}$$
$$\gg \begin{cases} \log n & \text{if } \sigma = 3/4,\\ n^{\sigma-3/4} & \text{if } 3/4 < \sigma < 1.\end{cases}$$

Our result for $1 - D_\sigma(u)$ follows after we replace n by $[e^u]$ if $\sigma = 3/4$ and by $[u^{4/(4\sigma-3)}]$ if $3/4 < \sigma < 1$. The case of $D_\sigma(-u)$ can be proved in the same way.

To derive the upper estimates, we need a result on the Laplace transform of limiting distribution functions [13, Lemma 3.1].

Lemma 5.2. *Let X be a real random variable with the probability distribution $D(x)$. Suppose $D(x) > 0$ for any $x > 0$. For the two cases: (i) $\psi(x) = x\log x$ and $\phi(x) = \log x$, or (ii) $\psi(x) = x^{4/(7-4\sigma)}$ and $\phi(x) = x^{(4\sigma-3)/4}$, there exist two positive numbers L and L' such that*

(a) *if* $\limsup_{\lambda\to\infty}\psi(\lambda)^{-1}\log E(\exp(\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} x^{-1}\log(1 - D(\phi(x))) \le -L',$$

(b) *if* $\limsup_{\lambda\to\infty}\psi(\lambda)^{-1}\log E(\exp(-\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} x^{-1}\log D(-\phi(x)) \le -L'.$$

Proof of upper bounds in Theorem 3. We take $N=\lambda$ if $\sigma=3/4$, and $N=\lambda^{4/(7-4\sigma)}$ if $3/4<\sigma<1$. When $n\le N$, we use

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp(\lambda A\frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}}).$$

Recall that $A=\sqrt{2}\sum_{r=1}^{\infty}\sigma_{1-2\sigma}(r^2)r^{7/2-2\sigma}$. Now consider $n>N$. If $\lambda A\sigma_{1-2\sigma}(n)<n^{7/4-\sigma}$, then by the inequality $e^x\le 1+x+x^2$ for $x\le 1$, and $\int_0^1 a_n(t)\,dt=0$, we have

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp((\lambda A)^2\frac{\sigma_{1-2\sigma}(n)^2\mu(n)^2}{n^{7/2-2\sigma}}).$$

Otherwise, $\lambda A\sigma_{1-2\sigma}(n)\ge n^{7/4-\sigma}$, it is obvious that

$$\begin{aligned}\int_0^1 \exp(\pm\lambda a_n(t))\,dt &\le \exp(\lambda A\frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}})\\ &\le \exp((\lambda A)^2\frac{\sigma_{1-2\sigma}(n)^2\mu(n)^2}{n^{7/2-2\sigma}}).\end{aligned}$$

Therefore, $\log E(\exp(\pm\lambda X))$ is

$$\begin{aligned}&\le \lambda A\sum_{n\le N}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}+(\lambda A)^2\sum_{n>N}\frac{\sigma_{1-2\sigma}(n)^2}{n^{7/2-2\sigma}}\\ &\ll \begin{cases}\lambda\log\lambda & \text{if } \sigma=3/4,\\ \lambda^{4/(7-4\sigma)} & \text{if } 3/4<\sigma<1.\end{cases}\end{aligned}$$

The proof is complete with Lemma 5.2.

6. $\Omega_{\pm}$-results

This section is devoted to prove Theorem 4. We apply the methods in [2] or [7], but beforehand, we transform $G_\sigma(t)$ into a simple finite series by convolution with the kernel

$$K(u)=2B\left(\frac{\sin 2\pi Bu}{2\pi Bu}\right)^2.$$

Similarly to [15], we have, for $1\ll B\ll L^{1/4}\ll T^{1/16}$,

$$t^{2\sigma-5/2}\int_{-L}^{L} G_\sigma(2\pi(t+u)^2)K(u)\,du=S_B(t)+O(B^{4\sigma-5}) \tag{6.1}$$

where

$$S_B(t)=\sqrt{2}\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(4\pi\sqrt{n}t-\frac{\pi}{4}). \tag{6.2}$$

To prove the Ω_--result, we use Dirichlet's Theorem to align the angles. More specifically, for any small $\delta>0$, we can find $l\in[T^{1/10},$

$(1+\delta^{-B^2})T^{1/10}]$ such that $\|l\sqrt{n}\| < \delta$. Taking $B \ll \delta\sqrt{\log T}$, we have $l \in [T^{1/10}, T^{1/5}]$ and

$$S_B(l) = -\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} + O\Big(\delta\sum_{n\le B^2}(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\Big). \tag{6.3}$$

A simple calculation shows that

$$2^{2\sigma}\cdot\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} = \sum_{n\le B^2}(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}.$$

We thus infer $S_B(t) = \Omega_-(\log\log t)$ if $\sigma = 3/4$, and $\Omega_-((\log t)^{\sigma-3/4})$ if $3/4 < \sigma < 1$.

We proceed to prove the Ω_+-result with the method in [2]. Take $x = \delta\log\log T\log\log\log T$ and $B = T^{1/100}$ $(L = B^4)$ for a small number $\delta > 0$. We consider the convolution of $S_B(t)$ with a kernel involving the function

$$T_x(u) = \prod_{q\in\mathbf{Q}_x}(1+\cos(4\pi\sqrt{q}u)) = \prod_{q\in\mathbf{Q}_x}\left(1+\frac{e^{4\pi i\sqrt{q}u}+e^{-4\pi i\sqrt{q}u}}{2}\right)$$

where $\mathbf{Q}_x$ is the set of positive squarefree integers whose prime factors are odd and smaller than x. The convolution will pick out terms with the desired frequencies,

$$\epsilon\int_{-\infty}^{\infty}S_B(t+u)T_x(u)\left(\frac{\sin\epsilon\pi u}{\epsilon\pi u}\right)^2 du = \sqrt{2}\sum_{\substack{n\le B^2\\ n\in\mathbf{Q}_x}}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(4\pi\sqrt{n}t-\frac{\pi}{4}).$$

To maximize the right-hand side, we apply Dirichlet's theorem again to find a number $l \in [T^{1/10}, (1+\delta^{-|\mathbf{Q}_x|})T^{1/10}]$ so that the right-side is

$$\gg \prod_{p\in\mathbf{Q}_x}(1+\frac{\sigma_{1-2\sigma}(p)}{p^{7/4-\sigma}}) \gg \begin{cases}\log x & \text{if } \sigma = 3/4,\\ \exp(cx^{\sigma-3/4}/(\log x)) & \text{if } \sigma > 3/4.\end{cases}$$

This follows from the the estimation of $\sum_{p\le x}p^{\sigma-7/4}$ for $\sigma = 3/4$ and $\sigma > 3/4$ respectively. The cardinality of $\mathbf{Q}_x$ is $O(\exp(cx/\log x))$ for some positive constant c. Our choice of x ensures that l is of a size of a small power of T. Consequently, we obtain

$$\sup_{T^{1/10}\ll u\ll T^{1/4}} S_B(u) \gg \begin{cases} \log\log\log T & \text{if } \sigma = 3/4, \\ \exp(c(\log\log T)^{\sigma-3/4}(\log\log\log T)^{\sigma-7/4}) & \text{if } \sigma > 3/4. \end{cases}$$

7. Occurrence of large values

Proof of Theorem 5. Define $K_\tau(u) = (1-|u|)(1+\tau\sin(4\pi\alpha u))$ where $\tau = -1$ or $+1$ and α is a large constant. Following the argument in [4], we derive that

$$\int_{-1}^{1}(t+u)^{2\sigma-5/2}\int_{-L}^{L} G_\sigma(2\pi(t+u+v)^2)K(v)\,dvK_\tau(u)\,du = \frac{\tau}{2}(1-B^{-1})\cos(4\pi t-\pi/4) + O(\alpha^{-2}) + O(B^{4\sigma-5}).$$

where $\delta_{1,n} = 1$ if $n = 1$ and 0 otherwise. Our assertion follows by choosing B and α ($L = B^4$) sufficiently large, and $\|4t\| \le 1/8$ with $t \in [\sqrt{T}, \sqrt{T}+1]$. (Note that τ can be $+1$ or -1 at our disposal.)

To prove Theorem 6, we need the next lemma which is the key.

Lemma 7.1. *For $T^{5/12} \le H \le T^{1/2}$,*

$$\int_T^{2T} \max_{0\le h\le H}(G_\sigma(t+h)-G_\sigma(t))^2\,dt \ll TH^{5-4\sigma}$$

where the implied constant depends on σ.

Proof. Following the arguments in [8], we have

$$\int_T^{2T}(G_\sigma(t+h)-G_\sigma(t))^2\,dt \ll Th^{5-4\sigma}\min((\sigma-3/4)^{-1},\log(T/h^2)) \tag{7.1}$$

where $\log^2 T \le h \le \sqrt{T}$. Let $b = T^{1/24}$ and $H = 2^\lambda b$. Then, as in [4], we can show

$$\max_{0\le h\le H}|G_\sigma(t+h)-G_\sigma(t)| \le \max_{1\le j\le 2^\lambda}|G_\sigma(t+jb)-G_\sigma(t)| + O(T^{2(1-\sigma)/3+\epsilon}b$$

for any fixed t. Let us take $1 \le j_0 = j_0(t) \le 2^\lambda$ such that

$$|G_\sigma(t+j_0b)-G_\sigma(t)| = \max_{1\le j\le 2^\lambda}|G_\sigma(t+jb)-G_\sigma(t)|.$$

Then we can express $j_0 = 2^\lambda\sum_{\mu\in S_t}2^{-\mu}$ for some set S_t of non-negative integers. Hence,

$$G_\sigma(t+j_0b)-G_\sigma(t) = \sum_{\mu\in S_t} G_\sigma(t+(\nu+1)2^{\lambda-\mu}b) - G_\sigma(t+\nu 2^{\lambda-\mu}b)$$

where $0 \le \nu = \nu_{t,\mu} < 2^{\mu}$ is an integer. By Cauchy-Schwarz's inequality and inserting the remaining ν's, we get

$$\begin{aligned}
&(G_\sigma(t+j_0b) - G_\sigma(t))^2 \\
&\le \Big(\sum_{\mu\in S_t} 2^{-(1-\sigma)\mu}\Big) \sum_{\mu\in S_t} 2^{(1-\sigma)\mu}(G_\sigma(t+(\nu+1)2^{\lambda-\mu}b) - G_\sigma(t+\nu 2^{\lambda-\mu}b))^2 \\
&\ll \sum_{\mu\in S_t}\sum_{0\le\nu<2^{\mu}} 2^{(1-\sigma)\mu}(G_\sigma(t+(\nu+1)2^{\lambda-\mu}b) - G_\sigma(t+\nu 2^{\lambda-\mu}b))^2
\end{aligned}$$

as $\sum_{\mu\in S_t} 2^{-(1-\sigma)\mu} \ll 1$. Integrating over $[T, 2T]$ and using (7.1), we see that

$$\begin{aligned}
&\int_T^{2T} \max_{0\le h\le H} (G_\sigma(t+h) - G_\sigma(t))^2\, dt \\
&\ll \sum_{\mu\in S_t}\sum_{0\le\nu<2^{\mu}} 2^{(1-\sigma)\mu} \int_{T+\nu 2^{\lambda-\mu}b}^{2T+\nu 2^{\lambda-\mu}b} (G_\sigma(t+2^{\lambda-\mu}b) - G_\sigma(t))^2\, dt + T^{17/12+\epsilon} \\
&\ll TH^{5-4\sigma} \sum_{\mu\in S_t}\sum_{0\le\nu<2^{\mu}} 2^{-(4-3\sigma)\mu} \\
&\ll TH^{5-4\sigma}.
\end{aligned}$$

This complete the proof of Lemma 7.1.

Proof of Theorem 6. Define $G_\sigma^{\pm}(t) = \max(0, \pm G_\sigma(t))$. By Theorem 2 (c), we have $\int_T^{2T} |G_\sigma(t)|^3\, dt \ll T^{1+3(5/4-\sigma)}$. Hence, Cauchy-Schwarz inequality gives

$$\left(\int_T^{2T} G_\sigma(t)^2\, dt\right)^2 \le \int_T^{2T} |G_\sigma(t)|\, dt \int_T^{2T} |G_\sigma(t)|^3\, dt.$$

we have $\int_T^{2T} |G_\sigma(t)|\, dt \gg T^{1+(5/4-\sigma)}$. Together with (2.1), $\int_T^{2T} G_\sigma^{\pm}(t)\, dt \ge c_{12}\int_T^{2T} t^{5/4-\sigma}\, dt$.

Consider $K^{\pm}(t) = G_\sigma^{\pm}(t) - (c_{12}-\epsilon)t^{5/4-\sigma}$ where $\epsilon = \delta^{5/2-2\sigma}$, we have

$$\int_T^{2T} K^{\pm}(t)\, dt \ge \epsilon \int_T^{2T} t^{5/4-\sigma}\, dt$$

and $K^{\pm}(t+h) - K^{\pm}(t) = G_\sigma^{\pm}(t+h) - G_\sigma^{\pm}(t) + O(T^{1/4-\sigma}h)$. Since $|G_\sigma^{\pm}(t+h) - G_\sigma^{\pm}(t)| \le |G_\sigma(t+h) - G_\sigma(t)|$, it follows that together with Lemma 7.1,

$$\int_T^{2T} \max_{h\le H} |K^{\pm}(t+h) - K^{\pm}(t)|\, dt \ll TH^{5/2-2\sigma} + T^{5/4-\sigma}H.$$

Define $\omega^{\pm}(t) = K^{\pm}(t) - \max_{h\le H}|K^{\pm}(t+h) - K^{\pm}(t)|$. Taking $H = c'\epsilon^{2/(5-4\sigma)}\sqrt{T}$ $(= c'\delta\sqrt{T})$ for some sufficiently small constant $c' > 0$, we have

$$\int_T^{2T} \omega^{\pm}(t)\,dt \geq \epsilon \int_T^{2T} t^{5/4-\sigma}\,dt - \int_T^{2T} \max_{h\leq H} |K^{\pm}(t+h) - K^{\pm}(t)|\,dt$$
$$\gg \epsilon T^{1+(5/4-\sigma)}.$$

Let $\mathcal{I}^{\pm} = \{t \in [T, 2T] : \omega^{\pm}(t) > 0\}$. Then

$$\int_T^{2T} \omega^{\pm}(t)\,dt \leq \int_{\mathcal{I}^{\pm}} \omega^{\pm}(t)\,dt \leq \int_{\mathcal{I}^{\pm}} K^{\pm}(t)\,dt$$
$$\leq \left(\int_{\mathcal{I}^{\pm}} dt\right)^{1/2} \left(\int_T^{2T} K^{\pm}(t)^2\,dt\right)^{1/2}.$$

We infer $|\mathcal{I}^{\pm}| \gg \epsilon^2 T$ as $\int_T^{2T} K^{\pm}(t)^2\,dt \ll \int_T^{2T} G_\sigma(t)^2\,dt + T^{7/2-2\sigma}$. When $t \in \mathcal{I}^{\pm}$, we have $K^{\pm}(t) \geq \max_{h\leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \geq 0$. Hence, $K^{\pm}(u) \geq 0$ for all $u \in [t, t+H]$, i.e. $G_\sigma^{\pm}(t) \geq (c_{12} - \epsilon)t^{5/4-\sigma}$. The number of such intervals is not less than $|\mathcal{I}^{\pm}|/H \gg c_{13}\delta^{4(1-\sigma)}\sqrt{T}$.

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