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Van der Corput sequences towards general (0,1)–sequences in base b

par Henri FAURE

Résumé. A la suite de travaux récents sur les suites à faible discrépance unidimensionnelles, on peut affirmer que les suites de van der Corput originales sont les plus mal distribuées pour diverses mesures d'irrégularités de distribution parmi deux grandes familles de $(0,1)$–suites, et même parmi toutes les $(0,1)$–suites pour la discrépance à l'origine D^*. Nous montrons ici que ce n'est pas le cas pour la discrépance extrême D en produisant deux types de suites qui sont les plus mal distribuées parmi les $(0,1)$–suites, avec une discrépance D essentiellement deux fois plus grande. En outre, nous donnons une présentation unifiée pour les deux généralisations connues des suites de van der Corput.

Abstract. As a result of recent studies on unidimensional low discrepancy sequences, we can assert that the original van der Corput sequences are the worst distributed with respect to various measures of irregularities of distribution among two large families of $(0,1)$–sequences, and even among all $(0,1)$–sequences for the star discrepancy D^*. We show in the present paper that it is not the case for the extreme discrepancy D by producing two kinds of sequences which are the worst distributed among all $(0,1)$–sequences, with a discrepancy D essentially twice greater. In addition, we give an unified presentation for the two generalizations presently known of van der Corput sequences.

1. Introduction

The van der Corput sequence is the prototype of large families of multidimensional sequences with very low discrepancy widely used in Monte Carlo and quasi-Monte Carlo methods, especially in numerical integration. The precursors are Halton (1960), using arbitrary coprime bases for each dimension, and Sobol' (1966) using only the base 2 with the action of primitive polynomials for each dimension. Next, the author (1982) proposed constructions with prime bases using powers of the Pascal matrix

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for each dimension and finally Niederreiter (from 1987, see [12]) worked out a general family of sequences including Sobol' and Faure constructions, but not Halton one, which he named (t,s)–sequences, with reference to the dimension s and to a quality parameter $t \geq 0$ (the lowest t, the lowest discrepancy). Due to the research of improvements for numerical applications (speeding up of convergence), a lot of people have recently concentrated on the onedimensional case ([2], [3], [6]-[10], [15]) obtaining precise results, mainly in base 2, which permit a better understanding of the behavior of $(0,1)$–sequences.

It is the aim of this article to continue this effort and to go farther in the comparison with the original van der Corput sequences in arbitrary bases. At the outset is a question of the referee for [7] who asked whether the van der Corput sequence is the worst distributed among general $(0,1)$–sequences with respect to usual distribution measures. In [7], we showed it is the worst with respect to D^*, D, T^*, T (see Section 2 for definitions) among two large families of special $(0,1)$–sequences, the so-called permuted van der Corput sequences and NUT digital $(0,1)$–sequences (see Sections 3 and 4). Moreover, Kritzer [8] has just proved that it is also the worst with respect to D^* among general $(0,1)$–sequences in the narrow sense. Therefore, it seemed natural it should be also the worst with respect to D. We show in Theorems 2 and 3 (Section 5) that it is not the case and we find sequences which are the worst with respect to D among general $(0,1)$–sequences. Before, in Theorem 1, we extend slightly the result of Kritzer to general $(0,1)$–sequences in the broad sense; this subtle difference was introduced by Niederreiter and Xing [13] to make quite clear the definition of (t,s)–sequences for their complex constructions. It permits also (see the proposition in Section 3.3) an unified presentation of the two main families in one dimension, the permuted and the digital $(0,1)$–van der Corput sequences .

2. Discrepancies

Let $X = (x_n)_{n\geq 1}$ be an infinite sequence in $[0,1]$, $N \geq 1$ an integer and $[\alpha,\beta[$ a sub-interval of $[0,1]$; the *error* to ideal distribution is the difference

$$E([\alpha,\beta[;N;X) = A([\alpha,\beta[;N;X) - N\lambda([\alpha,\beta[)$$

where $A([\alpha,\beta[;N;X)$ is the number of indices n such that $1 \leq n \leq N$ and $x_n \in [\alpha,\beta[$ and where $\lambda([\alpha,\beta[)$ is the length of $[\alpha,\beta[$.

To avoid any ambiguity, recall that $[\alpha,\beta[= [0,\beta[\cup[\alpha,1]$ if $\alpha > \beta$ ([1], p.105), so that $\lambda([\alpha,\beta[) = 1-\alpha+\beta$ and $E([\alpha,\beta[;k;X) = -E([\beta,\alpha[;k;X)$.

Definition of the L_∞–*discrepancies*:

$$D(N,X) = \sup_{0\le\alpha<\beta\le 1} |E([\alpha,\beta[;N;X)|,$$
$$D^*(N,X) = \sup_{0\le\alpha\le 1} |E([0,\alpha[;N;X)|,$$
$$D^+(N,X) = \sup_{0\le\alpha\le 1} E([0,\alpha[;N;X),$$
$$D^-(N,X) = \sup_{0\le\alpha\le 1} (-E([0,\alpha[;N;X)).$$

Usually, D is called the extreme discrepancy and D^* the star discrepancy; D^+ and D^- are linked to the preceding one's by

$$D(N,X) = D^+(N,X) + D^-(N,X)$$

and

$$D^*(N,X) = \max(D^+(N,X), D^-(N,X)).$$

Note that $D^* \le D \le 2D^*$.

Definition of the L_2–*discrepancies*:

$$T^*(N,X) = \Big(\int_0^1 E^2\big([0,\alpha[;N;X\big)d\alpha\Big)^{\frac{1}{2}},$$
$$T(N,X) = \Big(\int_0^1\int_0^1 E^2\big([\alpha,\beta[;N;X\big)d\alpha\ d\beta\Big)^{\frac{1}{2}}.$$

We have chosen these definitions, introduced in [11], because they show the parallel with the L_∞–discrepancies, but usually T^* is called the L_2–discrepancy (and is denoted by T). In dimension one, $\pi\sqrt{2}\ T = F$ where F is the diaphony introduced by Zinterhof :

$$F(N,X) = \left(2\sum_{m=1}^{\infty}\frac{1}{m^2}\left|\sum_{n=1}^{N}\exp(2i\pi m x_n)\right|^2\right)^{\frac{1}{2}}.$$

3. Generalized van der Corput sequences

3.1. Permuted van der Corput sequences. These sequences have been introduced by the author in [4], where an extention in variable bases is also proposed; we restrict here to fixed bases b.

Let $b\ge 2$ be an integer. For integers n and N with $n\ge 1$ and $1\le N\le b^n$, write

$$N-1=\sum_{r=0}^{\infty} a_r(N)\ b^r$$

in the b–adic system (so that $a_r(N)=0$ if $r\ge n$) and let $\Sigma=(\sigma_r)_{r\ge 0}$ be a sequence of permutations of $\{0,1,\ldots,b-1\}$.

Then the *permuted van der Corput sequence* S_b^Σ in base b associated with Σ is defined by

$$S_b^\Sigma(N) = \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}.$$

If $(\sigma_r) = (\sigma)$ is constant, we write $S_b^\Sigma = S_b^\sigma$.

The *original van der Corput sequence* in base b, S_b^I, is obtained with the identical permutation I.

3.2. Digital $(0,1)$–van der Corput sequences. To simplify, we deal only with *prime* bases b. Instead of permutations, we consider here the action on the digits of infinite $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{F}_b$. The following definition is the extension to bases b of the definition in base two given by Larcher and Pillichshammer in [9] and [15].

Let $C = (c_k^r)_{r \geq 0, k \geq 0}$ be an infinite matrix with entries $c_k^r \in \mathbb{F}_b$ such that, for any integer $m \geq 1$, every left upper $m \times m$ submatrix is non singular.

Then the *digital* $(0,1)$*–van der Corput sequence* X_b^C in base b associated with C is defined by

$$X_b^C(N) = \sum_{r=0}^{\infty} \frac{x_{N,r}}{b^{r+1}} \quad \text{in which} \quad x_{N,r} = \sum_{k=0}^{\infty} c_r^k a_k(N),$$

where the $a_k(N)$ are defined as in 3.1.

Note that the second summation is finite and performed in $\mathbb{F}_b$, but the first one can be infinite and is performed in the reals, with the possibility that $x_{N,r} = b-1$ for all but finitely many r.

An important particular case is the case of non singular upper triangular (NUT) matrices C, for which the first summation is finite. These sequences are called NUT digital $(0,1)$–sequences.

Of course, we obtain the original van der Corput sequence S_b^I with the identity matrix (see the beginning of Subsection 4.2.2 for more information).

3.3. General $(0,1)$–sequences in base b. The concept of (t,s)–sequences has been introduced by Niederreiter (see for instance [12]) to give a general framework for various constructions of s–multidimensional low discrepancy sequences and to obtain further constructions. Smaller values of the integer $t \geq 0$ give smaller discrepancies. But, in order to give sense to new important constructions, Niederreiter and Xing ([13] and [14] where many other references are given) as well as Tezuka [16] have been led to slightly generalize the original definition by using the so-called truncation we introduce now, restricting ourselves to the one-dimensional case we are only interested with in the present study.

Truncation : Let $x = \sum_{i=1}^{\infty} x_i b^{-i}$ be a b–adic expansion of $x \in [0,1]$, with the possibility that $x_i = b-1$ for all but finitely many i. For every integer $m \geq 1$, define $[x]_{b,m} = \sum_{i=1}^{m} x_i b^{-i}$ (depending on x via its expansion).

An *elementary interval in base* b is an interval in the shape of $[\frac{a}{b^d}, \frac{a+1}{b^d}[$ with integers a, d such that $d \geq 0$ and $0 \leq a < b^d$.

A sequence $(x_N)_{N \geq 1}$ (with prescribed b-adic expansions for each x_N) is a $(t,1)$*–sequence in base* b *(in the broad sense)* if for all integers $l, m \geq t$, every elementary interval E with $\lambda(E) = b^{t-m}$ contains exactly b^t points of the point set

$$\{[x_N]_{b,m}\ ; lb^m + 1 \leq N \leq (l+1)b^m\}.$$

The original definition of $(t,1)$–sequences was the same with x_N instead of $[x_N]_{b,m}$ in the definition of the point set above. These sequences are now called $(t,1)$–sequences *in the narrow sense* and the others just $(t,1)$–sequences (Niederreiter-Xing [13], Definition 2 and Remark 1); in this paper, we use intentionally the expression (in the broad sense) to emphasize the difference.

In the following, we deal only with the most interesting case of $(0,1)$–sequences.

Proposition 3.1. *The two generalizations of van der Corput sequences defined in subsections 3.1 and 3.2 are* $(0,1)$*–sequences in base* b *(in the broad sense).*

Remark. Here, the truncation is required for the sequences S_b^{Σ} when $\sigma_r(0) = b-1$ for all sufficiently large r $\left(\text{recall that } S_b^{\Sigma}(N) = \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}\right)$ and for the sequences X_b^C when the matrix C gives digits $x_{N,r} = b-1$ for all sufficiently large r (recall that $X_b^C(N) = \sum_{r=0}^{\infty} \frac{x_{N,r}}{b^{r+1}}$ with $x_{N,r} = \sum_{k=0}^{\infty} c_r^k a_k(N)$).

Proof. For digital $(0,1)$–sequences X_b^C, this property is already known. Indeed, the concept of (t,s)–sequence has been worked out from the first constructions of digital (t,s)–sequences by Sobol' and Faure in which the matrices C were NUT.

For permuted sequences S_b^{Σ}, the property has never been pointed out. We outline the proof hereafter.

According to the definitions, we have to prove that for arbitrary integers $l, m \geq 0$, every elementary interval $[ab^{-m}, (a+1)b^{-m}[$, with $0 \leq a < b^m$, contains one and only one point of the point set $\{[x_N]_{b,m}\ ; lb^m + 1 \leq N \leq (l+1)b^m\}$, in which $x_N = S_b^{\Sigma}(N)$.

Let $N-1 = \sum_{r=0}^{\infty} a_r(N)\ b^r$ be the b–adic expansion of $N-1$. The condition $lb^m+1 \leq N \leq (l+1)b^m$ implies that $a_r(N)$ is uniquely determined for all $r \geq m$.

Now, write that $[x_N]_{b,m} \in [ab^{-m}, (a+1)b^{-m}[$:

$$ab^{-m} \leq \sum_{r=0}^{m-1} \sigma_r\big(a_r(N)\big) b^{-(r+1)} < (a+1)b^{-m},$$

which is equivalent to

$$a \leq \sigma_0\big(a_0(N)\big)b^{m-1} + \sigma_1\big(a_1(N)\big)b^{m-2} + \cdots + \sigma_{m-1}\big(a_{m-1}(N)\big) < a+1,$$

that is

$$a = \sigma_0\big(a_0(N)\big)b^{m-1} + \sigma_1\big(a_1(N)\big)b^{m-2} + \cdots + \sigma_{m-1}\big(a_{m-1}(N)\big),$$

which determine uniquely the $\sigma_r\big(a_r(N)\big)$'s for $0 \leq r \leq m-1$.

Finally, since the σ_r's are bijections, we have found the remaining digits of $N-1$ and the given data l, m, a of our assumption determine an unique point belonging to the elementary interval. $\square$

4. Overview of previous results

4.1. Results on permuted van der Corput sequences. These results come from preceding studies ([4] and [1]); we recall them to show the parallel with digital (0,1)–sequences and for future use. We are only concerned by the exact formulas; we refer to the papers above for the various asymptotic behaviours resulting from these formulas. Until now this family gives the best sequences with respect to low discrepancies, the L_∞ one (see [4], [5]) as well as the L_2 one (see [1]).

Functions $\varphi_{b,h}^{\sigma}$ related to a pair (b, σ).

Set $Z_b^{\sigma} = \Big(\dfrac{\sigma(0)}{b}, \cdots, \dfrac{\sigma(b-1)}{b}\Big)$. For any integer h with $0 \leq h \leq b-1$, the real function $\varphi_{b,h}^{\sigma}$ is defined as follows:

Let k be an integer with $1 \leq k \leq b$; then for every $x \in [\frac{k-1}{b}, \frac{k}{b}[$ we set:

$$\varphi_{b,h}^{\sigma}(x) = A\left([0, \frac{h}{b}[; k; Z_b^{\sigma}\right) - hx \quad \text{if } 0 \leq h \leq \sigma(k-1) \quad \text{and}$$

$$\varphi_{b,h}^{\sigma}(x) = (b-h)x - A\left([\frac{h}{b}, 1[; k; Z_b^{\sigma}\right) \text{ if } \sigma(k-1) < h < b;$$

finally the function $\varphi_{b,h}^{\sigma}$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^{\sigma} = 0$.

The functions $\varphi_{b,h}^{\sigma}$ give rise to other functions, depending only on (b,σ), in accordance with the notion of discrepancy we deal with. Set

$$\psi_b^{\sigma,+} = \max_{0\le h\le b-1}(\varphi_{b,h}^{\sigma})\,, \qquad \psi_b^{\sigma,-} = \max_{0\le h\le b-1}(-\varphi_{b,h}^{\sigma})\,,$$

$$\psi_b^{\sigma} = \psi_b^{\sigma,+} + \psi_b^{\sigma,-}\,, \qquad \chi_b^{\sigma} = \sum_{0\le h<h'\le b-1}(\varphi_{b,h}^{\sigma} - \varphi_{b,h'}^{\sigma})^2.$$

Then, for arbitrary bases $b\ge 2$,

$$D^+(N,S_b^{\Sigma}) = \sum_{j=1}^{\infty}\psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right)\,, \qquad D^-(N,S_b^{\Sigma}) = \sum_{j=1}^{\infty}\psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right)\,,$$

$$D(N,S_b^{\Sigma}) = \sum_{j=1}^{\infty}\psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \quad\text{and}\quad T^2(N,S_b^{\Sigma}) = \frac{2}{b^2}\sum_{j=1}^{\infty}\chi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\,.$$

There is also an analogous but more complex formula for T^* resulting from other functions deduced from the $\varphi_{b,h}^{\sigma}$; we don't give it since we are mainly concerned with D and D^* in the sequel.

4.2. Results on digital $(0,1)$–van der Corput sequences.

4.2.1. *In base 2.* By Pillichshammer [15] following a study on twodimensional digital point sets by Larcher and Pillichshammer [10]:

For any digital $(0,1)$–van der Corput sequence X_2^C

$$D^*(N,X_2^C) \le D^*(N,S_2^I) \quad\text{and}$$

$$.28 \approx \frac{1}{5\log 2} \le \overline{\lim}\frac{D^*(N,X_2^{C_1})}{\log N} \le \frac{5099}{22528\log 2} \approx .32,$$

where C_1 is the NUT matrix whose all entries are 1. It is conjectured that the lower bound should be the exact value of the limit superior and that the matrix C_1 should give the minimum over all digital $(0,1)$–van der Corput sequences in base 2 (see [10] p.406 for the analog with $(0,m,2)$–nets in base 2).

There are other results in base 2 concerning the L_2–discrepancy T^* of symmetrisized digital $(0,1)$–sequences by Larcher and Pillichshammer [9] and also a precise study of a family of digital $(0,1)$–sequences generated by special matrices including I and C_1 by Drmota, Larcher and Pillichshammer [3]. But there is nothing in their studies concerning the extreme discrepancies D and T.

4.2.2. *In arbitrary prime base b, with NUT matrices.* By the author [6], followed by a comparative study with van der Corput sequences [7]:

If the generator matrix C is diagonal, we have

$$X_b^C = S_b^{\Delta} \quad\text{with}\quad \Delta = (\delta_r)_{r\ge 0} \quad\text{and}\quad \delta_r(i) = c_r^r i,$$

where δ_r is the multiplication in $\mathbb{F}_b$ by the diagonal entry c_r^r.

Now, if C, NUT, is not diagonal, the diagonal entries still determine the same δ_r, but the permutations σ_r in the exact formulas for D^+, D^-, D^* and T^* are translated permutations of the δ_r's depending on the entries strictly above the diagonal. To save room, we don't give these complex formulas and deal only with D and T.

$$D(N, X_b^C) = \sum_{j=1}^{\infty} \psi_b^{\delta_{j-1}}\left(\frac{N}{b^j}\right) = D(N, S_b^{\Delta})$$

$$\text{and} \quad T^2(N, X_b^C) = \frac{2}{b^2}\sum_{j=1}^{\infty} \chi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = T^2(N, S_b^{\Delta}).$$

Therefore $D(N, X_b^C)$ and $T(N, X_b^C)$ depend only on the entries on the diagonal and their behaviour is the same as permuted van der Corput sequences with the sequence of permutations Δ.

As a consequence of our preceding studies (see [7]), we can assert that for arbitrary NUT matrices C we have

$$D^*(N, X_b^C) \leq D(N, X_b^C) \leq D^*(N, S_b^I) = D(N, S_b^I),$$

$$T^*(N, X_b^C) \leq T^*(N, S_b^I) \quad \text{and} \quad T(N, X_b^C) \leq T(N, S_b^I).$$

In other words *the original van der Corput sequences* S_b^I *are the worst distributed sequences among the permuted ones and among the NUT digital ones (with b prime) with respect to the four measures* D^*, D, T^* *and* T.

Moreover our formulas should permit improvements and comparisons, at least for D and T, between NUT digital sequences as we did for permuted sequences (see [7] for a first approach).

4.3. Result on $(0,1)$–sequences in arbitrary base b. By Kritzer [8], following a study with Dick on the star discrepancy of $(t, m, 2)$–nets [2] :

He shows that *the original van der Corput sequences are the worst distributed with respect to the star discrepancy among all* $(0,1)$*–sequences* X_b in the narrow sense *and in arbitrary base* $b \geq 2$, that is

$$D^*(N, X_b) \leq D^*(N, S_b^I) = D(N, S_b^I).$$

This result extends to $(0,1)$–sequences in base b in the broad sense (see below).

5. New results for extreme discrepancies

5.1. On the star discrepancy of $(0,1)$–sequences in base b. The following theorem extends the result of Kritzer [8] on $(0,1)$–sequences in the narrow sense to $(0,1)$–sequences (in the broad sense).

Theorem 5.1. *The original van der Corput sequences are the worst distributed with respect to the star discrepancy among all* $(0,1)$*–sequences* X_b *(in the broad sense) and in arbitrary base* $b \geq 2$*, that is*

$$D^*(N, X_b) \leq D^*(N, S_b^I) = D(N, S_b^I).$$

Proof. Let be given an integer $m \geq 0$ and the collection of elementary intervals of length b^{-m}. By construction (see [4] Property 3.1.2 with $\sigma_{n-1} = I$), the van der Corput sequence S_b^I is the unique $(0,1)$–sequence whose points are in increasing order the most left at the origins of the elementary intervals and, of course, satisfy the elementary interval property.

Let τ be the permutation defined by $\tau(k) = b - 1 - k$ for $0 \leq k \leq b - 1$. Then, by construction (see [4] Property 3.1.2 with $\sigma_{n-1} = \tau$), S_b^τ is the unique $(0,1)$–sequence (strictly in the broad sense) whose points are in increasing order the most right at the extremities of the above elementary intervals and satisfy the elementary interval property.

Therefore, by the elementary interval property for X_b, for any real $\alpha \in [0,1]$ and any integer $N \geq 1$ we have

$$A([0,\alpha[; N; X_b) \leq A([0,\alpha[; N; S_b^I) \text{ and } A([0,\alpha[; N; X_b) \geq A([0,\alpha[; N; S_b^\tau).$$

Now, from the definitions of D^+ and D^-, we get

$$D^+(N, X_b) \leq D^+(N, S_b^I) \quad \text{and} \quad D^-(N, X_b) \leq D^-(N, S_b^\tau).$$

Finally, recalling from [4] (proof of Lemma 4.4.1) that $\psi_b^{\tau,+} = \psi_b^{I,-}$ and $\psi_b^{\tau,-} = \psi_b^{I,+}$, we obtain $D^-(N, S_b^\tau) = D^+(N, S_b^I)$, so that

$$D^*(N, X_b) = \max(D^+(N, X_b), D^-(N, X_b)) \leq D^+(N, S_b^I) = D^*(N, S_b^I).$$

Note that $\psi_b^{I,-} = 0$ ([4], 5.5.1) which implies that $D^-(N, S_b^I) = 0$ and $D^+(N, S_b^\tau) = 0$, so that $D^*(N, S_b^I) = D(N, S_b^I) = D^*(N, S_b^\tau) = D(N, S_b^\tau)$. $\square$

Remark. 1. The main idea of the proof is in the first sentence; it was already used by Dick and Kritzer [2] in the context of Hammersley two-dimensional point sets. But in his paper [8], Kritzer considers only $(0,1)$–sequences in the narrow sense and his proof is longer. The use of the sequence S_b^τ, whose functions $\psi_b^{\tau,+}$ and $\psi_b^{\tau,-}$ are exchanged with the functions $\psi_b^{I,-}$ and $\psi_b^{I,+}$ of S_b^I, together with the good control of discrepancy by means of functions ψ allows a shorter proof.

2. It should be noted that, until now, the concept of $(0,1)$–sequences in the broad sense had not been considered, except in the general context of multidimensional sequences by Tezuka and Niederreiter and Xing. For instance, in [9] and [15], Larcher and Pillchshammer define implicitly digital $(0,1)$–sequences in the broad sense (in base 2) by obliging the generator

matrices to fulfil precise conditions of regularity, but without link to the extension of Niederreiter and Xing ([13], p. 271).
3. In the broad sense, we can say that there are two worst sequences with respect to D^*, S_b^I and S_b^τ, while in the narrow sense, there is only one, S_b^I, since S_b^τ is not a sequence in the narrow sense.
4. As to the extreme discrepancy D of $(0,1)$–sequences in the broad sense, we have $D(N, X_b) \leq 2D(N, S_b^I)$. We shall see in 5.3 that this upper bound is reached up to an additive constant by special, but not digital, $(0,1)$–sequences in the broad sense.

5.2. On the extreme discrepancy of digital $(0,1)$–sequences in base 2. We show here that the van der Corput sequence in base 2, S_2^I is not the worst distributed sequence with respect to D among the digital $(0,1)$-sequences in base 2.

Theorem 5.2. *Let C_2 be the matrix*

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \textit{ Then, for all } N,$$

$$2D(N, S_2^I) - \frac{5}{2} \leq D(N, X_2^{C_2}) \leq 2D(N, S_2^I) \quad \textit{and}$$

$$D^*(N, S_2^I) - \frac{3}{2} \leq D^*(N, X_2^{C_2}) \leq D^*(N, S_2^I) = D(N, S_2^I).$$

Moreover the sequence $X_2^{C_2}$ is the worst distributed among all $(0,1)$–sequences in base 2 (in the broad sense) with respect to D.

Proof. We obtain directly the upper bounds from Theorem 1 and Remark 4 above with $b = 2$. The lower bounds need more explanation. Computing the 2–adic digits of $X_2^{C_2}$ as indicated in the definition 3.2, we obtain for all $N \geq 1$

$$X_2^{C_2}(2N-1) = S_2^I(2N-1) \in [0, \frac{1}{2}[\quad \text{and} \quad X_2^{C_2}(2N) = S_2^\tau(2N-1) \in]\frac{1}{2}, 1],$$

where τ is defined as in 5.1 for $b = 2$.

First, we concentrate on the discrepancy D^+. For every integer $N \geq 1$, we have (since $S_2^I(2N) \in [\frac{1}{2}, 1[$)

$$D^+(N, X_2^{C_2}) \geq \sup_{0 \leq \alpha \leq \frac{1}{2}} E([0, \alpha[; N; X_2^{C_2}) = \sup_{0 \leq \alpha \leq \frac{1}{2}} E([0, \alpha[; N; S_2^I).$$

Therefore, we have only to deal with S_2^I, which we know well. From the block construction of that sequence, we get for $0 \le \alpha \le \frac{1}{2}$

$$A([0,\alpha[;2N;S_2^I) = A([0,2\alpha[;N;S_2^I) \quad \text{and}$$
$$A([0,\alpha[;2N-1;S_2^I) = A([0,2\alpha[;N;S_2^I),$$

so that

$$E([0,\alpha[;2N;S_2^I) = E([0,2\alpha[;N;S_2^I) \quad \text{and}$$
$$E([0,\alpha[;2N-1;S_2^I) = E([0,2\alpha[;N;S_2^I) + \alpha,$$

which in turn, taking the supremum over all $\alpha \in [0, \frac{1}{2}]$, gives

$$D^+(2N, X_2^{C_2}) \ge D^+(N, S_2^I) \quad \text{and} \quad D^+(2N-1, X_2^{C_2}) \ge D^+(N, S_2^I).$$

Now, for S_2^I we also have the recursion formulas (for all integers $M \ge 1$)

$$D(2M, S_2^I) = D(M, S_2^I) \quad \text{and}$$
$$D(2M+1, S_2^I) = \frac{1}{2}(D(M+1, S_2^I) + D(M, S_2^I) + 1),$$

from which we deduce that $|D(M+1, S_2^I) - D(M, S_2^I)| \le 1$ (indeed, the property is true for $1 \le M \le 2$ and if it is true for all $2^{n-1} \le M \le 2^n$, then

$$0 \le D(2M+1, S_2^I) - D(2M, S_2^I) = \frac{1}{2}(D(M+1, S_2^I) - D(M, S_2^I) + 1) \le 1 \text{ and}$$

$$0 \le D(2M+1, S_2^I) - D(2M+2, S_2^I) = \frac{1}{2}(D(M, S_2^I) - D(M+1, S_2^I) + 1) \le 1;$$

note that according to the parity of M, one of the differences is less than $\frac{1}{2}$).

Thanks to this property, we get $D^+(N, S_2^I) = D(N, S_2^I) = D(2N, S_2^I) \ge D(2N-1, S_2^I) - 1$, so that $D^+(2N-1, X_2^{C_2}) \ge D(2N-1, S_2^I) - 1$ and therefore for all N, odd or even, we still have $D^+(N, X_2^{C_2}) \ge D(N, S_2^I) - 1$.

Secondly, we deal with the discrepancy D^-. For every integer $N \ge 1$, we have

$$D^-(N, X_2^{C_2}) = \sup_{0 \le \alpha \le 1} (-E([0,\alpha[;N;X_2^{C_2})) \ge \sup_{\frac{1}{2} \le \alpha \le 1} (-E([0,\alpha[;N;X_2^{C_2})).$$

Since $X_2^{C_2}(2N) = S_2^{\tau}(2N-1) = 1 - S_2^I(2N-1) \in]\frac{1}{2}, 1]$ and $X_2^{C_2}(2N-1) \in [0, \frac{1}{2}[$, for $\frac{1}{2} \le \alpha \le 1$, we first have

$$-E([0,\alpha[;2N;X_2^{C_2}) = E([0,1-\alpha[;2N;S_2^I).$$

Then, $D^-(1, X_2^{C_2}) = 0$ (since $X_2^{C_2}(1) = 0$) and for all $N \geq 2$ we obtain

$$\begin{aligned} -E([0,\alpha[;2N-1;X_2^{C_2}) &= E([0,1-\alpha[;2N-2;S_2^I) - (1-\alpha) \\ &\geq E([0,1-\alpha[;2N-2;S_2^I) - \frac{1}{2} \end{aligned}$$

(since here, we deal with the shifted sequence of S_2^I on $[0, \frac{1}{2}[$, that is $S_2^I(N-1)$ instead of $S_2^I(N)$ in the case of D^+ just above).

And as before, from the block construction of S_2^I, we get

$$D^-(2N, X_2^{C_2}) \geq D(N, S_2^I) \quad \text{and} \quad D^-(2N-1, X_2^{C_2}) \geq D(N-1, S_2^I) - \frac{1}{2}.$$

The end of the proof is the same as for D^+, with the same property of S_2^I, and leads to $D^-(N, X_2^{C_2}) \geq D(N, S_2^I) - \frac{3}{2}$ for all integers N.

Finally, we obtain the lower bounds with $D = D^+ + D^-$ and $D^* = \max(D^+, D^-)$. The last assertion results from the inequalities (the second one from Theorem 1)

$$D(N, X_2) \leq 2D^*(N, X_2) \leq 2D^*(N, S_2^I) = 2D(N, S_2^I) \leq D(N, X_2^{C_2}) + \frac{5}{2}$$

for any $(0,1)$–sequence X_2 in base 2. $\square$

Remark. 1. The sequence $X_2^{C_2}$ has already been considered by Larcher and Pillichshammer [9] to show that there exist symmetrisized versions of digital $(0,1)$–sequences in base 2 which do not have optimal order of L_2–discrepancy T^*. We see here that $X_2^{C_2}$ has about the same star discrepancy as S_2^I, but its extreme discrepancy D is about twice and it is the worst among $(0,1)$–sequences in base 2. Thus, this sequence appears to be really a bad one (recall that the symmetrisized sequence of S_2^I has optimal order of L_2–discrepancy T^*).
2. The constants $\frac{5}{2}$ and $\frac{3}{2}$ in Theorem 2 are not optimal. A deeper analysis of the remainder $E([0,\alpha[;N;X_2^{C_2})$ by means of the method we used in [6] should give exact formulas for its discrepancies D^+ and D^- in relation with $D(N, S_2^I)$ (for instance we claim that $D(2N, X_2^{C_2}) = 2D(2N, S_2^I)$).

5.3. On the extreme discrepancy of $(0,1)$–sequences in base b. The generator matrix C_2 of Theorem 2 can be taken in arbitrary (prime) base b to produce examples of digital $(0,1)$–sequences (in the broad sense) with extreme discrepancy D greater than that of S_b^I, but its study is more complicated and do not give the twice. Moreover, in base b, there is no reason to deal only with the digit 1, at least on the first column. We intend to explore later the family of sequences generated by such matrices. For the present, we shall give a simple construction, inspired by the proof of Theorem 2, which permits to obtain the same result.

Theorem 5.3. *Let $b \geq 3$ be an integer. Define the sequence $X_b^{I\tau} = (x_N)_{N\geq 1}$ by*

$$x_{bk+1} = S_b^I(bk+1) \quad , \quad x_{bk+2} = S_b^\tau(bk+1) \quad and$$
$$x_{bk+l} = S_b^I(bk+l-1) \quad if\ 3 \leq l \leq b, \quad for\ all\ k \geq 0.$$

Then, the sequence $X_b^{I\tau}$ is a $(0,1)$–sequence (not digital and not in the narrow sense), which satisfies

$$2D(N, S_b^I) - 2(b-1) \leq D(N, X_b^{I\tau}) \ \leq 2D(N, S_b^I) \quad and$$
$$D^*(N, S_b^I) - (b-1) \leq D^*(N, X_b^{I\tau}) \leq D^*(N, S_b^I) = D(N, S_b^I).$$

Moreover the sequence $X_b^{I\tau}$ is the worst distributed among all $(0,1)$–sequences in base b (in the broad sense) with respect to D.

Remark. The choice $x_{bk+l} = S_b^I(bk+l-1)$ for $3 \leq l \leq b$ is arbitrary. It suffices to take the points in the interval $[\frac{1}{b}, \frac{b-1}{b}]$ in order to keep the property of elementary intervals (for instance any sequence S_b^σ is convenient). Moreover, the constant $2(b-1)$ is not optimal and a slightly different choice for x_{bk+1} and x_{bk+2} should probably permit $(b-1)$ for $b \geq 4$; such refinements seem not very interesting in our eyes.

Proof. Like in Theorem 2, the upper bounds are straightforward with Theorem 1 and Remark 4. As to the lower bounds, we proceed in the same way, but here with the extremal intervals $[0, \frac{1}{b}[$ and $]\frac{b-1}{b}, 1]$.

Since $x_{bk+1} = S_b^I(bk+1) \in [0, \frac{1}{b}[$, $x_{bk+2} = S_b^\tau(bk+1) \in]\frac{b-1}{b}, 1]$ and for all $3 \leq l \leq b$ $x_{bk+l} = S_b^I(bk+l-1) \in [\frac{1}{b}, \frac{b-1}{b}]$, for every integer $N \geq 1$ we obtain

$$D^+(N, X_b^{I\tau}) \geq \sup_{0\leq\alpha\leq\frac{1}{b}} E([0,\alpha[; N; X_b^{I\tau}) = \sup_{0\leq\alpha\leq\frac{1}{b}} E([0,\alpha[; N; S_b^I) \quad \text{and}$$

$$D^-(N, X_b^{I\tau}) \geq \sup_{\frac{b-1}{b}\leq\alpha\leq 1} (-E([0,\alpha[; N; X_b^{I\tau})) = \sup_{\frac{b-1}{b}\leq\alpha\leq 1} (-E([0,\alpha[; N; S_b^\tau)).$$

¿From the block construction of S_b^I, for $0 \leq \alpha \leq \frac{1}{b}$ we have

$$A([0,\alpha[; bN-l; S_b^I) = A([0, b\alpha[; N; S_b^I) \quad \text{for all } 0 \leq l \leq b-1, \text{ so that}$$

$$E([0,\alpha[; bN-l; S_b^I) = E([0,\alpha[; bN; S_b^I) + l\alpha = E([0, b\alpha[; N; S_b^I) + l\alpha.$$

According to the first inequality above, we deduce, for all $0 \leq l \leq b-1$, that

$$D^+(bN-l, X_b^{I\tau}) \geq \sup_{0\leq\alpha\leq\frac{1}{b}} E([0, b\alpha[; N; S_b^I) = D^+(N, S_b^I).$$

Now, concerning D^-, we first note that $S_b^\tau(N) = 1 - S_b^I(N)$ but again, by definition of $X_b^{I\tau}$, we deal actually with the shifted sequence of S_b^I on $[0, \frac{1}{b}[$, $S_b^I(N-1)$ instead of $S_b^I(N)$.

For multiples of b, it's no matter: for $\frac{b-1}{b} \leq \alpha \leq 1$, we have for all $N \geq 1$

$$-E([0,\alpha[; bN; X_b^{I\tau}) = E([0, 1-\alpha[; bN; S_b^I).$$

For integers of the form $bN - l$ with $1 \leq l \leq b-2$, it's also no matter: we obtain

$$\begin{aligned} -E([0,\alpha[; bN-l; X_b^{I\tau}) &= E([0,1-\alpha[; bN; S_b^I) + l(1-\alpha) \\ &\geq E([0,1-\alpha[; bN; S_b^I). \end{aligned}$$

But for the integers of the form $bN - b + 1$, first $D^-(1, X_b^{I\tau}) = 0$ and for $N \geq 2$, we get

$$\begin{aligned} -E([0,\alpha[; bN-b+1; X_b^{I\tau}) &= E([0,1-\alpha[; bN-b; S_b^I) - (1-\alpha) \\ &\geq E([0,1-\alpha[; bN-b; S_b^I) - \frac{1}{b}. \end{aligned}$$

Again, from the block construction of S_b^I, we deduce that

$$E([0,1-\alpha[; bN; S_b^I) = E([0, b(1-\alpha)[; N; S_b^I)$$

which gives for all $0 \leq l \leq b-2$

$$D^-(bN-l, X_b^{I\tau}) \geq \sup_{\frac{b-1}{b} \leq \alpha \leq 1} E([0, b(1-\alpha)[; N; S_b^I) = D^+(N, S_b^I) \quad \text{and}$$

$$\begin{aligned} D^-(bN-b+1, X_b^{I\tau}) &\geq \sup_{\frac{b-1}{b} \leq \alpha \leq 1} E([0, b(1-\alpha)[; N-1; S_b^I) - \frac{1}{b} \\ &= D^+(N-1, S_b^I) - \frac{1}{b}. \end{aligned}$$

In arbitrary base b, we do not have nice recursion formulas for $D(N, S_b^I)$ as for $D(N, S_2^I)$ and we need another trick to achieve the proof. It rests on the general properties of functions ψ, here $\psi_b^I = \psi_b^{I,+}$, since $\psi_b^{I,-} = 0$. It is the purpose of the following lemma.

Lemma 5.4. *For every integer $M \geq 1$, we have*

$$D(bM, S_b^I) = D(M, S_b^I) \quad \text{and} \quad |D(M+1, S_b^I) - D(M, S_b^I)| \leq 1.$$

Proof. Recall that (with $1 \leq M \leq b^n$ for the last term)

$$D(M, S_b^I) = \sum_{j=1}^{\infty} \psi_b^I\left(\frac{M}{b^j}\right) = \sum_{j=1}^{n} \psi_b^I\left(\frac{M}{b^j}\right) + \frac{M}{b^n}.$$

¿From the properties of functions ψ (see [4] 3.2), we know that ψ_b^I is continuous, 1-periodic and maximum of piecewise affine functions with coefficients less than $b-1$ in absolute value. Thus, $f_n(x) = \displaystyle\sum_{j=1}^{n} \psi_b^I\left(\frac{x}{b^j}\right)$ is b^n–periodic and piecewise affine with coefficients bounded in absolute value by $(b-1)(\frac{1}{b} + \cdots + \frac{1}{b^n}) = 1 - \frac{1}{b^n}$.

Therefore, $|D(M+1, S_b^I) - D(M, S_b^I)| = |f_n(M+1) - f_n(M) + \frac{1}{b^n}| \leq 1 - \frac{1}{b^n} + \frac{1}{b^n} = 1$.

The first formula is obvious with the sum up to infinity and $\psi_b^I(0) = 0$.

□

Coming back to $D^+(bN-l, X_b^{I\tau})$ and $D^-(bN-l, X_b^{I\tau})$ and applying l times our Lemma (only once for the last case), we obtain successively (with $D^+(N, S_b^I) = D(N, S_b^I)$): for all $0 \leq l \leq b-1$

$$D^+(bN-l, X_b^{I\tau}) \geq D(N, S_b^I) = D(bN, S_b^I) \geq D(bN-l, S_b^I) - l,$$

for all $0 \leq l \leq b-2$

$$D^-(bN-l, X_b^{I\tau}) \geq D(N, S_b^I) = D(bN, S_b^I) \geq D(bN-l, S_b^I) - l \quad \text{and}$$

$$\begin{aligned} D^-(bN-b+1, X_b^{I\tau}) \geq D(N-1, S_b^I) - \frac{1}{b} &= D(bN-b, S_b^I) - \frac{1}{b} \\ &\geq D(bN-b+1, S_b^I) - \frac{1}{b} - 1\cdot \end{aligned}$$

Finally, taking the worst lower bound to gather all cases together, we obtain

$$\begin{aligned} &D^+(bN-l, X_b^{I\tau}) \geq D(N, S_b^I) - (b-1) \quad \text{and} \\ &D^-(bN-l, X_b^{I\tau}) \geq D(N, S_b^I) - (b-1) \quad \text{since } b \geq 3 \end{aligned}$$

which give the lower bounds of Theorem 3 with $D = D^+ + D^-$ and $D^* = \max(D^+, D^-)$.

If $b = 2$, we must keep $\frac{1}{b} + 1 = \frac{3}{2}$ for D^- and we obtain the result of Theorem 2.

Again the last assertion results from

$$\begin{aligned} D(N, X_b) \leq 2D^*(N, X_b) &\leq 2D^*(N, S_b^I) \\ &= 2D(N, S_b^I) \leq D(N, X_b^{I\tau}) + 2(b-1). \end{aligned}$$

□

Questions. 1. The sequence $X_b^{I\tau}$ is not digital except for $b = 2$, in which case we have $X_2^{I\tau} = X_2^{C_2}$. Are there digital $(0,1)$–sequences X_b^C having the worst distribution with respect to D like $X_b^{I\tau}$, i.e. $D(N, X_b^C) \geq 2D(N, S_b^I) - c$, at least for infinitely many N ?

2. The question of finding the worst $(0, 1)$–sequences with respect to D^* and D is solved by the result of Kritzer [8] and by Theorems 1 and 3 (apart from an additive constant for D). What's about for T^* and T ? This question seems harder to handle, because the L_2–discrepancies involve integrals instead of extrema.

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Henri Faure
Institut de Mathématiques de Luminy, U.M.R. 6206 CNRS
163 avenue de Luminy, case 907
13288 Marseille Cedex 09, France
E-mail : `faure@iml.univ-mrs.fr`