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Substitutions on two letters, cutting segments and their projections

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Résumé. Dans cet article on considère la structure des projections des segments de coupure correspondant aux substitutions unimodulaires sur un alphabet binaire. On montre qu'une telle projection est un bloc de lettres si et seulement si la substitution est sturmienne. Une double application de ce procédé à une substitution de Christoffel donne la substitution originelle. On obtient ainsi une dualité sur l'ensemble des substitutions de Christoffel.

Abstract. In this paper we study the structure of the projections of the finite cutting segments corresponding to unimodular substitutions over a two-letter alphabet. We show that such a projection is a block of letters if and only if the substitution is Sturmian. Applying the procedure of projecting the cutting segments corresponding to a Christoffel substitution twice results in the original substitution. This induces a duality on the set of Christoffel substitutions.

1. Introduction

The history of Sturmian words goes back to J. Bernoulli in 1772 and Christoffel [2] (1875). The first in depth study of Sturmian words was made by Morse and Hedlund [6], [7] in 1938 and 1940. A Sturmian word induces a broken half-line, the so-called cutting line, which approximates a half-line through the origin quite well. See Series [14] (1985). We call a substitution σ Sturmian if σ maps every Sturmian word to a Sturmian word. In 1991 Séébold [13] showed that Sturmian substitutions that have a fixed point are exactly those substitutions that have Sturmian words as fixed points. For further information on Sturmian words and substitutions we refer to Lothaire [5], Ch. 2 and Pytheas Fogg [8], Ch. 6.

In 1982 Rauzy [9] introduced a fractal which is defined as the closure of the projection of the cutting line corresponding to the Tribonacci substitution $0 \to 01,\ 1 \to 02,\ 2 \to 0$. The analogues of this so-called Rauzy fractal have been studied for many other substitutions, see [5], [8]. An important

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question is whether the projection of the cutting line generates a tiling or not. This question was the motivation for the author and Tijdeman [12] to have a closer look at the structure of the projections of the finite cutting segments corresponding to $\sigma^n(0)$ in case σ is the Fibonacci or Tribonacci substitution. They found in the Fibonacci case that the projections of the integer points of the cutting line generate a two-sided Fibonacci word. In the Tribonacci case they found a close connection with number systems. The latter connection was generalized by Fuchs and Tijdeman in [3]. In the present paper we generalize the former property to unimodular substitutions defined over two letters.

We examine for which substitutions the projected points form a (doubly-infinite) word and which properties these words have. In Section 2 we start with some notation and definitions. Next in Section 3 we give some properties of the points that we get after projecting the cutting segment corresponding to a substitution. We show that the order of the projected points of $\sigma^n(0)$ is preserved in the projection of $\sigma^{n+1}(0)$. Then in Section 4 we define Sturmian substitutions, Sturmian matrices and prove some of their properties.

In Section 5 we consider substitutions for which the incidence matrix is unimodular, and we show that the projected points form a central word if and only if the substitution is Sturmian. We show how the number of 0's and 1's in these central words can be calculated from the incidence matrix of the original substitution.

In the final Section 6 we consider a special class of Sturmian substitutions, that we call Christoffel substitutions. We show that when one starts with a Christoffel substitution, the projected points form Christoffel words, and that the relation between these words is again given by a Christoffel substitution. Moreover, if one applies the procedure of projecting the cutting segment corresponding to these Christoffel words again, the result will be the original substitution. This induces a duality on the set of Christoffel substitutions.

2. Notations and definitions

An *alphabet* $\mathcal{A}$ is a finite set of elements that are called *letters*. In this article we always assume $\mathcal{A} = \{0, 1\}$. A *word* is a function u from a finite or infinite block of integers to $\mathcal{A}$. If this block of integers contains 0 we call u a *central word*. If $a \in \mathcal{A}$ and $u(k) = a$ we say u has the letter a at position k, denoted by $u_k = a$. If the block of integers is finite we call u a finite word, otherwise it is an infinite word. If $v = v_0 \dots v_m$ is a finite word and if $u = u_0 u_1 \dots$ is a finite or infinite word, and there exists a k such

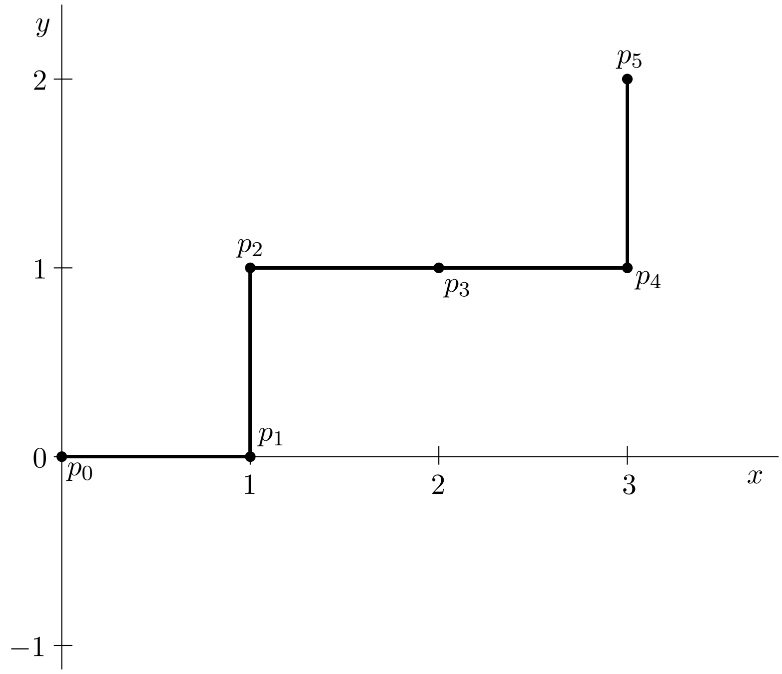

FIGURE 1. The cutting segment corresponding to $u = 01001$.

that $v_l = u_{k+l}$ for $l = 0, \ldots m$, then v is called a *subword* of u. If a word u is finite, we denote by $|u|$ the number of letters in u, and by $|u|_a$ the number of occurrences of the letter a in u.

A *substitution* σ is an application from an alphabet $\mathcal{A}$ to the set of finite words. It extends to a morphism by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$. It also extends in a natural way to a map over infinite words u. A *fixed point* of a substitution σ is an infinite word u with $\sigma(u) = u$.

A substitution over the alphabet $\mathcal{A}$ is *primitive* if there exists a positive integer k such that, for every a and b in $\mathcal{A}$, the letter a occurs in $\sigma^k(b)$.

We denote the largest integer y such that $y \leq x$ by $\lfloor x \rfloor$, the smallest integer y such that $y \geq x$ by $\lceil x \rceil$ and we put $\{x\} = x - \lfloor x \rfloor$.

Definition. Let $u = u_0 \ldots u_{m-1}$ be a finite word. The *cutting segment* in the x-y-plane corresponding to u consists of $m + 1$ integer points p_i given by $p_i = (|u_0 \ldots u_{i-1}|_0, |u_0 \ldots u_{i-1}|_1)$ for $i = 0, \ldots, m$, connected by line segments of lengths 1.

In Figure 1 we show the cutting segment corresponding to $u = 01001$.

Let $u = u_0 \ldots u_{m-1}$ be a finite word containing at least one zero. Consider the cutting segment corresponding to u, and draw the line through

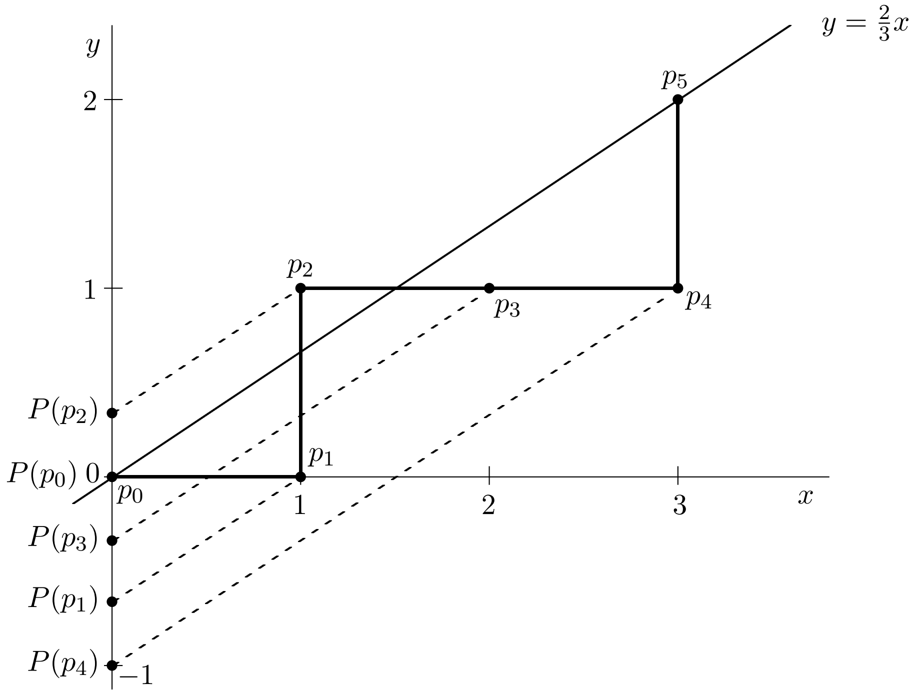

FIGURE 2. Projecting the cutting segment corresponding to $u = 01001$.

the origin and the end point of the segment, given by $y = \frac{|u|_1}{|u|_0}x$. We project each integer point p_i on the cutting segment parallel to this line to the y-axis. By $P(p_i)$ we denote the second coordinate of the projection of p_i. It is clear that $P(p_0) = P(p_m)$. See Figure 2 for an example.

Lemma 2.1. *Let $u = u_0 \dots u_{m-1}$ be a finite word containing at least one zero and let $P(p_i)$ for $i = 0, \dots, m$ be defined as above. Then for every $i = 0, \dots, m$ we have $P(p_i) = (|u_0 \dots u_{i-1}|_1|u|_0 - |u_0 \dots u_{i-1}|_0|u|_1)/|u|_0 \in \mathbb{Z}/|u|_0$.*

Proof. We use induction on i. Obviously $P(p_0) = 0$. Assume the lemma is valid for $i \geq 0$. If $u_i = 0$ then $P(p_{i+1}) = P(p_i) - |u|_1/|u|_0 = (|u_0 \dots u_i|_1|u|_0 - |u_0 \dots u_i|_0|u|_1)/|u|_0$, and if $u_i = 1$ then $P(p_{i+1}) = P(p_i) + 1 = (|u_0 \dots u_i|_1|u|_0 - |u_0 \dots u_i|_0|u|_1)/|u|_0$. $\square$

The following lemma says that when the numbers of 0's and 1's in u are relatively prime, the points p_i are projected to distinct points.

Lemma 2.2. *Let $u = u_0 \dots u_{m-1}$ be a finite word containing at least one zero, let $|u|_1 > 0$ if $m > 1$, let $\gcd(|u|_0, |u|_1) = 1$ and let $P(p_i)$ for $i = 0, \dots, m$ be defined as above. Then for $i, j \in \{0, \dots, m-1\}$ and $i \neq j$ we have $P(p_i) \neq P(p_j)$.*

Proof. Suppose $P(p_i) = P(p_j)$. Put $x = |u_0 \dots u_{i-1}|_1$, $y = |u_0 \dots u_{i-1}|_0$ and $x' = |u_0 \dots u_{j-1}|_1$, $y' = |u_0 \dots u_{j-1}|_0$. Then $P(p_i) = (x|u|_0 - y|u|_1)/|u|_0$ and $P(p_j) = (x'|u|_0 - y'|u|_1)/|u|_0$, hence $x|u|_0 - y|u|_1 = x'|u|_0 - y'|u|_1$. Since $\gcd(|u|_0, |u|_1) = 1$ and $|x - x'| \le |u|_1, |y - y'| \le |u|_0$ with at least one strict inequality, we have $x = x'$, $y = y'$, hence $i = j$. □

Let $D = \left\{ -|u|_0 P(p_i) \middle| i \in \{0, \dots, m-1\} \right\}$, hence D is a subset of $\mathbb{Z}$ containing m elements under the hypotheses of Lemma 2.2. We define the *central function* $w : D \to \{0, \dots, m-1\}$ as follows. If $P(p_i) = k/|u|_0$ then $w(-k) = i$. We say w has number i at position $-k$. If $k_1 < k_2 < k_3$ are integers and w has numbers at positions k_1, k_3 but not at k_2, we say w has a gap at k_2. If the central function w has no gap we call w a *central block.* Note that a central function always has the number 0 at position 0. By $|w|$ we mean the number of positions on which w is defined, hence $|w| = |u|$ if w is the central function corresponding to u. The following lemma shows that $w(k+1) - w(k)$ is constant modulo $|u|$.

Lemma 2.3. *Let u be a finite word with* $\gcd(|u|_0, |u|_1) = 1$ *and w the central function corresponding to u. If w has a number at position k, then* $w(k) = k|u|_1^{-1} \pmod{|u|}$*, where the inverse is taken modulo $|u|$.*

Proof. Suppose w has the number i at position k. Then we obtain successively from $|u|_0 + |u|_1 = |u|$ that

$$|u_0 \dots u_{i-1}|_0 |u|_1 - |u_0 \dots u_{i-1}|_1 |u|_0 = k,$$

$$|u_0 \dots u_{i-1}|_0 |u|_1 + |u_0 \dots u_{i-1}|_1 |u|_1 \equiv k \pmod{|u|},$$

$$i|u|_1 \equiv k \pmod{|u|}.$$

□

Remark. If $w(k) = i$ then $w(k + |u|_1) = i + 1$ in case $u_i = 0$, and $w(k - |u|_0) = i + 1$ in case $u_i = 1$. We say that to move from number i to $i+1$ in w we either "jump" $|u|_1$ positions to the right or $|u|_0$ positions to the left. It follows that the $|w|$ positions on which w is defined, represent exactly the cosets modulo $|w|$.

Example 1. If $u = 01001$, the central function w associated with u is a central block given by $w = 2\underline{0}314$, where we have underlined the number at position 0. See Figure 2.

3. The central functions w_n

In this section for every n we define a central function w_n corresponding to $u_n = \sigma^n(0)$, where σ is a substitution. We give conditions on σ so that the construction of w_n is well-defined, and we show that the order of the numbers in w_n is preserved in w_{n+1}.

Let σ be a primitive substitution that has a fixed point. Without loss of generality we may assume the fixed point starts with 0. Let $M_\sigma = \begin{pmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{pmatrix}$ be its incidence matrix. Define $u_n = \sigma^n(0)$ for $n \in \mathbb{Z}_{\geq 0}$ where we use the convention that $\sigma^0(v) = v$ for every word v. We assume $\gcd(|u_n|_0, |u_n|_1) = 1$ for every $n \in \mathbb{Z}_{\geq 0}$ and consider for each u_n the corresponding central function w_n.

Example 2. Let ϕ be the substitution defined by $\phi(0) = 01001$, $\phi(1) = 01$. If we start with 0 and repeatedly apply ϕ we get successively

$$\begin{aligned} u_0 &= 0 \\ u_1 &= 01001 \\ u_2 &= 0100101010010100101 \\ &\dots \end{aligned}$$

This yields the following table of central functions w_n, where we have underlined the number at position 0. Note that the central functions are central blocks, that is, have no gap.

(3.1)

n	w_n																		
0					$\underline{0}$														
1				2	$\underline{0}$	3	1	4											
2	9	2	14	7	$\underline{0}$	12	5	17	10	3	15	8	1	13	6	18	11	4	16

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We use the following lemmas to derive conditions on the incidence matrix M_σ under which the central functions w_n exist for all n.

Lemma 3.1. *If M is a 2×2-matrix with integer coefficients and $\det(M) \neq 0$, then $\operatorname{trace}(M^n) - (\operatorname{trace}(M))^n$ is divisible by $\det(M)$ for every $n > 0$.*

Proof.

$$\begin{aligned} \operatorname{trace}(M^n) - (\operatorname{trace}(M))^n &= \lambda_1^n + \lambda_2^n - (\lambda_1 + \lambda_2)^n \\ &= -\lambda_1\lambda_2 \sum_{k=1}^{n-1} \binom{n}{k} \lambda_1^{k-1}\lambda_2^{n-k-1} \end{aligned}$$

with $\det(M) = \lambda_1\lambda_2$. $\square$

Lemma 3.2. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with $a, b, c, d \in \mathbb{R}$. Then $M^n = (a+d)M^{n-1} - (ad-bc)M^{n-2}$ for $n \geq 2$.*

Proof. We have $M^2 = (a+d)M - (ad-bc)M^0$. $\square$

Proposition 3.1. *Let σ be a primitive substitution with a fixed point starting with 0, let $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its incidence matrix and let $u_n = \sigma^n(0)$. Then* $\gcd(|u_n|_0, |u_n|_1) = 1$ *for every $n \in \mathbb{Z}_{\geq 0}$ if and only if* $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$.

Proof. Assume $\gcd(|u_n|_0, |u_n|_1) = 1$ for every $n \in \mathbb{Z}_{\geq 0}$. We define a_n, b_n, c_n, d_n by $M_\sigma^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for $n \in \mathbb{Z}_{\geq 0}$. Since $M_{\sigma^n} = M_\sigma^n$, we have $|u_n|_0 = a_n, |u_n|_1 = b_n$. Hence $\gcd(a, b) = \gcd(|u_1|_0, |u_1|_1) = 1$. By the previous lemma, $|u_2|_0 = (a+d)a-(ad-bc), |u_2|_1 = (a+d)b$. Since $\gcd(|u_2|_0, |u_2|_1) = 1$, we have $\gcd(a + d, ad - bc) = 1$.

We now prove the other implication. Assume that $\gcd(a, b) = \gcd(a + d, ad - bc) = 1$. From Lemma 3.1 and our assumption that $\gcd(\mathrm{trace}(M_\sigma), \det(M_\sigma)) = 1$ it follows that

(3.2)
$$\gcd(a_n + d_n, a_1d_1 - b_1c_1) = \gcd(\mathrm{trace}(M_\sigma^n), \det(M_\sigma)) = 1 \text{ for every } n > 0.$$

We prove by induction on n that $\gcd(a_n, b_n) = 1$. We know that $\gcd(a_1, b_1) = 1$. Assume $\gcd(a_m, b_m) = 1$ for $m = 1, \ldots, n$. Note that

$$M_\sigma^{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa_n + cb_n & ba_n + db_n \\ ac_n + cd_n & bc_n + dd_n \end{pmatrix}.$$

Assume p is prime and $p|a_{n+1}$ and $p|b_{n+1}$. Then $p|ab_{n+1} - ba_{n+1} = aba_n + adb_n - aba_n - bcb_n = b_n(ad - bc)$. Hence $p|b_n$ or $p|(ad - bc)$. Suppose $p|b_n$. From $p|b_{n+1} = ba_n + db_n$ it follows that $p|b$ and from $p|a_{n+1} = aa_n + cb_n$ that $p|a$, but $\gcd(a, b) = 1$. Thus $p \nmid b_n$ and

$$p|(ad - bc) = \det(M). \tag{3.3}$$

By (3.2) this gives $p \nmid a_{n+1} + d_{n+1}$, hence $p \nmid d_{n+1}$. By (3.3) we have $p|\det(M^n) = a_nd_n - b_nc_n$. Thus $p|b(a_nd_n - b_nc_n)$ and because $p|d_nb_{n+1} = ba_nd_n + db_nd_n$ we obtain $p|db_nd_n + bb_nc_n = b_nd_{n+1}$. This contradiction implies that $\gcd(a_{n+1}, b_{n+1}) = 1$. □

Consider the sequence of words w_n in (3.1) from Example 2. We see that the numbers $0, \ldots, 4$ in w_1 are ordered in the same way as the numbers $0, 5, 7, 12, 17$ in w_2 directly below them. This observation illustrates the following proposition.

Proposition 3.2. *Let σ be a primitive substitution with incidence matrix $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a fixed point starting with 0, let* $\gcd(a, b) = \gcd(a + d, ad-bc) = 1$ *and let w_n be the corresponding central functions for $n \in \mathbb{Z}_{>0}$.*

If w_n has numbers at positions k and l then w_{n+1} has numbers at positions $\det(M_\sigma)\cdot k$ and $\det(M_\sigma)\cdot l$. If the number at position k of w_n is larger than the number at position l of w_n then the number at position $\det(M_\sigma)\cdot k$ of w_{n+1} is larger than the number at position $\det(M_\sigma)\cdot l$ of w_{n+1}.

Proof. First note that since

$$|u_{n+1}|_0 = a|u_n|_0 + c|u_n|_1$$
$$|u_{n+1}|_1 = b|u_n|_0 + d|u_n|_1$$

we have $a|u_{n+1}|_1 - b|u_{n+1}|_0 = (ad-bc)|u_n|_1 = \det(M_\sigma)|u_n|_1$, and similarly $c|u_{n+1}|_1 - d|u_{n+1}|_0 = -\det(M_\sigma)|u_n|_0$. We prove the proposition by induction. Since the first letter of u_n is a 0, the first jump in w_n starting at position 0 is to the right and places the number 1 at position $|u_n|_1$. When going from u_n to u_{n+1} this 0 is replaced by a 0's and b 1's. Therefore the number $a+b$ in w_{n+1} is placed at position $a|u_{n+1}|_1 - b|u_{n+1}|_0 = \det(M_\sigma)|u_n|_1$. We see that the statements of the proposition hold for the positions of the numbers 0 and 1 in w_n. Assume that in w_n the number m is placed at position k and in w_{n+1} the number m' is placed at position $\det(M_\sigma)k$. The next jump is either to the right or to the left, depending on the value of the $(m+1)$-th letter in u_n. If the next jump in w_n is to the right, then the number $m+1$ is placed at position $k+|u_n|_1$ in w_n, and it follows that the number $m'+a+b$ is placed at position $\det(M_\sigma)(k+|u_n|_1)$ in w_{n+1}. If the next jump in w_n is to the left, then the number $m+1$ is placed at position $k-|u_n|_0$ in w_n, and it follows that the number $m'+c+d$ is placed at position $\det(M_\sigma)(k-|u_n|_0)$ in w_{n+1}. The assertions of the proposition follow now easily. □

Example 3. Let the substitution σ be given by $\sigma(0) = 011$, $\sigma(1) = 0$ so that it has incidence matrix $M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\det(M_\sigma) = -2$. We get the following table for $u_n = \sigma^n(0)$:

$$\begin{aligned} u_0 &= 0 \\ u_1 &= 011 \\ u_2 &= 01100 \\ u_3 &= 01100011011 \\ &\dots \end{aligned}$$

This gives the following sequence of w_n's, where $\sqcup$ indicates a gap.

n					w_n														
0					$\underline{0}$														
1					$\underline{0}$	2	1												
2	3	$\sqcup$	4	2	$\underline{0}$	$\sqcup$	1												
3	3	$\sqcup$	$\sqcup$	$\sqcup$	$\underline{0}$	2	4	$\sqcup$	8	10	1	$\sqcup$	5	7	9	$\sqcup$	$\sqcup$	$\sqcup$	6

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Comparing w_2 with w_3 illustrates Proposition 3.2.

4. Sturmian matrices and Sturmian substitutions

In this section we define Sturmian matrix, Sturmian substitution, reduced matrix, dual matrix and we list some properties of these objects.

A word u is called *balanced* if $||v|_0 - |w|_0| < 2$ for all subwords v, w of equal length. A finite word u is called *strongly balanced* if u^2 is balanced. Here u^2 is the concatenation of u with u. It is easy to see that a finite word u is strongly balanced if and only if u^n is balanced for some $n \geq 2$. A one-sided infinite word is *Sturmian* if it is balanced and not ultimately periodic. We call a 2×2-matrix *Sturmian* if it has determinant equal to ± 1 and has entries in $\mathbb{Z}_{\geq 0}$. We call a substitution σ over two letters *Sturmian* if $\sigma(u)$ is a Sturmian word for every Sturmian word u. [5] Th 2.3.7 says that a substitution σ is Sturmian if and only if there exists a Sturmian word v such that $\sigma(v)$ is Sturmian. It follows that a primitive substitution σ is Sturmian if and only if its fixed point is Sturmian.

Lemma 4.1. *A Sturmian substitution maps every finite balanced word to a finite balanced word.*

Proof. See [10] Cor.9. $\square$

The set of Sturmian substitutions is generated by the following three substitutions, cf. [5] Th 2.3.7.

$$E : \begin{cases} 0 & \to & 1 \\ 1 & \to & 0 \end{cases} \qquad \phi : \begin{cases} 0 & \to & 01 \\ 1 & \to & 0 \end{cases} \qquad \tilde{\phi} : \begin{cases} 0 & \to & 10 \\ 1 & \to & 0 \end{cases} .$$

Their incidence matrices are $M_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_\phi = M_{\tilde{\phi}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, respectively. From this it follows that each Sturmian substitution has a Sturmian matrix as incidence matrix. On the other hand we have the following result.

Theorem 4.1. *If M is a Sturmian matrix then there exists a Sturmian substitution that has M as its incidence matrix.*

Proof. Let M be a Sturmian matrix. It can be written as a product of factors M_E and M_ϕ, cf. [8] Sec.6.5.5. By replacing each matrix with the corresponding substitution the composition of these substitutions is a Sturmian substitution that has M as incidence matrix. $\square$

Lemma 4.2. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix.*

(i) *If $a + b = c + d$, then M is the identity matrix or M_E.*
(ii) *If $a > c$ and $b < d$, then M is equal to the identity matrix.*
(iii) *If $a < c$ and $b > d$, then $M = M_E$.*

Proof. (i) Assume $a+b=c+d$. Then $M=\begin{pmatrix} a & b \\ a-\delta & b+\delta \end{pmatrix}$ for some $\delta\in\mathbb{Z}$. Since $\det(M)=(a+b)\delta=\pm1$ we have $a+b=1$ and $\delta=\pm1$. It follows that M is of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(ii) Assume $a>c$ and $b<d$. Using that $ad-bc=\pm1$ we get $bc\leq(d-1)(a-1)\leq bc-(a+d)+2$. It follows that $a+d\leq 2$, hence $a=d=1$ and $b=c=0$.
(iii) The proof is similar to the proof of (ii). □

Corollary 4.1. *Let* $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix. If* M *is an upper matrix then* $a\geq c$ *and* $b\geq d$ *with at least one of the two inequalities strict. If* M *is a lower matrix then* $a\leq c$ *and* $b\leq d$ *with at least one of the two inequalities strict.*

Proof. This follows immediately from Lemma 4.2. □

Definition. Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix. If $a+b>c+d$, then we call M an *upper matrix*, if $a+b<c+d$ a *lower matrix*.

Corollary 4.2. *Let* $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix with* $ab\neq0$. *Then there exists a non-negative integer* k *such that* $\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$ *is an upper matrix.*

Proof. Let $k=\min(\lfloor c/a\rfloor, \lfloor d/b\rfloor)$ and let $N=\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$. Then $\det N=\det M$ hence N is Sturmian. Since $c-ka<a$ or $d-kb<b$ it follows from Lemma 4.2 that N is an upper matrix. □

Definition. We call the upper matrix that is constructed from M as described in Corollary 4.2 the *reduced matrix* of M.

It follows directly from Corollary 4.1 that this definition of reduced matrix is unique.

Lemma 4.3. *Let* $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an upper matrix. Then there exists a unique upper matrix* $N=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *such that* $\det(N)=\det(M)$, $\alpha+\beta=a+b$ *and* $\alpha=c+d$. *Moreover* $\gamma+\delta=a$ *and* $trace(M)=trace(N)$.

Proof. Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an upper matrix. We assume $\det(M)=1$, the case for $\det(M)=-1$ is similar. We define $N:=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}=$

$\begin{pmatrix} c+d & a+b-(c+d) \\ c & a-c \end{pmatrix}$. We have to show that N has nonnegative entries. This is clear for α, β, γ. Assume $a < c$. Then $b = (ad-1)/c < d$. This implies $c+d \leq a+b-1 < c+d$. It follows that $\delta \geq 0$. Next we show that N is unique. Because $\alpha + \beta = a + b$ and $\alpha = c + d$, the variables α and β are uniquely defined. It is well known from number theory that the equation $\alpha x - \beta y = 1$ in unknowns x, y has a unique solution in nonnegative integers x, y with $x + y < \alpha + \beta$. Therefore γ and δ are also uniquely defined. For $\alpha + \beta > \gamma + \delta$, observe that if $b = 0$ then $a = d = 1$, hence $a + b = 1 \leq c + d$. The other properties are obvious. □

Definition. For an upper matrix M we call the matrix N as defined in the previous lemma the *dual matrix* of M. If $N = M$ we say that M is *self-dual.*

It follows directly from the proof of Lemma 4.3 that if N is the dual matrix of M, then M is the dual matrix of N, and that an upper Sturmian matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is self-dual if and only if $a = c + d$.

Lemma 4.4. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix with* $a, b > 1$. *Then* $\lfloor c/a \rfloor = \lfloor d/b \rfloor$.

Proof. From $ad - bc = \pm 1$ it follows that $\frac{d}{b} - \frac{c}{a} = \pm \frac{1}{ab}$. From $\frac{c+\frac{1}{b}}{a} = \frac{d}{b}$ and $b > 1$ we get $\left\lfloor \frac{c}{a} \right\rfloor = \left\lfloor \frac{d}{b} \right\rfloor$. From $\frac{d+\frac{1}{a}}{b} = \frac{c}{a}$ and $a > 1$ we also get $\left\lfloor \frac{c}{a} \right\rfloor = \left\lfloor \frac{d}{b} \right\rfloor$. □

The statement of Lemma 4.4 is not valid if $a = 1$ and $ad - bc = -1$ and if $b = 1,\ ad - bc = 1$.

Lemma 4.5. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a Sturmian matrix with* $ab \neq 0$ *and let* $k = \min(\lfloor c/a \rfloor, \lfloor d/b \rfloor)$. *Put*

$$M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \text{ and } N_n = \begin{pmatrix} a_n & b_n \\ c_n - ka_n & d_n - kb_n \end{pmatrix}$$

for $n > 0$. *Then* N_n *is the reduced matrix of* M^n.

Proof. The case where $n = 1$ has been proven in Corollary 4.2. Assume N_n is an upper matrix for $n > 1$. Then it follows from Corollary 4.1 that $c_n \leq (k+1)a_n,\ d_n \leq (k+1)b_n$ with at least one of the inequalities strict. From this we derive $ac_n + cd_n \leq (k+1)(aa_n + cb_n)$ and $bc_n + dd_n \leq (k+1)(ba_n + db_n)$. Since $M^{n+1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa_n + cb_n & ba_n + db_n \\ ac_n + cd_n & bc_n + dd_n \end{pmatrix}$ it follows that N_{n+1} is an upper matrix. □

5. Unimodular substitutions

We call a substitution σ over 2 letters *unimodular* if its incidence matrix is Sturmian. We assume that all substitutions that we consider are unimodular, primitive and have a fixed point starting with 0. Note that for a unimodular substitution $\gcd(a,b) = \gcd(a+d, ad-bc) = 1$, so that it follows from Proposition 3.1 that the construction of w_n is well-defined for all $n > 0$.

Lemma 5.1. *Let $u = 0u_1u_2\ldots$ be the fixed point of a primitive unimodular substitution. Then u is not eventually periodic.*

Proof. Let ψ be the primitive unimodular substitution with fixed point u. Then ψ has a Sturmian incidence matrix M. It follows from Theorem 4.1 that there exists a Sturmian substitution σ with incidence matrix M. Let $u_n = \psi^n(0)$ and $u'_n = \sigma^n(0)$. Because the numbers of 0's and 1's in u_n and u'_n are completely determined by M, we have $|u_n|_0 = |u'_n|_0$ and $|u_n|_1 = |u'_n|_1$ for every $n \in \mathbb{Z}_{\geq 0}$. Therefore $\lim_{n\to\infty} |u_n|_0/|u_n|_1 = \lim_{n\to\infty} |u'_n|_0/|u'_n|_1$ and this limit is irrational since it is the fixed point of a Sturmian substitution ([5] Sec.2.1.1). It follows that u is not eventually periodic. $\square$

For a finite word u we denote by $f_u^a = \frac{|u|_a}{|u|}$ the frequency of the letter a in u.

Lemma 5.2. *Let u be a finite word that is not balanced. Then u has a subword v with $2|v| \leq |u|$ such that $|v|_0 > \lceil |v| f_u^0 \rceil$ or $|v|_1 > \lceil |v| f_u^1 \rceil$.*

Proof. Since u is not balanced, it has subwords v, w of equal length such that $|v|_0 - |w|_0 \geq 2$. Without loss of generality we may assume that v and w are disjoint, so that $|v| = |w| \leq |u|/2$. Assume $|v|_0 \leq \lceil |v| f_u^0 \rceil$. Then $|w|_1 = |w| - |w|_0 \geq |w| + 2 - |v|_0 \geq |w| + 2 - \lceil |v| f_u^0 \rceil = |w| + 2 - \lceil |w|(1 - f_u^1) \rceil \geq 1 + \lceil |w| f_u^1 \rceil$. $\square$

Lemma 5.3. *Let u be a finite word with $\gcd(|u|_0, |u|_1) = 1$. Then u is strongly balanced if and only if for every subword v of u we have $|v|_0 = \lfloor |v| f_u^0 \rfloor$ or $|v|_0 = \lceil |v| f_u^0 \rceil$.*

Proof. Assume u is not strongly balanced. Put $u = u_0 \ldots u_m$. Since u^2 is not balanced, there exists a subword v of u^2 with $|v| \leq |u|$ and $|v|_0 > \lceil |v| f_u^0 \rceil$ or $|v|_1 > \lceil |v| f_u^1 \rceil$, according to the previous lemma. It is obvious that $|v| \neq |u|$. If v is a subword of u we are done. Otherwise v is of the form $u_p \ldots u_m u_0 \ldots u_q$ with $q < p - 1$. Without loss of generality we assume $|v|_1 > \lceil |v| f_u^1 \rceil$, hence $-|v|_0 > \lceil -|v| f_u^0 \rceil$. Define $v' = u_{q+1} \ldots u_{p-1}$. This is a subword of u with $|v| + |v'| = |u|$. We get

$$|v'|_0 = |u|_0 - |v|_0 > \lceil |u|_0 - |v| f_u^0 \rceil = \lceil |v'| f_u^0 \rceil .$$

Assume there is a subword v of $u = u_0 \dots u_m$ with $|v| < |u|$ and $|v|_0 < \lfloor |v| f_u^0 \rfloor$ or $|v|_0 > \lceil |v| f_u^0 \rceil$. We assume the second case, the proof for the fist case is similar. Put $n = |v|$, hence $n < |u| = m + 1$. Consider the $m + 1$ subwords of u^2 of length n that start with the letters $u_0, u_1, \dots, u_m$. Each letter of u occurs in exactly n of these subwords, hence on average each of these subwords contains nf_u^0 zeros. Because $\gcd(|u|_0, |u|) = 1$ we know $nf_u^0 \notin \mathbb{Z}$. Assume all these subwords contain $\lceil nf_u^0 \rceil$ or more zeros. Then $|u|_0 \geq (m+1) \lceil nf_u^0 \rceil / n > (m+1) f_u^0 = |u|_0$. This contradiction implies that there exists a subword v' of u^2 of length n with fewer than $\lceil nf_u^0 \rceil$ zeros, and it follows that u^2 is not balanced. $\square$

Proposition 5.1. *Let u be a finite word with* $\gcd(|u|_0, |u|_1) = 1$ *and let w be the corresponding central function. Then w forms a central block if and only if u is strongly balanced.*

Proof. Assume that u is not strongly balanced. Then using the previous lemma we may assume without loss of generality that there exists a subword v of u with $|v|_0 > \lceil |v| f_u^0 \rceil$, since if $|v|_0 < \lceil |v| f_u^0 \rceil$ then $|v|_1 > \lceil |v| f_u^1 \rceil$. Because of the way w is defined, there are two letters in w at positions that are $|u|_1 |v|_0 - |u|_0 |v|_1$ apart. We get

$$\begin{aligned} |u|_1 |v|_0 - |u|_0 |v|_1 &\geq |u|_1 \left(\lceil f_u^0 |v| \rceil + 1 \right) - |u|_0 \left(|v| - \lceil f_u^0 |v| \rceil - 1 \right) \\ &\geq |u| f_u^0 |v| + |u| - |u|_0 |v| = |u|. \end{aligned}$$

Hence w contains a gap.

Assume w contains a gap. Then w has two numbers at positions that are $|w|$ or more apart. Call the smallest of these two numbers a and the largest b. Let v be the subword of u that starts at the $(a + 1)$-th letter and ends with the b-th letter. There are two possibilities, $|u|_0 |v|_1 - |u|_1 |v|_0 \geq |u|$ or $|u|_1 |v|_0 - |u|_0 |v|_1 \geq |u|$, depending on whether the position of a in w is left or right of the position of b. We will assume the second inequality holds; the proof for the first case is similar. We get, successively,

$$\begin{aligned} |u|_1 |v|_0 &\geq |u| + |u|_0 (|v| - |v|_0), \\ |v|_0 &\geq 1 + f_u^0 |v| > \lceil f_u^0 |v| \rceil . \end{aligned}$$

It follows from the previous lemma that u_n is not strongly balanced. $\square$

Remark. If we replace the condition in Proposition 5.1 that u is strongly balanced by the condition that u is balanced the statement does not hold. Take for example $u = 0010100$ so that $\gcd(|u|_0, |u|_1) = 1$, u is balanced (but u^2 is not balanced). Then $w = 5 \sqcup 6\ 3\ \underline{0}\ 4\ 1 \sqcup 2$ contains gaps.

Theorem 5.1. *Let σ be a primitive unimodular substitution with fixed point starting with 0. Then σ is Sturmian if and only if w_n forms a central block for every $n > 0$.*

Proof. Because σ^n is a Sturmian substitution it follows from Corollary 4.1 that $u_n^2 = \sigma^n(00)$ is balanced. Applying Proposition 5.1 gives that w_n has no gap for any $n > 0$.

Assume w_n has no gap for any $n > 0$. Proposition 5.1 implies that u_n is balanced for every n, and therefore the fixed point of σ is balanced. Because it is not eventually periodic according to Lemma 5.1, it is a Sturmian word. Hence σ is a Sturmian substitution. $\square$

Example 2 (continued). For $n > 0$ we define v_n as follows. Let g_n be the number at position -1 of w_n. Then v_n is obtained by replacing every number in w_n that is smaller than g_n by 0, and every other number by 1. We get the following table of central words over a two-letter alphabet.

n					v_n														
0					$\underline{0}$														
1				1	$\underline{0}$	1	0	1											
2	1	0	1	1	$\underline{0}$	1	0	1	1	0	1	1	0	1	0	1	1	0	1

......

This example illustrates the following definition. Let σ be a primitive unimodular substitution with fixed point starting with 0. For $n > 0$ put $g_n = -\det^n(M_\sigma)|u_n|_1^{-1} \pmod{|u_n|}$, where the inverse is taken with respect to $|u_n|$. Let $w'_n = w_n$ for each n if $\det(M_\sigma) = 1$, otherwise let $w'_n = w_n$ and reflect each w'_n in the origin for n odd. Note that it follows from Lemma 2.3 that g_n is the number at position -1 of w'_n if there is one.

Definition. Let σ be a primitive Sturmian substitution with fixed point starting with 0 and g_n, w'_n as defined above. Then for $n > 0$ we get the *central word* v_n by replacing every number in w'_n that is smaller than g_n by 0, and every other number by 1.

We already noted in Section 2 that w_n has in each coset modulo $|w_n|$ exactly one position on which it is defined. This means we can extend w'_n to a bi-infinite central block $\widehat{w'_n}$ with period $|w_n|$. It follows from the definition of v_n that $\widehat{w'_n}(i) < \widehat{w'_n}(i+1) \iff \widehat{w'_n}(i) < \widehat{w'_n}(-1) \iff v_n(i) = 0$. The next corollary follows directly from Proposition 3.2.

Corollary 5.1. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and assume that v_p has a letter at position -1. Then for $n \geq p$ if v_n has a letter at position k then v_{n+1} has the same letter at position k.*

We see that in this case $v := \lim_{n\to\infty} v_n$ is an infinite central word. The only primitive Sturmian substitutions with fixed point starting with 0 for which there does not exist an integer p such that v_p has a letter at position -1 are the Christoffel substitutions defined in Section 6.

Example 4. Let the primitive Sturmian substitution ψ be given by $\psi(0) = 01$, $\psi(1) = 011$. Then the statement of Corollary 5.1 does not hold since $v_1(1) = 1$ and $v_2(1) = 0$, as we see in the table below. Note that there is no letter at position -1.

n	u_n	$w_n = w'_n$	v_n
0	0		
1	0 1	$\underline{0}$ 1	$\underline{0}$ 1
2	0 1 0 1 1	$\underline{0}$ 2 4 1 3	$\underline{0}$ 0 1 0 1

Theorem 5.2. *Let σ be a primitive Sturmian substitution with fixed point starting with 0. If v_n has a letter at position i then it equals* 0 *if $\{-ig_n/|v_n|\} \in [0, g_n/|v_n|)$ and* 1 *otherwise. Moreover, v_n is strongly balanced for every $n > 0$.*

Proof. The first statement follows directly from the definition of v_n and from Lemma 2.3 which says $w_n(i) = i|u_n|_1^{-1} \pmod{|u_n|}$. It follows from [5] Sec.2.1.2. that words defined in this way are strongly balanced (so-called rotation words). ☐

Remark. The central words v_n need not be balanced if we would not require that g_n is the number at position 1 or -1 in w_n. We illustrate this using the substitution ϕ from Example 2 and choosing in w_2 the number at position -2 as g_2. Then

$$v_2 = 0010\underline{0}00100100001001.$$

Clearly v_2 is not balanced, since it contains subwords of length 4 that contain two 0's and subwords of length 4 that contain four 0's.

Theorem 5.3. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and M_σ as its incidence matrix. Then $|v_n|_0$ and $|v_n|_1$ are given by the top left and top right entry, respectively, of the dual matrix of the reduced matrix of M_σ^n for $n > 0$.*

Proof. We know that $M_\sigma^n = \begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c_n & d_n \end{pmatrix}$ with $c_n, d_n \in \mathbb{Z}_{\geq 0}$ for $n > 0$. We call its reduced matrix $\begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c'_n & d'_n \end{pmatrix}$. It follows from the definition of w_n that $w'_n(-1) = x_n + y_n$ where (x_n, y_n) is the unique solution of the equation $|u_n|_0 x - |u_n|_1 y = \det(M_\sigma^n)$ with $0 \leq x_n \leq |u_n|_1$ and $0 \leq y_n \leq |u_n|_0$. Therefore we must have that $(c'_n, d'_n) = (y_n, x_n)$. From the definition of v_n we get immediately that $|v_n|_0 = g_n = w'_n(-1) = x_n + y_n$. This means that the dual matrix of the reduced matrix of M_σ^n has $|v_n|_0$ as top left entry, and because the sum of the elements of the top row is the same for the dual matrix, it has $|v_n|_1$ as top right entry. ☐

Corollary 5.2. *Let σ be a primitive Sturmian substitution with fixed point starting with 0, M_σ its incidence matrix and let v_n be the central word corresponding to u_n for some $n > 0$. Shift v_n a number of positions to the right so that the left most letter is at position 0. In the same way as before we can project the cutting segment constructed from the shifted word v_n to form the central block t_n. By looking at the incidence matrix of M_σ we get t'_n from t_n in the same way as we construct w'_n from w_n. Finally we construct a new central word z_n from t'_n. Then $|z_n|_0 = |u_n|_0$ and $|z_n|_1 = |u_n|_1$.*

Proof. Let N_n be the dual matrix of the reduced matrix of M_σ^n. We know from the previous theorem that N_n is of the form $\begin{pmatrix} |v_n|_0 & |v_n|_1 \\ c_n & d_n \end{pmatrix}$ for some $c_n, d_n \in \mathbb{Z}_{\geq 0}$ and it is an upper matrix. This means that the solution (x_n, y_n) of the equation $|v_n|_0 x - |v_n|_1 y = \det(M_\sigma^n)$ is given by the bottom entries of N_n. Therefore $|z_n|_0$ and $|z_n|_1$ are given by the top left and top right entry of the dual matrix of N_n, respectively, which is the reduced matrix of M_σ^n, since the dual matrix of the dual matrix is the original matrix. Of course the reduced matrix of M_σ^n has the same top row as M_σ^n itself. $\square$

Let σ be a primitive Sturmian substitution with fixed point starting with 0 and $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its incidence matrix. For $n > 0$ we define (x_n, y_n) as the unique solution of the equation $|u_n|_0 x - |u_n|_1 y = \det(M_\sigma^n)$ satisfying $0 \leq x_n \leq |u_n|_1$ and $0 \leq y_n \leq |u_n|_0$. Since $M_\sigma^n = \begin{pmatrix} |u_n|_0 & |u_n|_1 \\ c_n & d_n \end{pmatrix}$, it follows that $|u_n|_0 d_n - |u_n|_1 c_n = \det(M_\sigma^n)$. Hence $x_n = d_n - k|u_n|_1$, $y_n = c_n - k|u_n|_0$ for some suitable $k \in \mathbb{Z}_{\geq 0}$. From Lemma 4.5 we know that the value of k is independent of n. Since according to Lemma 3.2 both $(c_n), (d_n)$ and $(|u_n|_0), (|u_n|_1)$ satisfy the recurrence relation $p_n = (a+d)p_{n-1} - (ad - bc)p_{n-2}$ it follows that (x_n) and (y_n) satisfy the same recurrence relation. By going backwards we define x_0 and y_0. Since $|u_0|_0 x_0 - |u_0|_1 y_0 = 1$ and $|u_0|_0 = 1, |u_0|_1 = 0$, we see that $x_0 = 1$, but y_0 is not determined by this equation.

We mentioned in the proof of Theorem 5.3 that $|v_n|_0 = x_n + y_n$. Hence we see that $(|v_n|_0)$ and $(|v_n|_1)$ satisfy the same recurrence relation. By going backwards we can now define $|v_0|_0$ and $|v_0|_1$, and we see that also $|v_0|_0 = x_0 + y_0$. Note that v_0 has no meaning in terms of projecting a cutting sequence, and is only formally defined. We will see that $|v_0|_0$ and $|v_0|_1$ can take negative values, but their sum is $|u_0| = 1$. Theorem 5.3 can be used to find $|v_n|_0$ and $|v_n|_1$ for $n > 0$. In order to calculate $|v_0|_0$ and $|v_0|_1$ we prove the following lemma.

Lemma 5.4. *Let σ be a primitive Sturmian substitution with fixed point starting with 0 and $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its incidence matrix. Let k be the non-negative integer such that $\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$ is an upper matrix. Then $|v_0|_1 = k$ and $|v_0|_0 = 1 - |v_0|_1 = 1-k$.*

Proof. The reduced matrix of M_σ equals $\begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$ where $c' = c - ka, d' = d - kb$ and $ad' - bc' = \det M_\sigma$. Recall $\det M_\sigma = \pm 1$. In the table below we know $|u_0|_0 = 1, |u_0|_1 = 0, |u_1|_0 = a, |u_1|_1 = b$. Using the recurrence relation on $(|u_n|_0), (|u_n|_1)$ we compute $|u_2|_0 = a(a+d) \mp 1, |u_2|_1 = b(a+d)$. We know that $x_0 = 1$, $x_1 = d'$ and $y_1 = c'$. Next we get $x_2 = (a+d)d' \mp 1$ from the recurrence relation on (x_n). Recall that $a + d \neq 0$ and $b \neq 0$. We use the equation $|u_2|_0 x_2 - |u_2|_1 y_2 = 1$ to find y_2 as follows.

$$b(a+d)y_2 = a(a+d)^2 d' \mp a(a+d) \mp (a+d)d';$$

$$by_2 = a(a+d)d' \mp (a+d') = (bc' \pm 1)(a+d) \mp (a+d') = bc'(a+d) \pm kb;$$

$$y_2 = c'(a+d) \pm k.$$

Finally we find $y_0 = -k$ from the recurrence relation on (y_n) and then $|v_0|_0 = x_0 + y_0 = 1-k, |v_0|_1 = 1 - |v_0|_0 = k$. See the table below.

n	$\vert u_n\vert_0$	$\vert u_n\vert_1$	x_n	y_n	$\vert v_n\vert_0$	$\vert v_n\vert_1$
0	1	0	1	$-k$	$1-k$	k
1	a	b	d'	c'		
2	$a(a+d) \mp 1$	$b(a+d)$	$(a+d)d' \mp 1$	$c'(a+d) \pm k$		

□

Consider a primitive Sturmian substitution σ with fixed point starting with 0, with incidence matrix M_σ and u_n, v_n defined as before. As mentioned before we have for $(|v_n|_0)$ and $(|v_n|_1)$ the same recurrence relation as for $(|u_n|)$. The following theorem shows that there is a Sturmian matrix that we call M_τ such that $(|v_n|_0, |v_n|_1) \cdot M_\tau = (|v_{n+1}|_0, |v_{n+1}|_1)$ for every $n \in \mathbb{Z}_{\geq 0}$. Thus M_τ has the same trace and determinant as M_σ.

Theorem 5.4. *Let σ be a primitive Sturmian substitution with fixed point starting with 0, with incidence matrix $M_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $(v_n)_{n=0}^{\infty}$ be the corresponding central words. Let k be the non-negative integer such that $\begin{pmatrix} a & b \\ c-ka & d-kb \end{pmatrix}$ is an upper matrix. Then $(|v_n|_0, |v_n|_1) \cdot M_\tau =$*

$(|v_{n+1}|_0, |v_{n+1}|_1)$ *for every* $n \in \mathbb{Z}_{\geq 0}$ *where* $M_\tau = M'_\sigma + k \begin{pmatrix} a'-c' & b'-d' \\ a'-c' & b'-d' \end{pmatrix}$ *and* $M'_\sigma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *is the dual matrix of the reduced matrix of* M_σ.

Proof. Let σ be a Sturmian substitution. We write its incidence matrix as $M_\sigma = \begin{pmatrix} a & b \\ c+ak & d+bk \end{pmatrix}$ so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the reduced matrix of M_σ. Then $M_\tau = \begin{pmatrix} c+d(k+1) & a+b-(c+d)+k(b-d) \\ c+kd & a-c+k(b-d) \end{pmatrix}$. Using the previous lemma we see that $|v_0|_1$ is k and we can make the following table.

n	$\lvert u_n\rvert_0$	$\lvert u_n\rvert_1$	x_n	y_n	$\lvert v_n\rvert_0$	$\lvert v_n\rvert_1$
0	1	0	1	$-k$	$1-k$	k
1	a	b	d	c	$c+d$	$a+b-(c+d)$

We see that

$$\begin{aligned}(|v_0|_0, |v_0|_1) \cdot M_\tau &= (1-k, k) \cdot \begin{pmatrix} c+d(k+1) & a+b-(c+d)+k(b-d) \\ c+kd & a-c+k(b-d) \end{pmatrix} \\ &= (c+d, a+b-(c+d)) = (|v_1|_0, |v_1|_1).\end{aligned}$$

Straightforward calculations show that $\mathrm{trace}(M_\tau) = a+d+bk = \mathrm{trace}(M_\sigma)$ and $\det(M_\tau) = ad-bc = \det(M_\sigma)$. Since $(|v_n|_0)$ and $(|v_n|_1)$ satisfy the same recurrence relation $p_n = \mathrm{trace}(M_\tau)p_{n-1} - \det(M_\tau)p_{n-2}$ as $(|u_n|_0)$, $(|u_n|_1)$ and since $\mathrm{trace}(M)M - \det(M)Id = M^2$ for any matrix M, it follows that $(|v_n|_0, |v_n|_1) \cdot M_\tau = (|v_{n+1}|_0, |v_{n+1}|_1)$ for every $n \in \mathbb{Z}_{\geq 0}$. □

6. Christoffel substitutions

In this section we consider a special class of unimodular substitutions that we will call Christoffel substitutions.

Let u be a finite word. We call u a *Lyndon word* if $u = vw$ implies $u < wv$ in the lexicographical order. Let $0 < p < q$ and $\gcd(p, q) = 1$. The finite word $u = u_0u_1 \ldots u_{q-1}$ is called a *lower Christoffel word* or just *Christoffel word* if

$$u_i = \left\lfloor (i+1)\frac{p}{q} \right\rfloor - \left\lfloor i\frac{p}{q} \right\rfloor,$$

cf. [5] Sect.2.1.2. Note that p equals the number of 1's in u. It follows from the definition that for each $a, b \in \mathbb{Z}_{>0}$ with $\gcd(a, b) = 1$ there exists a unique Christoffel word containing a zeros and b ones. We also see that every Christoffel word starts with 0 and ends with 1.

Lemma 6.1. *A finite non-constant word is a Christoffel word if and only if it is a balanced Lyndon word.*

Proof. See [1] Sec.4. □

This means Christoffel words are the finite balanced words in which the 0's are placed "as far as possible" to the left.

Definition. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Sturmian matrix with determinant 1 and $abcd \neq 0$. We call the unique substitution σ that has M as incidence matrix and for which $\sigma(0)$ and $\sigma(1)$ are Christoffel words the *Christoffel substitution* corresponding to M.

Lemma 6.2. *Let σ be a substitution with $\sigma(0), \sigma(1)$ Lyndon words, and $\sigma(0) < \sigma(1)$ in the lexicographical order, and let u be a Lyndon word. Then $\sigma(u)$ is a Lyndon word.*

Proof. See [11]. □

Let M be a matrix with determinant 1. M can be written in a unique way as the product of some occurrences of the matrices $X_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $X_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We associate to each occurrence of X_0 the substitution $\psi_0 : \begin{cases} 0 & \to & 0 \\ 1 & \to & 01 \end{cases}$ and to each occurrence of X_1 the substitution $\psi_1 : \begin{cases} 0 & \to & 01 \\ 1 & \to & 1 \end{cases}$. Let σ be the inverse product of the associated substitutions. It is clear that ψ_0 and ψ_1 are Sturmian substitutions, therefore σ is a Sturmian substitution. Hence $\sigma(0)$ and $\sigma(1)$ are balanced. Also it follows from Lemma 6.2 that they are Lyndon words. Now we conclude from Lemma 6.1 that σ is a Christoffel substitution.

Since for each matrix with determinant 1 and positive integer entries there exists a unique Christoffel substitution, we get the following result.

Lemma 6.3. *Christoffel substitutions are Sturmian substitutions.*

Lemma 6.4. *Let M_σ be a Sturmian matrix with positive entries and determinant equal to 1 and let σ be the Christoffel substitution belonging to M_σ. If M_σ is an upper matrix there exist possibly empty words u and v such that $\sigma(0) = u0v$ and $\sigma(1) = u1$. If M_σ is a lower matrix there exist possibly empty words u and v such that $\sigma(0) = u0v$ and $\sigma(1) = (u0v)^k u1$ for some positive integer k.*

Proof. If M_σ can be written as the product of two matrices X_0, X_1, we have either $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\sigma : \begin{cases} 0 & \to & 001 \\ 1 & \to & 01 \end{cases}$ or $M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\sigma : \begin{cases} 0 & \to & 01 \\ 1 & \to & 011 \end{cases}$ and we see that the statement holds. Assume the statement of the lemma is true for incidence matrices that can be written as the product of n matrices X_0, X_1. Let M_τ be a Sturmian matrix with positive entries and determinant equal to 1 that can be written as the product of $n+1$ matrices X_0, X_1, and let τ be the Christoffel substitution belonging to M_τ. Then $\tau = \sigma'\psi_0$ or $\tau = \sigma'\psi_1$, where σ' is of the form stated in the lemma. It is easy to check that τ is also of this form. $\square$

Corollary 6.1. *Let σ be a Christoffel substitution. Then $\sigma^n(0)$ is a Christoffel word for every $n > 0$.*

Proof. Because according to Lemma 6.3 a Christoffel substitution is a Sturmian substitution, $\sigma^n(0)$ is balanced. It follows from Lemma 6.4 that $\sigma(0) < \sigma(1)$ in the lexicographical order, and applying Lemma 6.2 gives that $\sigma^n(0)$ is a Lyndon word. According to Lemma 6.1, $\sigma^n(0)$ is a Christoffel word. $\square$

Proposition 6.1. *Let u be a Christoffel word, let $w' = w$ be the central block that we get after projecting the cutting segment of u, and let v be the corresponding central word, as defined in Section 5. Then v is a Christoffel word.*

Proof. According to [1] Sec.4 the cutting segment of a Christoffel word lies below the line connecting the origin and the endpoint of the cutting segment. It follows that when the integer points on the cutting segment are projected parallel to this line, they all lie below the origin, except the origin which is projected to itself. Since Christoffel words are Sturmian words according to Lemma 6.3, there is no gap by Theorem 5.1. Hence the projected points form the word $w = w(0)\dots w(|u|-1)$ with $w(i) = i|u|_1^{-1} \pmod{|u|}$. Put $g = |u|_1^{-1} \pmod{|u|}$. Then

$$v(i) = 0 \iff w(i) < w(i+1) \iff ig \pmod{|u|} < (i+1)g \pmod{|u|}$$
$$\iff \left\lfloor i\frac{g}{|u|}\right\rfloor = \left\lfloor (i+1)\frac{g}{|u|}\right\rfloor$$

and we see that v is a Christoffel word containing $|u|_1^{-1} \pmod{|u|}$ ones. $\square$

Theorem 6.1. *Let σ be a Christoffel substitution. Let (v_n) be the words that we get by projecting the cutting segments of $u_n = \sigma^n(0)$ for $n > 0$. Then there exists a Christoffel substitution τ such that $\tau(v_n) = v_{n+1}$ for $n > 0$ and τ has M_τ as defined in Theorem 5.4 as incidence matrix.*

Proof. It follows from the previous proposition that the words (v_n) are Christoffel words. According to Theorem 5.4 the relation between the number of 0's and 1's in v_n and v_{n+1} is given by M_τ. Define τ as the Christoffel substitution that has M_τ as incidence matrix. Because τ is Sturmian, $\tau(v_n)$ is a balanced word. Because v_n is a Lyndon word, and τ is a Christoffel substitution with $\tau(0) < \tau(1)$ in the lexicographical order, $\tau(v_n)$ is also a

Lyndon word. Thus $\tau(v_n)$ is a Christoffel word. Because there is only one Christoffel word of length $|v_{n+1}|$ that contains $|v_{n+1}|_0$ zeros, it follows that $\tau(v_n) = v_{n+1}$. $\square$

Definition. Let σ be a Christoffel substitution with incidence matrix M_σ that is an upper matrix. Then we call τ as defined in Theorem 6.1 the *dual substitution* of σ.

Corollary 6.2. *Let τ be the dual substitution of σ. Then σ is the dual substitution of τ.*

Proof. Let σ be a Christoffel substitution and let its incidence matrix be an upper matrix. It follows from Lemma 5.4 that $|v_0|_0 = 1$ and $|v_0|_1 = 0$. Hence according to Theorem 6.1 we have $v_n = \tau^n(0)$. It follows from Theorem 5.4 that M_τ is an upper matrix. Define z_n as the central word that corresponds to v_n, as in Corollary 5.2 (note that the left most letter of v_n is already at position 0, so that v_n doesn't have to be shifted to the right). Then it follows from this corollary and the fact that τ is a Christoffel substitution that $z_n = u_n$ for $n \in \mathbb{Z}_{\geq 0}$. $\square$

Example 5. The duality relation exists between the following pairs σ, τ of Christoffel substitutions.

- $\sigma : \begin{cases} 0 & \rightarrow & 00101 \\ 1 & \rightarrow & 01 \end{cases}$ and $\tau : \begin{cases} 0 & \rightarrow & 01011 \\ 1 & \rightarrow & 011 \end{cases}$
- $\sigma : \begin{cases} 0 & \rightarrow & 0010101 \\ 1 & \rightarrow & 01 \end{cases}$ and $\tau : \begin{cases} 0 & \rightarrow & 0110111 \\ 1 & \rightarrow & 0111 \end{cases}$
- $\sigma : \begin{cases} 0 & \rightarrow & 0101011 \\ 1 & \rightarrow & 01011 \end{cases}$ and $\tau : \begin{cases} 0 & \rightarrow & 0001001 \\ 1 & \rightarrow & 001 \end{cases}$

The following Christoffel substitutions are self-dual (i.e. $\sigma = \tau$).

- $\sigma : \begin{cases} 0 & \rightarrow & 001 \\ 1 & \rightarrow & 01 \end{cases}$
- $\sigma : \begin{cases} 0 & \rightarrow & 0001 \\ 1 & \rightarrow & 001 \end{cases}$

The next lemma and theorem are thanks to a suggestion by Julien Cassaigne.

Lemma 6.5. *Let $z = z_0 z_1 \dots z_n$ be a finite word with $z_i \in \{0, 1\}$. Put $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = X_{z_0} X_{z_1} \dots X_{z_n}$. Then $X_{z_n} X_{z_{n-1}} \dots X_{z_0} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.*

Proof. It is easy to check that this holds for $n = 0, 1$. Assume it holds for n. Let $z = z_0 \dots z_{n+1}$ be given. Then $X_{z_0} \dots X_{z_{n+1}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X_{z_{n+1}}$. In case $z_{n+1} = 0$ we get $\begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}$ and in case $z_{n+1} = 1$ we get

$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$. In the same way we find $X_{z_{n+1}} \dots X_{z_0}$ equals $\begin{pmatrix} d & b \\ c+d & a+b \end{pmatrix}$ in case $z_{n+1} = 0$, and $\begin{pmatrix} c+d & a+b \\ c & a \end{pmatrix}$ in case $z_{n+1} = 1$. We see that the lemma also holds for $n+1$. $\square$

Let σ be a Christoffel substitution with incidence matrix M_σ that is an upper matrix. Hence there exists a finite word $z = z_0 z_1 \dots z_n$ so that $\sigma = \psi_{z_0} \dots \psi_{z_n} \psi_1$. If we write $M_\sigma = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$, we see that $X_{z_n} \dots X_{z_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence using the previous lemma we get $X_{z_0} \dots X_{z_n} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. We define the Christoffel substitution $\phi = \psi_{z_n} \dots \psi_{z_0} \psi_1$ and see that it has incidence matrix $M_\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} c+d & a+b \\ c & a \end{pmatrix}$. We find that M_ϕ is the dual matrix of M_σ and hence ϕ is the dual substitution of σ. This gives the following theorem.

Theorem 6.2. *Let σ be a Christoffel substitution with as incidence an upper matrix, so that we can write $\sigma = \psi_{z_0} \dots \psi_{z_n} \psi_1$, with $z_i \in \{0,1\}$. Then the dual substitution of σ equals $\psi_{z_n} \dots \psi_{z_0} \psi_1$.*

Example 5 (continued). If we write the substitutions σ and τ from Example 5 as products of ψ_0 and ψ_1 we get the following pairs of substitutions.

- $\sigma = \psi_0\psi_1\psi_1$ and $\tau = \psi_1\psi_0\psi_1$
- $\sigma = \psi_0\psi_1\psi_1\psi_1$ and $\tau = \psi_1\psi_1\psi_0\psi_1$
- $\sigma = \psi_1\psi_0\psi_0\psi_1$ and $\tau = \psi_0\psi_0\psi_1\psi_1$
- $\sigma = \tau = \psi_0\psi_1$
- $\sigma = \tau = \psi_0\psi_0\psi_1$

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