Journal de Théorie des Nombres
de Bordeaux **19** (2007), 547–560

Diophantine inequalities with power sums

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Résumé. On appelle somme de puissances toute suite $\alpha : \mathbb{N} \to \mathbb{C}$ de nombres complexes de la forme

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n,$$

où les $b_i \in \overline{\mathbb{Q}}$ et les $c_i \in \mathbb{Z}$ sont fixés. Soit $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ un polynôme unitaire, absolument irréductible, de degré au moins 2 en y. On démontre que les solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ de l'inégalité

$$|F(\alpha(n), y)| < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon}$$

sont paramétrées par un nombre fini de sommes de puissances. Par conséquent, on déduit la finitude des solutions de l'équation diophantienne

$$F(\alpha(n), y) = f(n),$$

où $f \in \mathbb{Z}[x]$ est un polynôme non constant et α est une somme de puissances non constante.

Abstract. The ring of power sums is formed by complex functions on $\mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n,$$

for some $b_i \in \overline{\mathbb{Q}}$ and $c_i \in \mathbb{Z}$. Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be absolutely irreducible, monic and of degree at least 2 in y. We consider Diophantine inequalities of the form

$$|F(\alpha(n), y)| < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon}$$

and show that all the solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ have y parametrized by some power sums in a finite set. As a consequence, we prove that the equation

$$F(\alpha(n), y) = f(n),$$

with $f \in \mathbb{Z}[x]$ not constant, F monic in y and α not constant, has only finitely many solutions.

Manuscrit reçu le 18 mars 2003.

The author was supported by Istituto Nazionale di Alta Matematica "Francesco Severi", grant for abroad Ph.D.

1. Introduction

The present paper deals with diophantine equations and inequalities involving certain *power sums*, i.e. functions of $n \in \mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n\ , \tag{1}$$

with $c_1 > c_2 > \ldots > c_h > 0$, where the b_i, called the *coefficients* of $\alpha(n)$, are (nonzero) algebraic numbers and the c_i, called the *roots* of $\alpha(n)$, are distinct integers or rationals. A power sum is *non-degenerate* if no quotient of two distinct roots is a root of unity. It is well known that such functions, even allowing the b_i to be polynomials in n and the c_i to be algebraic numbers, satisfy linear recurrence relations. Since long ago, a number of results concerning diophantine equations and inequalities with power sums have been proved. Among the recent ones, we may mention, for instance, the results by Kiss [9] who proved, under some assumptions on the absolute values of the roots of $\alpha(n)$, that the inequality $|sx^q - \alpha(n)| > e^{cn}$, where $\alpha(n)$ is a non-degenerate power sum with algebraic roots and polynomial coefficients, holds for integers s, $x > 1$ and q, provided that n and q are large enough. Shorey and Stewart [14] proved that for any fixed $\delta > 0$ the inequality $|sx^q - \alpha(n)| > |c_1|^{n(1-\delta)}$, where $\alpha(n)$ is non-degenerate with algebraic roots and constant coefficients, holds for all the non-zero integers s, x, for $n > 0$, and for every non-zero integer $q > q_o(\alpha, P)$, where P is the greatest prime factor of s, assuming that $sx^q \neq b_1 c_1^n$ and that in $\alpha(n)$ there is a root with largest absolute value. This result was proved using estimates for linear forms in logarithms due to Baker (see [1]). Pethö [10] proved for non-degenerate power sums with $h = 2$ and coprime coefficients that if $\alpha(n) = sx^q$ holds for integers $x \neq 0$, $q \geq 2$ and $n > 0$, then $\max\{|x|, q, n\}$ is bounded by an effectively computable number depending on the greatest prime divisor of s. Recently Corvaja and Zannier [2] have found new results concerning the inequality $|\alpha(n) - y^d| \ll |\alpha(n)|^\rho$, where $\alpha(n)$ has positive integral roots and rational coefficients, $d \geq 2$ and $\rho < 1 - 1/d$. Using the Schmidt Subspace Theorem (see [12]) they proved that if this inequality has infinitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$, then all the solutions, but finitely many, have y parametrized by some power sums in a finite set; also, the numbers n such that (n, y) is a solution, except finitely many, form a finite union of arithmetical progressions. As a consequence, for every $d \geq 2$ the equation $\alpha(n) = y^d$ has only finitely many solutions, if we suppose that $\alpha(n)$ has positive integral roots and that two roots with largest absolute value are coprime, apart from trivial cases, which are easy to classify. In [3], under some assumptions on the size of the roots of $\alpha(n)$ and allowing the coefficients and the roots of $\alpha(n)$ to be algebraic, they extended this result to the more general equation $F(\alpha(n), y) = 0$. This paper will not be concerned with quantitative aspects, though the methods allow to estimate the number of relevant solutions. In the context of the paper by Corvaja and Zannier ([3]), some extimates have been obtained by Fuchs [7], using a quantitative version of the Subspace Theorem due to Evertse (see [6]).

In this paper we first study lower bounds for the quantity $|F(\alpha(n),y)|$, and in particular the inequality $|F(\alpha(n),y)| < \left|\frac{\partial F}{\partial y}(\alpha(n),y)\right| \cdot |\alpha(n)|^{-\varepsilon}$ for power sums with integral roots and algebraic coefficients, where $F(x,y)$ is an absolutely irreducible polynomial monic in y. We shall obtain (Theorem 3.1) that all the solutions $(n,y)\in\mathbb{N}\times\mathbb{Z}$ have y parametrized by some power sums in a finite set. This conclusion is in a sense best possible, since the same result doesn't hold for $\varepsilon<0$. In fact, suppose that $F(\alpha(n),y)$ has a real zero y_n for all sufficiently large n. Setting $y(n)$ to be the nearest integer to y_n, we have (see [4])

$$|F(\alpha(n),y(n))| = |y(n)-y_n|\left|\frac{\partial F}{\partial y}(\alpha(n),\xi)\right| < \left|\frac{\partial F}{\partial y}(\alpha(n),\xi)\right| \ll \left|\frac{\partial F}{\partial y}(\alpha(n),y(n))\right|,$$

where $y(n)\le\xi\le y_n$ (or $y_n\le\xi\le y(n)$).

Our proof shall use a result concerning the inequality $|\alpha(n)-y| < e^{-n\varepsilon}$ derived by Corvaja and Zannier [2, Lemma 2] from Schmidt Subspace Theorem. From Theorem 3.1 follows (Corollary 3.2) the generalization of the result in [2, Theorem 3] to the inequality $|F(\alpha(n),y)| < |\alpha(n)|^{1-\frac{1}{d}-\varepsilon}$, under some assumptions on the Puiseux expansion at infinity of y as function of x under the relation $F(x,y)=0$. As a simple application (Corollary 3.3) we shall deduce the finiteness of the solutions of the equation $F(\alpha(n),y) = f(n)$, under the assumption that $f(n)$ is a non constant polynomial and that $\alpha(n)$ is not constant. This gives a generalization of the results in [2] and [3].

2. Notation

In the present paper we will denote by $\Sigma_{\mathbb{Q}}$ and $\Sigma_{\mathbb{Z}}$ the rings of functions on $\mathbb{N}$ of the form $\alpha(n) = b_1c_1^n + b_2c_2^n + \ldots + b_hc_h^n$, where the distinct roots $c_i\neq 0$ are in $\mathbb{Q}$ or in $\mathbb{Z}$ respectively, and the coefficients $b_i\in\mathbb{Q}^\star$. If $\mathbb{K}\subset\mathbb{C}$ is a number field, we will denote by $\mathbb{K}\Sigma_{\mathbb{Q}}$ and $\mathbb{K}\Sigma_{\mathbb{Z}}$ the ring of power sums with coefficients in $\mathbb{K}$.

The subrings of power sums with only positive roots will be denoted by $\mathbb{K}\Sigma_{\mathbb{Q}}^+$ and $\mathbb{K}\Sigma_{\mathbb{Z}}^+$. Working in this domain causes no loss of generality: the assumption of positivity of the roots may usually be achieved by writing $2n+r$ instead of n, and considering the cases of $r=0,1$ separately.

Note that every constant power sum, i.e. a power sum with only one root $c_1=1$, belongs to $\Sigma_{\mathbb{Z}}^+$. Power sums will be denoted by Greek letters.

3. Statements

Theorem 3.1. *Let $F \in \overline{\mathbb{Q}}[x, y]$ be absolutely irreducible, monic and of degree $d \geq 2$ in y; let $\alpha(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$, and let $\varepsilon > 0$ be fixed. Then there exists a finite set of power sums $\{\beta_1(n), \ldots, \beta_s(n)\} \subset \Sigma_{\mathbb{Z}}^+$ such that every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality*

$$|F(\alpha(n), y)| < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon} \tag{2}$$

satisfies $y = \beta_i(n)$, for a certain $i = 1, \ldots, s$.

The set $\{\beta_1(n), \ldots, \beta_s(n)\}$ contains at most d^2 non constant power sums. Moreover, the set of natural numbers n such that (n, y) is a solution of (2) is the union of a finite set and a finite number of arithmetic progressions.

For the formulation of Corollary 3.2 we need the following.

Definition. Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be monic in y and of degree $d \geq 2$ in y. Let $F(x, y) = (y - \varphi_1(x)) \cdot \ldots \cdot (y - \varphi_d(x))$ be the factorization of $F(x, y)$ in the ring of Puiseux series in x at infinity. Here, for each $j = 1, \ldots, d$, $\varphi_j(x) = \sum_{i=-k_j}^{+\infty} a_{ij}\, x^{-i/e_j}$, with $a_{-k_j j} \neq 0$ and for a real determination of x^{1/e_j}, is an expansion at infinity of y as function of x.

In the present paper we will call the polynomial $F(x, y)$ *"regular"* if for every $j, l = 1, \ldots, d$, with $j \neq l$, we have $k_j/e_j \neq k_l/e_l$ or $a_{-k_j j} \neq a_{-k_l l}$.

Corollary 3.2. *Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be monic in y, absolutely irreducible, regular, of degree $d \geq 2$ in y. Let $\alpha(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$; let $\varepsilon > 0$ and $c > 0$ be fixed. Then there exists a finite set of power sums $\{\beta_1(n), \ldots, \beta_s(n)\} \subset \Sigma_{\mathbb{Z}}^+$ such that every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality*

$$|F(\alpha(n), y)| < c \cdot |\alpha(n)|^{1 - \frac{1}{d} - \varepsilon} \tag{3}$$

satisfies $y = \beta_i(n)$ for a certain $i = 1, \ldots, s$.

The set $\{\beta_1(n), \ldots, \beta_s(n)\}$ contains at most d^2 non constant power sums. Moreover, the natural numbers n such that (n, y) is a solution of (3), except finitely many, make up a finite union of arithmetical progressions.

Corollary 3.3. *Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be monic in y, absolutely irreducible and of degree $d \geq 2$ in y; let $f(n) \in \mathbb{Z}[x]$ be a non constant polynomial; let $\alpha(n)$ be a non constant power sum with integral roots and algebraic coefficients. Then the equation*

$$F(\alpha(n), y) = f(n) \tag{4}$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

4. Auxiliary results

The following Lemma 4.1, proved in a more general version by Corvaja and Zannier (see [2, Lemma 2]) using a version of the Subspace Theorem due to H.P. Schlickewei (see [11], [12, Theorem 1, p. 178]), plays a crucial role throughout the paper, since it contains the fundamental information to prove Theorem 3.1.

Lemma 4.1. *Let* $\tau(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^{+}$, *and let* $\varepsilon > 0$ *be fixed. Then there exists a power sum* $\beta(n) \in \Sigma_{\mathbb{Z}}^{+}$ *such that for all but finitely many solutions* $(n, y) \in \mathbb{N} \times \mathbb{Z}$ *of the inequality*

$$|\tau(n) - y| < e^{-n\varepsilon}, \tag{5}$$

we have $y = \beta(n)$.

Moreover, the roots of $\beta(n)$ *are in the set of the roots of* $\tau(n)$.

For the proof of Theorem 3.1 we need also some standard results from the theory of algebraic functions fields, namely the theory of Puiseux expansions. We recall here a simple version of the Puiseux Theorem concerning the Puiseux expansions at the infinity for the polynomials of $\overline{\mathbb{Q}}(x)[y]$. More general versions can be found in [5] and [8].

Theorem 4.2 (Puiseux Theorem)**.** *Let* $F(x, y) \in \overline{\mathbb{Q}}(x)[y]$ *be an absolutely irreducible polynomial, monic and of degree* d *in* y. *Then for* $i = 1, \ldots, d$ *there exist* $e_i \in \mathbb{N}$, $1 \leq e_i \leq d$, *and Laurent series in* x^{-1/e_i}

$$\varphi_i(x) = \sum_{k=v_i}^{+\infty} a_{ik}\, x^{-k/e_i}, \quad i = 1, \ldots, d$$

with $v_i \leq 0$, *such that*

$$F(x, y) = \prod_{1=1,\ldots,d} (y - \varphi_i(x)).$$

The Laurent series $\varphi_1(x), \ldots, \varphi_d(x)$ *are convergent for* $|x|$ *large enough, and the coefficients* a_{ij} *are elements of a finite field extension* $\mathbb{K}$ *of* $\mathbb{Q}$.

The Laurent series $\varphi_1(x), \ldots, \varphi_d(x)$ coming from the Puiseux Theorem are called *Puiseux series* of the polynomial $F(x, y)$.

5. Proofs

Proof of Theorem 3.1. Plainly, we need to consider only the case that (2) has infinitely many solutions. We shall consider solutions with n larger than a certain constant N, since the finitely many solutions with $n \leq N$ can be considered as constant power sums. Finally, we can suppose $\alpha(n)$ not constant.

Let $F(x,y) = (y-\varphi_1(x)) \cdot \ldots \cdot (y-\varphi_d(x))$, where $\varphi_j(x) = \sum_{i=-k_j}^{+\infty} a_{ij}\, x^{-i/e_j}$, with $a_{-k_j j} \neq 0$ and $1 \leq e_j \leq d$ for $j = 1, \ldots, d$, are the series of the Puiseux expansion at infinity of y as function of x (see Theorem 4.2), i.e. $\varphi_j(x)$ are the solutions of the equation $F(x,y) = 0$ in the field of the Puiseux series.

Let us remark that by the Puiseux Theorem the series $\varphi_j(x)$ exist and the coefficients a_{ij} generate a finite field extension $\mathbb{K}$ of $\mathbb{Q}$.

We have $\frac{\partial F}{\partial y}(x,y) = \sum_{j=1}^{d} \frac{F(x,y)}{y-\varphi_j(x)}$, and so

$$F(x,y) = \frac{\partial F}{\partial y}(x,y)\Big(\sum_{j=1}^{d}\frac{1}{y-\varphi_j(x)}\Big)^{-1} \tag{6}$$

holds.

From (6) we obtain that for each solution (n,y) of (2) the inequality

$$\Big|\frac{\partial F}{\partial y}(\alpha(n),y)\Big| \cdot \Big|\sum_{j=1}^{d}(y-\varphi_j(\alpha(n)))^{-1}\Big|^{-1} < \Big|\frac{\partial F}{\partial y}(\alpha(n),y)\Big| \cdot |\alpha(n)|^{-\varepsilon}$$

holds. By (2), we can assume $\left|\frac{\partial F}{\partial y}(\alpha(n),y)\right| \neq 0$. It follows that

$$\Big|\sum_{j=1}^{d}(y-\varphi_j(\alpha(n)))^{-1}\Big| > |\alpha(n)|^{\varepsilon}$$

holds, and so for all the solutions of (2) we have

$$\sum_{j=1}^{d}\Big|y-\varphi_j(\alpha(n))\Big|^{-1} > |\alpha(n)|^{\varepsilon}.$$

Let $\varepsilon_1 = \dfrac{\varepsilon}{2}$. For n large enough the inequality $\sum_{j=1}^{d}|y-\varphi_j(\alpha(n))|^{-1} > d\cdot|\alpha(n)|^{\varepsilon_1}$ holds, and so for a certain $j = 1, \ldots, d$ we have $|y-\varphi_j(\alpha(n))|^{-1} > |\alpha(n)|^{\varepsilon_1}$. This means that for every solution (n,y) of (2) with n large enough the inequality

$$|y-\varphi_j(\alpha(n))| < |\alpha(n)|^{-\varepsilon_1} \tag{7}$$

is satisfied for a certain $j = 1, \ldots, d$, with j depending on n.

We shall prove that for given $j = 1, \ldots, d$ there exists a finite set $\{\beta_1(n), \ldots, \beta_t(n)\} \subset \Sigma_{\mathbb{Z}}$ such that every solution (n,y) of (7) has $y = \beta_i(n)$ for a certain $i = 1, \ldots, t$.

Once we prove this, the theorem will follow.

Define a partition $\{M_1, \ldots, M_d\}$ of the solutions (n, y) of (2) by prescribing that for every $(n, y) \in M_i$ we have

$$|y - \varphi_i(\alpha(n))| = \min_{1 \le j \le d} \big\{ |y - \varphi_j(\alpha(n))| \big\}.$$

We can consider separately the solutions in each subset M_i. It will suffice to deal with $i = 1$.

Let us write

$$\varphi_1(x) = \sum_{i=-k}^{+\infty} a_i x^{-i/e_1} = a_{-k} x^{k/e_1} + \ldots + a_{-1} x^{1/e_1} + a_0 + a_1 x^{-1/e_1} + \ldots, \tag{8}$$

for a real determination of x^{1/e_1}, where $k = k_1$ and $a_i = a_{i,1}$ for every $i \ge -k$.

Let $\alpha(n) = \sum_{j=1}^{h} b_j\, c_j^n$, with $c_j \in \mathbb{Z}$, $c_j \ne 1$ for some j and $b_j \in \overline{\mathbb{Q}}$ $\forall\, j = 1, \ldots, h$. We can suppose $c_1 > c_2 > \ldots > c_h > 0$.

For n large enough the series $\varphi_1(\alpha(n))$ converges, so we can write

$$\varphi_1(\alpha(n)) = \sum_{i=-k}^{0} a_i \alpha(n)^{-i/e_1} + O(\alpha(n)^{-1/e_1}). \tag{9}$$

Choosing $\varepsilon_2 > 0$ smaller than ε_1 and $1/e_1$, for n large enough each solution of $|y - \sum_{i=-k}^{+\infty} a_i \alpha(n)^{-i/e_1}| < |\alpha(n)|^{-\varepsilon_1}$ satisfies

$$|y - \sum_{i=-k}^{0} a_i \alpha(n)^{-i/e_1}| < |\alpha(n)|^{-\varepsilon_2}.$$

Put

$$\tilde{\varphi}_1(x) = \sum_{i=-k}^{0} a_i x^{-i/e_1}. \tag{10}$$

From now on we will consider the inequality

$$|y - \tilde{\varphi}_1(\alpha(n))| < |\alpha(n)|^{-\varepsilon_2} \tag{11}$$

instead of $|y - \varphi_1(\alpha(n))| < |\alpha(n)|^{-\varepsilon_1}$.

We can write $\alpha(n) = b_1 c_1^n (1 + \sigma(n))$, with $\sigma(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$, and $\sigma(n) = O((c_2/c_1)^n)$.

For every $l \in \mathbb{N}$ we have

$$\alpha(n)^{l/e_1} = b_1^{l/e_1} (c_1^n)^{l/e_1} (1 + \sigma(n))^{l/e_1}, \tag{12}$$

for a real determination (resp. real positive) of b_1^{l/e_1} (resp. c_1^{l/e_1}). We will fix this determination for the remaining part of the proof.

Expanding the function $t \mapsto (1+t)^{l/e_1}$ in Taylor series, we have for every $l \in \mathbb{N}$

$$(1+\sigma(n))^{l/e_1} = 1 + \sum_{j=1}^{m} B_{j,l}\ \sigma(n)^j + O(|\sigma(n)|^{m+1}), \tag{13}$$

where m is an integer to be chosen later and $B_{j,l}$, $j = 1, \dots, m$, $l \in \mathbb{N}$, are the Taylor coefficients $\binom{l/e_i}{j}$ of the function $t \mapsto (1+t)^{l/e_1}$.

From (12) and (13) we obtain

$$\alpha(n)^{l/e_1} = b_1^{l/e_1} c_1^{nl/e_1} \Big(1 + \sum_{j=1}^{m} B_{j,l}\ \sigma(n)^j\Big) + O\big(|\sigma(n)|^{m+1} \cdot c_1^{nl/e_1}\big). \tag{14}$$

Let us define, for every $l \in \mathbb{N}$,

$$\gamma_l(n) := \sum_{j=1}^{m} B_{j,l}\ \sigma(n)^j \ \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^{+}.$$

Since (14) holds, we can write

$$\alpha(n)^{l/e_1} = b_1^{l/e_1} c_1^{nl/e_1}\ \big(1 + \gamma_l(n)\big) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nl/e_1}\big). \tag{15}$$

From (10) and (15) we obtain

$$\tilde{\varphi}_1(\alpha(n)) = \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^n)^{-i/e_1} \big(1+\gamma_{-i}(n)\big)\Big) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big). \tag{16}$$

Let us write $n = n_1 e_1 + r$, with $0 \le r < e_1 \le d$. We can rewrite (16) as (17)

$$\tilde{\varphi}_1(\alpha(n)) = \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^r)^{-i/e_1} c_1^{-n_1 i} \big(1+\gamma_{-i}(n)\big)\Big) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big).$$

Since $\overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^{+}$ is a ring, we see that

$$\tau(n) := \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^r)^{-i/e_1} c_1^{-n_1 i} \big(1+\gamma_{-i}(n)\big)\Big)$$

is a power sum with rational positive roots and algebraic coefficients. Moreover, its roots lie in the multiplicative group generated by the real e_1-th roots (as determined above) of the roots of the power sum $\alpha(n)$.

We can write

$$\tilde{\varphi}_1(\alpha(n)) = \tau(n) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big). \tag{18}$$

So we have

$$|y - \tilde{\varphi}_1(\alpha(n))| = \big|y - \tau(n)\big| + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big)\ ,$$

and from (11) we obtain

$$(19)\qquad |y-\tau(n)| < |\alpha(n)|^{-\varepsilon_2} + O\big((c_2^n/c_1^n)^{m+1}\cdot c_1^{nk/e_1}\big)\ .$$

Let us notice that for a fixed m large enough $(c_2^n/c_1^n)^{m+1}c_1^{nk/e_1} < |\alpha(n)|^{-\varepsilon_2}$ holds for every n large enough. Choosing a suitable m large enough, every solution of (19) with n large enough is also a solution of

$$(20)\qquad |y-\tau(n)| < 2|\alpha(n)|^{-\varepsilon_2}.$$

Choosing $\varepsilon_3 > 0$ small enough, $2|\alpha(n)|^{-\varepsilon_2} < e^{-n\varepsilon_3}$ holds for n large enough, since $|\alpha(n)| \longrightarrow +\infty$ for $n \to +\infty$ (we are supposing $\alpha(n)$ not constant).

Thus the inequality (20) implies

$$(21)\qquad |y-\tau(n)| < e^{-n\varepsilon_3}.$$

Applying Lemma 4.1 we obtain that every solution of (21), with finitely many exceptions, has $y=\beta_1(n)$, where $\beta_1(n) \in \Sigma_{\mathbb{Z}}^+$. The roots of the power sum $\beta_1(n)$ are in the set of the roots of $\tau(n)$, and so in the multiplicative group generated by the real e_1-th roots of the roots of the power sum $\alpha(n)$.

Let us notice that the finitely many solutions (n,y) of (21) such that $y \neq \beta_1(n)$ can be considered as constant power sums $\beta_2(n),\dots,\beta_r(n) \in \Sigma_{\mathbb{Z}}^+$ with a single root 1.

This means that for $j=1$ every solution (n,y) of (7) has $y=\beta_i(n)$ for a certain $i \in \{1,\dots,t\}$, where $\{\beta_1,\dots,\beta_t\} \subset \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$, with $t \geq r$.

In a similar way this result can be obtained for $j=2,\dots,d$ in (7). So we have that every solution of (2) has $y=\beta_i(n)$ for a certain $i \in \{1,\dots,s\}$, where $\{\beta_1,\dots,\beta_s\} \subset \Sigma_{\mathbb{Z}}^+$, with $s \geq t$.

Since each of the Puiseux series $\varphi_j(x)$, $j=1,\dots,d$, gives rise to at most e_j non constant power sums (remember that we chose $0 \leq r < e_j$ in (17) and that $e_j \leq d$ for every $j=1,\dots,d$), the set $\{\beta_1(n),\dots,\beta_s(n)\}$ contains at most d^2 non constant power sums.

Finally, we note that the roots of the power sums $\beta_1(n),\dots,\beta_s(n)$ are positive integers lying in the multiplicative group generated by the real e-th roots, with $1 \leq e \leq d$, of the roots of the power sum $\alpha(n)$.

This proves the Theorem. □

Proof of Corollary 3.2 As in the proof of Theorem 3.1, we shall consider only solutions (n,y) of (3) with n larger than a certain constant N, since the solutions with $n \leq N$ are finite in number and can be considered as constant power sums.

Let $F(x,y) = (y-\varphi_1(x)) \cdot \ldots \cdot (y-\varphi_d(x))$, where

$\varphi_j(x) = \sum_{i=-k_j}^{+\infty} a_{ij}\, x^{-i/e_j}$, with $a_{-k_j j} \neq 0$ and $1 \leq e_j \leq d$ for $j = 1, \ldots, d$, are the series of the Puiseux expansion at infinity of y as function of x.

Let $\varepsilon_1 > 0$ to be chosen later. In the proof of Theorem 3.1 we have shown that there exists a finite set of power sums with positive integral roots and rational coefficients $\{\beta_1(n), \ldots, \beta_t(n)\}$ such that, for every $j = 1, \ldots, d$, every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality

$$|y - \varphi_j(\alpha(n))| < |\alpha(n)|^{-\varepsilon_1}$$

has $y = \beta_i(n)$ for a certain $i = 1, \ldots, t$. Moreover, the set $\{\beta_1(n), \ldots, \beta_t(n)\}$ contains at most d non constant power sums.

Let us consider sets $M_1, \ldots, M_d$ of pairs $(n, y) \in \mathbb{N} \times \mathbb{Z}$ such that for every $(n, y) \in M_i$ we have

$$|y - \varphi_i(\alpha(n))| = \min_{1 \leq j \leq d} \big\{ |y - \varphi_j(\alpha(n))| \big\}.$$

As before, we can consider separately each set, say M_1.

For every $i = 2, \ldots, d$, we have

$$|y - \varphi_i(\alpha(n))| \geq \frac{1}{2}\, |\varphi_i(\alpha(n)) - \varphi_1(\alpha(n))|. \tag{22}$$

Since the polynomial F is regular, we can have either that $k_i/e_i \neq k_1/e_1$, $\forall\, i = 2, \ldots, d$, or that there exist some $i \in \{2, \ldots, d\}$ such that $k_i/e_i = k_1/e_1$, but $a_{-k_i i} \neq a_{-k_1 1}$.

If $k_i/e_i \neq k_1/e_1 \quad \forall\, i = 2, \ldots, d$, for n large enough we have

$$\begin{aligned} |y - \varphi_i(\alpha(n))| \geq \frac{1}{2} |\varphi_1(\alpha(n)) - \varphi_i(\alpha(n))| &= \frac{1}{2} |(\varphi_1 - \varphi_i)(\alpha(n))| \\ &> a \cdot |\alpha(n)^{1/d}|, \end{aligned}$$

for a certain positive constant $a > 0$.

If there exist some $i \in \{2, \ldots, d\}$ such that $k_i/e_i = k_1/e_1$, but $a_{-k_i i} \neq a_{-k_1 1}$, since $k_1 \geq 1$ for these i, for n large enough we have

$$\begin{aligned} |y - \varphi_i(\alpha(n))| &> \frac{1}{2} |\varphi_1(\alpha(n)) - \varphi_i(\alpha(n))| \\ &> f \cdot |a_{-k_1 1} \alpha(n)^{k_1/e_1} - a_{-k_i i} \alpha(n)^{k_1/e_1}| \\ &= f \cdot |a_{-k_1 1} - a_{-k_i i}| \cdot |\alpha(n)^{k_1/e_1}| \\ &> g \cdot |\alpha(n)^{1/e_1}| \\ &\geq g \cdot |\alpha(n)^{1/d}|, \end{aligned}$$

for certain positive constants f and g.

Therefore, for every $i = 2, \ldots, d$, the inequality

$$|y - \varphi_i(\alpha(n))| \geq h \cdot |\alpha(n)^{1/d}|, \tag{23}$$

holds for a certain constant $h = \min\{a, g\}$.

From (23) it follows, with $b = h^{d-1}$, that the inequality

$$\begin{aligned} |F(\alpha(n), y)| &= |y - \varphi_1(\alpha(n))| \cdot |y - \varphi_2(\alpha(n))| \cdot \ldots \cdot |y - \varphi_d(\alpha(n))| \\ &> b \cdot |\alpha(n)^{(d-1)/d - \varepsilon_1}| \\ &= b \cdot |\alpha(n)^{1-\frac{1}{d}-\varepsilon_1}| \\ &= b \cdot |\alpha(n)|^{1-\frac{1}{d}-\varepsilon_1} \end{aligned}$$

holds for all pairs $(n, y) \in \mathbb{N} \times \mathbb{Z}$ with n large enough and $y \neq \beta_i(n)$ for every $i = 1, \ldots, t$.

Choosing $\varepsilon_1 > 0$ small enough we obtain, for n large enough

$$b \cdot |\alpha(n)|^{1-\frac{1}{d}-\varepsilon_1} > c \cdot |\alpha(n)|^{1-\frac{1}{d}-\varepsilon}.$$

Therefore the inequality

$$|F(\alpha(n), y)| > c \cdot |\alpha(n)|^{1-\frac{1}{d}-\varepsilon}$$

holds for all the pairs $(n, y) \in \mathbb{N} \times \mathbb{Z}$ with n large enough and $y \neq \beta_i(n)$ for every $i = 1, \ldots, t$.

This means that each solution of (3) has $y = \beta_i(n)$, for a certain $i = 1, \ldots, s$, with $s \geq t$.

As in the proof of Theorem 3.1, we can obtain that the natural numbers n such that (n, y) is a solution of the inequality, except finitely many, make up a finite union of arithmetical progressions and that the roots of the power sums $\beta_1(n), \ldots, \beta_s(n)$ are positive integers lying in the multiplicative group generated by the real e-th roots, with $1 \leq e \leq d$, of the roots of the power sum $\alpha(n)$.

Since the set $\{\beta_1(n), \ldots, \beta_t(n)\}$ contains at most d non constant power sums, and since we have d choices for the set M_i, the set $\{\beta_1(n), \ldots, \beta_s(n)\}$ contains at most d^2 non constant power sums. □

Remark 5.1. From Corollary 3.2 we can derive that if the inequality (3) has infinitely many solutions, then there exists at least one power sum $\beta(n) \in \Sigma_{\mathbb{Z}}^{+}$ not constant such that $(n, \beta(n))$ is a solution. Since $F(\alpha(n), \beta(n))$ is a power sum, to have infinitely many solutions to (3) the absolute value of the largest root of $F(\alpha(n), \beta(n))$ must be smaller than $|c_1|^{1-\frac{1}{d}-\varepsilon}$, where c_1 is the largest root of α. This means that to have infinitely many solutions the coefficients of the roots of the power sum $F(\alpha(n), \beta(n))$ with absolute value larger than $|c_1|^{1-\frac{1}{d}-\varepsilon}$ must vanish. This condition is easily verifiable in concrete cases with algebrical methods, so it is easy to decide wheather the inequality (3), with a particular power sum $\alpha(n)$, a particular polynomial F and a particular value of ε, has infinitely many solutions or not.

Remark 5.2. If the polynomial F is not regular, we can get a weaker result than that of Corollary 3.2. Using the same notations of Corollary 3.2, let

$$\overline{d} = \max_{i=1,\ldots,d} \left\{ \ |\{\varphi_j : k_j/e_j = k_i/e_i \quad \text{and} \quad a_{k_j j} = a_{k_i i}\}| \ \right\}. \tag{24}$$

If F is not regular we have $2 \leq \overline{d} \leq d$. Without losing generality, let $\varphi_1, \ldots, \varphi_{\overline{d}}$ be the $\overline{d}$ Puiseux series such that $k_1/e_1 = \ldots = k_{\overline{d}}/e_{\overline{d}}$o and $a_{k_1 1} = \ldots = a_{k_{\overline{d}} \overline{d}}$.

As in the proof of Corollary 3.2, we obtain that

$$\begin{aligned} |F(\alpha(n), y)| &= |y - \varphi_1(\alpha(n))| \cdot \ldots \cdot |y - \varphi_{\overline{d}}(\alpha(n))| \\ &\quad \cdot |y - \varphi_{\overline{d}+1}(\alpha(n))| \cdot \ldots \cdot |y - \varphi_d(\alpha(n))| \\ &> c \cdot |\alpha(n)|^{-\varepsilon_1 \overline{d}} \cdot \left(|\alpha(n)|^{1/d}\right)^{d-\overline{d}} \\ &= c \cdot |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon} \end{aligned}$$

holds for all the pairs (n, y) such that $y \neq \beta_i(n)$ for every $i = 1, \ldots, s$, where $\{\beta_1(n), \ldots, \beta_s(n)\}$ is a finite set of power sums with positive integral roots and rational coefficients.

So for every $c > 0$ and for every $\varepsilon > 0$ fixed, every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality

$$|F(\alpha(n), y)| < c \cdot |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon} \tag{25}$$

has $y = \beta_i(n)$, for a certain $i \in \{1, \ldots, s\}$.

Let us notice that if $\overline{d} \neq d$, there exist $\varepsilon > 0$ such that $1 - \frac{\overline{d}}{d} - \varepsilon > 0$.

Remark 5.3 If, under the notations of Corollary 3.2 and Remark 5.2, we have $\overline{d} = d$, with a proper substitution we can reduce the polynomial $F(x, y)$ to the cases considered above. Indeed, writing the series of the Puiseux expansion of $F(x, y)$ as

$$\varphi_j(x) = a_{-k}\, x^{k/e_j} + \ldots + a_{-g}\, x^{g/e_j} + \sum_{i=-g+1}^{+\infty} a_{ij}\, x^{-i/e_j},$$

with $j = 1, \ldots, d$, where a_{-g} is the last common term in every $\varphi_j(x)$, we have

$$F(x, y) = \prod_{j=1}^{d} \left(y - \sum_{i=-k}^{-g} a_i\, x^{-i/e_j} - \sum_{i=-g+1}^{+\infty} a_{ij}\, x^{-i/e_j} \right).$$

Applying the substitution

$$y - \sum_{i=-k}^{-g} a_i\, x^{-i/e_j} \longmapsto z,$$

we obtain a new polynomial $G(x, z)$ that, for the choice of the substitution, can either be regular, and so we can apply Corollary 3.2, or satisfy the hypothesis of Remark 5.2.

Proof of Corollary 3.3. Let $\overline{d}$ be defined as in (24). We can have that either the inequality

$$|F(\alpha(n), y)| < |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon}, \tag{26}$$

with $\varepsilon = \frac{1}{2d}$, has finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ or infinitely many.

If (26) has only finitely many solutions, let us observe that, since $\alpha(n)$ is not constant, for n large enough we have

$$2\ |f(n)|\ <\ |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon},$$

and so also the inequality $|F(\alpha(n), y)| < 2\ |f(n)|$ has finitely many solutions.

The solutions of $F(\alpha(n), y) = f(n)$ are contained in the set of solutions of $|F(\alpha(n), y)| < 2\ |f(n)|$, and so they are only finitely many.

If (26) has infinitely many solutions, from Theorem 3.1 (if $F(x, y)$ is regular), Remark 5.2 (if $\overline{d} < d$) and Remark 5.3 (if $\overline{d} = d$) we know that they all have $y = \beta_i(n)$, for $i = 1, \dots, s$, where $\{\beta_1, \dots, \beta_s\}$ is a set of power sums with rational coefficients and positive integral roots.

For every $i = 1, \dots, s$, $F(\alpha(n), \beta_i(n))$ is a power sum that may be constant.

If for a certain i $F(\alpha(n), \beta_i(n))$ is constant, we have

$$\frac{F(\alpha(n), \beta_i(n))}{f(n)} \xrightarrow{n\to\infty} 0.$$

If for a certain i $F(\alpha(n), \beta_i(n))$ is not constant, we have

$$\left|\frac{F(\alpha(n), \beta_i(n))}{f(n)}\right| \xrightarrow{n\to\infty} +\infty.$$

In both cases $F(\alpha(n), \beta_i(n))$ can not assume the values of $f(n)$ for infinitely many n, and so the equation $F(\alpha(n), y) = f(n)$ has only finitely many solutions. □

Remark 5.4. In Corollary 3.3 the assumption that $|\alpha(n)|$ is not constant is necessary. Consider e.g. the case $\alpha(n) = 1$, $F(x, y) = y^2 + x$, $f(n) = n^2 + 1$, that has as solutions the couples $(n, \pm n)$, $n \in \mathbb{N}$. In all the other statements of the present paper this assumption is not required.

Acknowledgements. The results contained in this paper are part of a Master thesis [13] prepared at Università degli Studi di Udine; the author thanks his supervisors Pietro Corvaja and Umberto Zannier for the important help given.

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