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# Sequences of algebraic integers and density modulo 1

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Résumé. Nous établissons la densité modulo 1 des ensembles de la forme

$$\{\mu^m \lambda^n \xi + r_m : n, m \in \mathbb{N}\},$$

où $\lambda, \mu \in \mathbb{R}$ sont deux entiers algébriques de degré $d \geq 2$, qui sont rationnellement indépendants et satisfont des hypothèses techniques supplémentaires, $\xi \neq 0$, et $r_m$ une suite quelconque de nombres réels.

Abstract. We prove density modulo 1 of the sets of the form

$$\{\mu^m \lambda^n \xi + r_m : n, m \in \mathbb{N}\},$$

where $\lambda, \mu \in \mathbb{R}$ is a pair of rationally independent algebraic integers of degree $d \geq 2$, satisfying some additional assumptions, $\xi \neq 0$, and $r_m$ is any sequence of real numbers.

## 1. Introduction

It is a very well known result in the theory of distribution modulo 1 that for every irrational $\xi$ the sequence $\{n\xi : n \in \mathbb{N}\}$ is dense modulo 1 (and even uniformly distributed modulo 1) [11].

In 1967, in his seminal paper [4], Furstenberg proved the following

**Theorem 1.1** (Furstenberg, [4, Theorem IV.1])**.** *If $p, q > 1$ are rationally independent integers (i.e., they are not both integer powers of the same integer) then for every irrational $\xi$ the set*

$$\{p^n q^m \xi : n, m \in \mathbb{N}\} \tag{1.2}$$

*is dense modulo 1.*

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One possible direction of generalizations is to consider $p$ and $q$ in Theorem 1.1 not necessarily integer. This was done by Berend in [3].

According to [10], Furstenberg conjectured that under the assumptions of Theorem 1.1, the set $\{(p^n + q^m)\xi : n, m \in \mathbb{N}\}$ is dense modulo 1. As far as we know, this conjecture is still open. However, there are some results concerning related questions. For example, B. Kra in [9], proved the following

**Theorem 1.3** (Kra, [9, Theorem 1.2 and Corollary 2.2])**.** *For $i = 1, 2$, let $1 < p_i < q_i$ be two rationally independent integers. Assume that $p_1 \neq p_2$ or $q_1 \neq q_2$. Then, for every $\xi_1, \xi_2 \in \mathbb{R}$ with at least one $\xi_i \notin \mathbb{Q}$, the set*

$$\{p_1^n q_1^m \xi_1 + p_2^n q_2^m \xi_2 : n, m \in \mathbb{N}\}$$

*is dense modulo* 1.

*Furthermore, let $r_m$ be any sequence of real numbers and $\xi \notin \mathbb{Q}$. Then, the set*

$$\{p_1^n q_1^m \xi + r_m : n, m \in \mathbb{N}\} \tag{1.4}$$

*is dense modulo* 1.

Inspired by Berend's result [3], we prove some kind of a generalization of the second part of Theorem 1.3 (some kind of an extension of the first part is given in [15]). Namely, we allow algebraic integers, satisfying some additional assumption, to appear in (1.4) instead of integers, and we prove the following

**Theorem 1.5.** *Let $\lambda, \mu$ be a pair of rationally independent real algebraic integers of degree $d \geq 2$, with absolute values greater than 1. Let $\lambda_2, \ldots, \lambda_d$ denote the conjugates of $\lambda = \lambda_1$. Assume that either $\lambda$ or $\mu$ has the property that for every $n \in \mathbb{N}$, its $n$-th power is of degree $d$, and that $\mu$ may be expressed in the form $g(\lambda)$, where $g$ is a polynomial with integer coefficients, i.e.,*

$$\mu = g(\lambda), \text{ for some } g \in \mathbb{Z}[x]. \tag{1.6}$$

*Assume further that*

$$\text{for each } i = 2, \ldots, d, \text{ either } |\lambda_i| > 1 \text{ or } |g(\lambda_i)| > 1, \tag{1.7}$$

*and*

$$\text{for each } i = 2, \ldots, d, \ |\lambda_i| \neq 1. \tag{1.8}$$

*Then for any non-zero $\xi$, and any sequence of real numbers $r_m$, the set*

$$\{\mu^m \lambda^n \xi + r_m : n, m \in \mathbb{N}\} \tag{1.9}$$

*is dense modulo* 1.

As an example illustrating Theorem 1.5 we can consider the following expressions

$$(\sqrt{23}+1)^n(\sqrt{23}+2)^m + 2^m\beta \text{ or } (3+\sqrt{3})^n(\sqrt{3})^m 5 + 7^m\beta,\ \beta \in \mathbb{R}.$$

**Remark.** We believe that assumption (1.6) is not necessary to conclude density modulo 1 of the sets of the form (1.9).

Another kind of a generalization of Furstenberg's Theorem 1.1, which we are going to use in the proof of our result, is to consider higher-dimensional analogues. A generalization to a commutative semigroup of non-singular $d \times d$-matrices with integer coefficients acting by endomorphisms on the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and to the commutative semigroups of continuous endomorphisms of other compact abelian groups was given by Berend in [1] and [2], respectively (see Sect. 2.3). Recently some results for non-commutative semigroups of endomorphisms of $\mathbb{T}^d$ have been obtained in [5, 6, 13].

The structure of the paper is as follows. In Sect. 2 we recall some notions and facts from ergodic theory and topological dynamics. Following Berend [1, 2], we recall the definition of an ID-semigroup of endomorphisms of the $d$-dimensional torus $\mathbb{T}^d$. Then we state Berend's theorem, [1], which gives conditions that guarantee that a given semigroup of endomorphisms of $\mathbb{T}^d$ is an ID-semigroup. This theorem is crucial for the proof of our main result. Finally in Sect. 3, using some ideas from [9, 3] we prove Theorem 1.5.

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## 2. Preliminaries

**2.1. Algebraic numbers.** We say that $P \in \mathbb{Z}[x]$ is *monic* if the leading coefficient of $P$ is one, and *reduced* if its coefficients are relatively prime. A *real algebraic integer* is any real root of a monic polynomial $P \in \mathbb{Z}[x]$, whereas an *algebraic number* is any root (real or complex) of a (not necessarily monic) non-constant polynomial $P \in \mathbb{Z}[x]$. The *minimal polynomial* of an algebraic number $\theta$ is the reduced element $Q$ of $\mathbb{Z}[x]$ of the least degree such that $Q(\theta) = 0$. If $\theta$ is an algebraic number, the roots of its minimal polynomial are simple. The *degree* of an algebraic number is the degree of its minimal polynomial.

Let $\theta$ be an algebraic integer of degree $n$ and let $P \in \mathbb{Z}[x]$ be the minimal polynomial of $\theta$. The $n-1$ other distinct (real or complex) roots $\theta_2, \ldots, \theta_n$ of $P$ are called *conjugates* of $\theta$.

**2.2. Topological transitivity, ergodicity and hyperbolic toral endomorphisms.** We start with some basic notions, [12, 7]. We consider a *discrete topological dynamical system* $(X, f)$ given by a compact metric space $X$ and a continuous map $f : X \to X$. We say that a topological dynamical system $(X, f)$ (or simply that a map $f$) is *topologically transitive* if for any two nonempty open sets $U, V \subset X$ there exists $n = n(U, V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. One can show that $f$ is topologically transitive if for every nonempty open set $U$ in $X$, $\bigcup_{n\geq 0} f^{-n}(U)$ is dense in $X$ (see [8] for other equivalent definitions). If there exists a point $x \in X$ such that its orbit $\{f^n(x) : n \in \mathbb{N}\}$ is dense in $X$, then we say that $x$ is a *transitive point*. Under some additional assumptions on $X$, the map $f$ is topologically transitive if and only if there is a transitive point $x \in X$. Namely, we have the following

**Proposition 2.1** ([14])**.** *If $X$ has no isolated point and $f$ has a transitive point then $f$ is topologically transitive. If $X$ is separable, second category and $f$ is topologically transitive then $f$ has a transitive point.*

Consider a probability space $(X, \mathcal{F}, \mu)$ and a continuous transformation $f : X \to X$. We say that the map $f$ is *measure preserving*, and that $\mu$ is *$f$-invariant*, if for every $A \in \mathcal{F}$ we have $\mu(f^{-1}(A)) = \mu(A)$. Recall that $f$ is said to be *ergodic* if every set $A$ such that $f^{-1}(A) = A$ has measure 0 or 1.

Let $L$ be a *hyperbolic matrix*, that is a $d \times d$-matrix with integer entries, with non-zero determinant, and without eigenvalues of absolute value 1. Then $L\mathbb{Z}^d \subset \mathbb{Z}^d$, so $L$ determines a map of the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Such a map is called a *hyperbolic toral endomorphism.* It is known (see e.g. [12]) that the Haar measure $m$ of $\mathbb{T}^d$ is invariant under surjective continuous homomorphisms. In particular, it is $L$-invariant. We state two propositions about toral endomorphisms. Their proofs can be found in [12].

**Proposition 2.2.** *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be a hyperbolic toral endomorphism. Then $L$ is ergodic.*

The next proposition gives an elementary and useful relation between ergodicity and topological transitivity.

**Proposition 2.3.** *Let $L$ be a continuous endomorphism of $\mathbb{T}^d$ which preserves the Haar measure $m$. If $L$ is ergodic then it is topologically transitive. In particular, if $L$ is a hyperbolic toral endomorphism then $L$ has a transitive point $t \in \mathbb{T}^d$, i.e., $\{L^n t : n \in \mathbb{N}\}$ is dense in $\mathbb{T}^d$.*

We will also need the following lemma about finite invariant sets of ergodic endomorphisms. For the proof see [1, Lemma 5.2].

**Lemma 2.4.** *Let $L : \mathbb{T}^d \to \mathbb{T}^d$ be an ergodic endomorphism. A finite $L$-invariant set is necessarily composed of torsion elements only.*

**2.3. ID semigroups of endomorphisms acting on $\mathbb{T}^d$.** Following [1, 2], we say that the semigroup $\Sigma$ of endomorphisms of a compact group

$G$ has the *ID-property* (or simply that $\Sigma$ is an *ID-semigroup*) if the only infinite closed $\Sigma$-invariant subset of $G$ is $G$ itself. (ID-property stands for *infinite invariant is dense.*) A subset $A \subset G$ is said to be *$\Sigma$-invariant* if $\Sigma A \subset A$.

We say, exactly like in the case of real numbers, that two endomorphisms $\sigma$ and $\tau$ are *rationally dependent* if there are integers $m$ and $n$, not both of which are 0, such that $\sigma^m = \tau^n$, and *rationally independent* otherwise.

Berend in [1] gave necessary and sufficient conditions in arithmetical terms for a commutative semigroup $\Sigma$ of endomorphisms of the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ to have the ID-property. Namely, he proved the following.

**Theorem 2.5** (Berend, [1, Theorem 2.1]). *A commutative semigroup $\Sigma$ of continuous endomorphisms of $\mathbb{T}^d$ has the ID-property if and only if the following hold:*

(i) *There exists an endomorphism $\sigma \in \Sigma$ such that the characteristic polynomial $f_{\sigma^n}$ of $\sigma^n$ is irreducible over $\mathbb{Z}$ for every positive integer $n$.*

(ii) *For every common eigenvector $v$ of $\Sigma$ there exists an endomorphism $\sigma_v \in \Sigma$ whose eigenvalue in the direction of $v$ is of norm greater than 1.*

(iii) *$\Sigma$ contains a pair of rationally independent endomorphisms.*

**Remark.** Let $\Sigma$ be a commutative ID-semigroup of endomorphisms of $\mathbb{T}^d$. Then the $\Sigma$-orbit of the point $x \in \mathbb{T}^d$ is finite if and only if $x$ is a rational element, i.e., $x = r/q$, $r \in \mathbb{Z}^d$, $q \in \mathbb{N}$ (see [1]).

## 3. Proof of Theorem 1.5

Let $\lambda > 1$ be a real algebraic integer of degree $d$ with minimal (monic) polynomial $Q_\lambda \in \mathbb{Z}[x]$,

$$Q_\lambda(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_1 x + c_0.$$

We associate with $\lambda$ the following *companion matrix* of $Q_\lambda$,

$$\sigma_\lambda = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -c_0 & -c_1 & -c_2 & \ldots & -c_{d-1} \end{pmatrix}.$$

**Remark.** We can think of $\sigma_\lambda$ as a matrix of multiplication by $\lambda$ in the algebraic number field $\mathbb{Q}(\lambda)$. Namely, if $x \in \mathbb{Q}(\lambda)$ has coordinates $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1})$ in the basis consisting of $1, \lambda, \ldots, \lambda^{d-1}$, then $\lambda x$ has coordinates $\alpha\sigma_\lambda$.

Let $\mu = g(\lambda)$, where $g \in \mathbb{Z}[x]$ is a polynomial with integer coefficients, and define the matrix $\sigma_\mu = g(\sigma_\lambda)$.

Denote by $\Sigma$ the semigroup of endomorphisms of $\mathbb{T}^d$ generated by $\sigma_\lambda$ and $\sigma_\mu$. The vector $v = (1, \lambda, \lambda^2, \dots, \lambda^{d-1})^t$ is an eigenvector of the matrix $\sigma_\lambda$ with an eigenvalue $\lambda$, that is $\sigma_\lambda v = \lambda v$. Since $\Sigma$ is a commutative semigroup, it follows that $v$ is a common eigenvector of $\Sigma$, in particular $\sigma_\mu v = g(\sigma_\lambda)v = g(\lambda)v = \mu v$.

Clearly, under the assumptions on $\lambda$ and $\mu$, the operators $\sigma_\lambda$ and $\sigma_\mu$ are rationally independent endomorphisms of $\mathbb{T}^d$ and the characteristic polynomial either of $\sigma_\lambda^n$ or $\sigma_\mu^n$ is irreducible over $\mathbb{Z}$ for every $n \in \mathbb{N}$. Furthermore, it follows from (1.7) that the condition (ii) of Theorem 2.5 is also satisfied. Thus we have proved the following

**Lemma 3.1.** *Let $\lambda$ and $\mu$ be as in Theorem 1.5. Let $\Sigma$ be the semigroup of endomorphisms of $\mathbb{T}^d$ generated by $\sigma_\lambda$ and $\sigma_\mu$. Then $\Sigma$ is the ID-semigroup.*

The next lemma is a generalization of [9, Lemma 2.1] to the higher-dimensional case. Let $X$ be a compact metric space with a distance $d$. Consider the space $\mathcal{C}_X$ of all closed subsets of $X$. The *Hausdorff metric* $d_H$ on the space $\mathcal{C}_X$ is defined as

$$d_H(A, B) = \max\{\max_{x\in A} d(x, B), \max_{x\in B} d(x, A)\},$$

where $d(x, B) = \min_{y\in B} d(x, y)$ is the distance of $x$ from the set $B$. It is known that if $X$ is a compact metric space then $\mathcal{C}_X$ is also compact.

**Lemma 3.2.** *Let $\sigma, \tau$ be a pair of rationally independent and commuting endomorphisms of $\mathbb{T}^d$. Assume that the semigroup $\Sigma = \langle \sigma, \tau \rangle$ generated by $\sigma$ and $\tau$ satisfies the conditions of Theorem 2.5, and $\sigma$ is a hyperbolic toral endomorphism of $\mathbb{T}^d$. Let $A$ be an infinite $\sigma$-invariant subset of $\mathbb{T}^d$. Then for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that the set $\tau^m A$ is $\varepsilon$-dense.*

*Proof.* It is clear that, taking the closure of $A$ if necessary, we can assume that $A$ is closed. We consider the space $\mathcal{C}_{\mathbb{T}^d}$ of all closed subsets of $\mathbb{T}^d$ with the Hausdorff metric $d_H$. Let

$$\mathcal{F} := \overline{\{\tau^n A : n \in \mathbb{N}\}} \subset \mathcal{C}_{\mathbb{T}^d}.$$

Since the set $A$ is $\sigma$-invariant, it follows that every element (set) $F \in \mathcal{F}$ is also $\sigma$-invariant. Define,

$$T = \bigcup_{F\in\mathcal{F}} F \subset \mathbb{T}^d.$$

Since $A$ is an infinite set and $A \subset T$, it follows that $T$ is infinite. Notice that $T$ is closed in $\mathbb{T}^d$, since $\mathcal{F}$ is closed in $\mathcal{C}_{\mathbb{T}^d}$. Moreover, $T$ is $\sigma$- and $\tau$-invariant. Hence, by Theorem 2.5, we get

$$T = \mathbb{T}^d.$$

Since $\sigma$ is a hyperbolic toral endomorphism, it follows by Proposition 2.3, that there exists $t \in T$ such that the orbit $\{\sigma^n t : n \in \mathbb{N}\}$ is dense in $\mathbb{T}^d$, i.e.,

$$\overline{\{\sigma^n t : n \in \mathbb{N}\}} = \mathbb{T}^d \tag{3.3}$$

Clearly, $t \in F$ for some $F \in \mathcal{F}$. By definition of $\mathcal{F}$, there is a sequence $\{n_k\} \subset \mathbb{N}$ such that $F = \lim_k \tau^{n_k} A$, and the limit is taken in the Hausdorff metric $d_H$. Since $t \in F$ and $F$ is $\sigma$-invariant, we get $F \supset \overline{\{\sigma^n t : n \in \mathbb{N}\}} = \mathbb{T}^d$ (see (3.3)). Hence, $F = \mathbb{T}^d$. Therefore, for sufficiently large $k$, $\tau^{n_k} A$ is $\varepsilon$-dense. □

Now we are ready to give

*Proof of Theorem 1.5.* Let $\alpha = \xi(1, \lambda, \lambda^2, \dots, \lambda^{d-1})^t \in \mathbb{R}^d$ be a common eigenvector of the semigroup $\Sigma$. Consider

$$A = \{\sigma_\lambda^n \pi(\alpha) : n \in \mathbb{N}\} = \{\pi(\lambda^n \xi, \lambda^{n+1}\xi, \dots, \lambda^{n+d-1}\xi)^t : n \in \mathbb{N}\},$$

where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection. By (1.8), $\sigma_\lambda$ is a hyperbolic toral endomorphism. In particular, by Proposition 2.2, $\sigma_\lambda$ is ergodic. Since $\pi(\alpha)$ is not a torsion element, it follows from Lemma 2.4 that $A$ is infinite. By Lemma 3.1, $\Sigma = \langle \sigma_\lambda, \sigma_\mu \rangle$ is the ID-semigroup of $\mathbb{T}^d$. Thus, by Lemma 3.2 applied to $\sigma_\lambda$ and $\sigma_\mu$, there exists $m \in \mathbb{N}$ such that $\sigma_\mu^m A$ is $\varepsilon$-dense. Let $v_m = \pi(r_m, 0, \dots, 0)^t$. Since

$$\sigma_\mu^m A + v_m = \{\pi(\mu^m \lambda^n \xi + r_m, \mu^m \lambda^{n+1}\xi, \dots, \mu^m \lambda^{n+d-1}\xi)^t : n \in \mathbb{N}\}$$

is a translate of an $\varepsilon$-dense set, it is also $\varepsilon$-dense. Now, taking the projection of the set $\sigma_\mu^m A + v_m$ on the first coordinate we get the result. □

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