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## L<sup>1</sup>-CONVERGENCE OF MODIFIED COMPLEX TRIGONOMETRIC SUMS

(submitted by F. G. Avkhadiev)

ABSTRACT. In this paper we study  $L^1$ -convergence of modified complex trigonometric sums introduced by Ram and Kumari [2] and obtain a necessary and sufficient condition for  $L^1$ -convergence of Fourier series under a new class  $K^*$  of coefficients.

### 1. Introduction

It is well known that if a trigonometric series converges in  $L^1$ -metric to a function  $f \in L^1$ , then it is the Fourier series of the function  $f$ . Riesz [1, Vol. II, Ch. VIII § 22] gave a counter example showing that in a metric space  $L^1$  we cannot expect the converse of the above said result to hold true. This motivated the various authors to study  $L^1$ -convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in  $L^1$ -metric to the sum of the trigonometric series whereas the classical series itself may not.

Let the partial sums of the complex trigonometric series

$$\sum_{|n| < \infty} c_n e^{int}, t \in \mathbf{T} = \frac{R}{2\pi Z}$$

be denoted by  $S_n(c, t) = \sum_{|k| < n} c_k e^{ikt}$ .

If the trigonometric series is the Fourier of some  $f \in L^1(T)$ , where  $f = \lim_{n \rightarrow \infty} S_n(t)$  we shall write  $c_n = \hat{f}(n)$  for all  $n$ , and  $S_n(c, t) = S_n(f, t) = S_n(f)$ .

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Concerning the  $L^1$ -convergence of the Fourier series Goldberg and Stanojević [3] proved that if

$$\{(\hat{f}(n) - \hat{f}(-n)) \log n\} \quad (1.1)$$

is a null sequence of bounded variation and for some  $1 < p \leq 2$ ,

$$\frac{1}{n} \sum_{k=n}^{2n} k^p \left| \Delta \hat{f}(k) \right|^p = o(1) (n \rightarrow \infty), \quad (1.2)$$

then  $\|S_n(f, t) - f(t)\| = o(1)$  ( $n \rightarrow \infty$ ), is equivalent to  $\hat{f}(n) \log |n| = o(1)$  ( $|n| \rightarrow \infty$ ).

In this paper instead of (1.1), a weaker condition will be assumed, i.e.,

$$\frac{1}{[\lambda n]} \sum_{k=1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) k \log k = o(1), \quad (n \rightarrow \infty)$$

$$\lim_{\lambda \downarrow 1} \lim_{n \rightarrow \infty} \sum_{k=n}^{[\lambda n]} \Delta \left( \frac{c_k - c_{-k}}{k} \right) k \log k = 0,$$

and (1.2) will be relaxed as follows

$$\lim_{\lambda \downarrow 1} \lim_{n \rightarrow \infty} \sum_{k=n}^{[\lambda n]} k^{p-1} \left| \Delta \left( \frac{c_k}{k} \right) \right|^p = 0$$

for some  $1 < p \leq 2$ .

Kumari and Ram [2] introduced modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^n \sum_{j=k}^n \Delta \left( \frac{a_j}{j} \right) k \cos kx$$

$$g_n(x) = \sum_{k=1}^n \sum_{j=k}^n \Delta \left( \frac{a_j}{j} \right) k \sin kx$$

and studied their  $L^1$ -convergence under different sets of conditions on the coefficients  $\{a_n\}$ . Also they deduced as corollaries the  $L^1$ -convergence of the trigonometric cosine and sine series.

The object of this paper is to study the  $L^1$ -convergence of the complex form of the above defined sums Kumari and Ram with coefficients belonging to a new class  $\mathbf{K}^*$  of sequence.

Let  $D_n(t)$ ,  $\tilde{D}_n(t)$  denote the Dirichlet kernel and conjugate Dirichlet kernel respectively and we shall use the following relations:

$$D'_n(t) = E'_n(t) + E'_{-n}(t) \quad 2i\tilde{D}'_n(t) = E'_n(t) - E'_{-n}(t),$$

where  $E'_n(t)$  denotes the first differential of  $E_n(t) = \sum_{k=0}^n e^{ikt}$ .

The complex form of the above defined modified sums is

$$g_n(c, t) = S_n(c, t) + \frac{i}{n+1} [c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)] \quad (1.3)$$

We introduce here a new class  $\mathbf{K}^*$  of sequence as follows.

**Definition.** A null sequence of complex numbers belongs to the class  $K^*$ , if for some  $1 < p \leq 2$ ,

$$\frac{1}{[\lambda n]} \sum_{k=1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) k \log k = o(1), \quad (n \rightarrow \infty) \quad (1.4)$$

$$\lim_{\lambda \downarrow 1} \lim_{n \rightarrow \infty} \sum_{k=n}^{[\lambda n]} \Delta \left( \frac{c_k - c_{-k}}{k} \right) k \log k = 0, \quad (1.5)$$

$$\lim_{\lambda \downarrow 1} \lim_{n \rightarrow \infty} \sum_{k=n}^{[\lambda n]} k^{p-1} \left| \Delta \left( \frac{c_k}{k} \right) \right|^p = 0. \quad (1.6)$$

## 2 Lemmas

The following lemmas will be useful for the proof of our result, of which the first two are due to Sheng [47]:

**Lemma 2.1.**  $\|D'_n(t)\| = \frac{4}{\pi}(n \log n) + O(n)$

**Lemma 2.2.**  $\|\tilde{D}'_n(t)\| = O(n \log n)$

**Lemma 2.3.** For each non-negative integer  $n$ , there holds

$$\left\| \frac{i}{n+1} (c_{n+1} E'_n(t) - c_{-(n+1)} E'_{-n}(t)) \right\| = o(1), \quad (n \rightarrow \infty)$$

if and only if  $c_{n+1} \log |n| = o(1)$ , ( $|n| \rightarrow \infty$ ), where  $\{c_n\}$  is a complex null sequence.

This Lemma is just the slight modification of the following result of Sheng [47]. For each non-negative integer  $n$ , there holds

$$\|c_n E'_n(t) + c_{-n} E'_{-n}(t)\| = o(1), \quad (n \rightarrow \infty)$$

if and only if  $nc_n \log |n| = o(1)$  ( $|n| \rightarrow \infty$ ), where  $\{c_n\}$  is a complex null sequence.

## 3 Main Result

The main result of this paper is the following theorem:

**Theorem 3.1.** Let  $\{c_n\} \in \mathbf{K}^*$ , then for  $f \in L^1(T)$ ,

(i)  $\|g_n(f, t) - f(t)\| = o(1)(n \rightarrow \infty)$ .

(ii)  $\|S_n(f, t) - f(t)\| = o(1)(n \rightarrow \infty)$  if and only if  $c_{n+1} \log |n| = o(1)$  ( $|n| \rightarrow \infty$ ).

**Proof of Theorem.** Let  $\lambda > 1$  and  $n > 1$ , then we have

$$V_n^\lambda(f, t) - f(t) = \frac{[\lambda n] + 1}{[\lambda n] - n} (\sigma_{[\lambda n]}(f, t) - f(t)) - \frac{n + 1}{[\lambda n] - n} (\sigma_n(f, t) - f(t)),$$

where  $V_n^\lambda(f, t) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} S_k(f, t)$  is the truncated Cesàro means, and

$$\sigma_n(f, t) = \frac{1}{n+1} \sum_{k=0}^n S_k(f, t). \text{ Also, } V_n^\lambda(f, t) - S_n(f, t) = \sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n] - |k| + 1}{[\lambda n] - n} \hat{f}(k) e^{ikt}$$

With the use of (1.3) we obtain

$$\begin{aligned} g_n(f, t) - V_n^\lambda(f, t) \\ = - \sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n] - |k| + 1}{[\lambda n] - n} \hat{f}(k) e^{ikt} + \frac{i}{n+1} (c_{n+1} E_n(t) + c_{-(n+1)} E_{-n}(t)) \end{aligned}$$

Using summation by parts, we get

$$\begin{aligned} - \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k + 1}{[\lambda n] - n} c_k e^{ikt} \\ = i \left\{ \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k}{k} \right) E'_k(t) + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) E'_k(t) \right. \\ \left. - \left( \frac{c_{n+1}}{n+1} \right) E'_n(t) \right\} \end{aligned}$$

Similarly, we have

$$\begin{aligned} - \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k + 1}{[\lambda n] - n} c_{-k} e^{-ikt} \\ = -i \left\{ \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_{-k}}{k} \right) E'_{-k}(t) + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_{-k}}{k} \right) E'_{-k}(t) \right. \\ \left. - \left( \frac{c_{-(n+1)}}{n+1} \right) E'_{-n}(t) \right\} \end{aligned}$$

Therefore,

$$\begin{aligned} g_n(f, t) - V_n^\lambda(f, t) \\ = i \left( \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k}{k} \right) E'_k(t) - \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_{-k}}{k} \right) E'_{-k}(t) \right) \\ + i \left( \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) E'_k(t) - \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_{-k}}{k} \right) E'_{-k}(t) \right) \end{aligned}$$

Since  $E'_n(t) = 2i\tilde{D}'_n(t) + E'_{-n}(t)$

Therefore

$$g_n(f, t) - V_n^\lambda(f, t)$$

$$\begin{aligned}
 &= i \left( \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k}{k} \right) 2i \tilde{D}'_k(t) + \sum_{k=n+1}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right) \\
 &+ i \left( \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) 2i \tilde{D}'_k(t) + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right)
 \end{aligned}$$

$$\begin{aligned}
 (3.1) \quad &\|g_n(f, t) - f(t)\| \leq \|g_n(f, t) - V_n^\lambda(f, t)\| + \|V_n^\lambda(f, t) - f(t)\| \\
 &\leq \frac{[\lambda n] + 1}{[\lambda n] - n} \|(\sigma_{[\lambda n]}(f, t) - f(t))\| + \frac{n + 1}{[\lambda n] - n} \|(\sigma_n(f, t) - f(t))\| \\
 &+ \left\| \frac{2}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\| \\
 &+ \left\| 2 \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) + \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\| \\
 &= \frac{[\lambda n] + 1}{[\lambda n] - n} \|(\sigma_{[\lambda n]}(f, t) - f(t))\| + \frac{n + 1}{[\lambda n] - n} \|(\sigma_n(f, t) - f(t))\| \\
 &+ \mathbf{I}_1 + \mathbf{I}_2
 \end{aligned}$$

For the estimate of  $\mathbf{I}_1$ , we have

$$\begin{aligned}
 \mathbf{I}_1 &= \left\| \frac{2}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) + \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\| \\
 &\leq \frac{2}{[\lambda n] - n} \left\| \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) \right\| + \frac{1}{[\lambda n] - n} \left\| \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\| \\
 &= \mathbf{J}_{11} + \frac{1}{[\lambda n] - n} \left\| \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\|
 \end{aligned}$$

where

$$\mathbf{J}_{11} = \frac{2}{[\lambda n] - n} \left\| \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) \right\|$$

Since

$$\begin{aligned}
 \tilde{D}'_k(t) &= (k + 1)D_k(t) - (k + 1)K_k(t) \\
 (k + 1)K_k(t) &= (k + 1)D_k(t) - \tilde{D}'_k(t),
 \end{aligned}$$

where  $K_k(t) = \frac{1}{k + 1} \sum_{j=0}^k D_j(t)$  is the Fejér kernel

As  $K_k(t) \geq 0$

Therefore we have

$$\tilde{D}'_k(t) \leq (k + 1)D_k(t)$$

This gives

$$\mathbf{J}_{11} = \frac{2}{[\lambda n] - n} \left\| \sum_{k=n+1}^{[\lambda n]} \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) \right\|$$

$$\begin{aligned} &\leq \frac{2}{[\lambda n] - n} \left\| \sum_{k=n}^{[\lambda n]} (k+1) \left(\frac{c_k}{k}\right) D_k(t) \right\| \\ &\leq \frac{2([\lambda n] + 1)}{[\lambda n] - n} \left\| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) D_k(t) \right\| \end{aligned}$$

Now

$$\begin{aligned} &\int_0^\pi \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) D_k(t) \right| dt \\ &= \int_0^{\frac{\pi}{[\lambda n]}} \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) D_k(t) \right| dt + \int_{\frac{\pi}{[\lambda n]}}^\pi \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) D_k(t) \right| dt \\ &= \mathbf{I}_n + \mathbf{J}_n \\ \mathbf{I}_n &\leq C_p \sum_{k=n}^{[\lambda n]} \left| \left(\frac{c_k}{k}\right) \right| \end{aligned}$$

$$\leq C_p [\lambda n] \left( \frac{1}{[\lambda n]} \sum_{k=n}^{[\lambda n]} \left| \left(\frac{c_k}{k}\right) \right|^p \right)^{\frac{1}{p}}$$

Therefore

$$\mathbf{J}_{11} \leq C_p [\lambda n] \left( \frac{1}{[\lambda n]} \sum_{k=n}^{[\lambda n]} \left| \left(\frac{c_k}{k}\right) \right|^p \right)^{\frac{1}{p}}$$

where  $C_p$  is an absolute constant

$$\begin{aligned} \mathbf{J}_n &= \int_{\frac{\pi}{[\lambda n]}}^\pi \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) D_k(t) \right| dt \\ &= \int_{\frac{\pi}{[\lambda n]}}^\pi \frac{1}{\sin \frac{t}{2}} \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) \sin\left(k + \frac{1}{2}\right)t \right| dt \end{aligned}$$

After applying Holder-inequality

$$\mathbf{J}_n \leq \left[ \int_{\frac{\pi}{[\lambda n]}}^\pi \left( \frac{1}{\sin \frac{t}{2}} \right)^p dt \right]^{\frac{1}{p}} \left[ \int_0^\pi \left| \sum_{k=n}^{[\lambda n]} \left(\frac{c_k}{k}\right) \sin\left(k + \frac{1}{2}\right)t \right| dt \right]^{\frac{1}{q}}$$

and then the Hausdorff-Young inequality, we have

$$\begin{aligned} \mathbf{J}_n &\leq C_p [\lambda n]^{\frac{1}{q}} \left( \sum_{k=n}^{[\lambda n]} \left| \left(\frac{c_k}{k}\right) \right|^p \right)^{\frac{1}{p}} \\ &= C_p [\lambda n] \left( \frac{1}{[\lambda n]} \sum_{k=n}^{[\lambda n]} \left| \left(\frac{c_k}{k}\right) \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

where  $C_p$  is an absolute constant depending on  $p$ , and  $\frac{1}{p} + \frac{1}{q} = 1$

Also Lemma 1 and Lemma 2 imply that

$$\|E'_{-k}(t)\| = O(k \log k)$$

Hence

$$\begin{aligned} \mathbf{I}_1 &\leq C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{k=1}^{[\lambda n]} \left| \left( \frac{c_k}{k} \right) \right|^p \right)^{\frac{1}{p}} \\ &+ C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{k=n}^{[\lambda n]} \left| \left( \frac{c_k - c_{-k}}{k} \right) \right| k \log k \right) \end{aligned}$$

where  $C_1$  and  $C_2$  are absolute constants.

Similarly the last term of (3.1) is

$$\begin{aligned} \mathbf{I}_2 &= \left\| 2 \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k}{k} \right) \tilde{D}'_k(t) + \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta \left( \frac{c_k - c_{-k}}{k} \right) E'_{-k}(t) \right\| \\ &\leq C_3 \sum_{k=n}^{[\lambda n]} \left| \Delta \left( \frac{c_k - c_{-k}}{k} \right) \right| k \log k \\ &+ C_4 \left( \sum_{k=n}^{[\lambda n]} k^{p-1} \left| \left( \frac{c_k}{k} \right) \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

where  $C_3$  and  $C_4$  are absolute constants.

Combining the above results, we get

$$\begin{aligned} \|g_n(f, t) - f(t)\| &\leq \frac{[\lambda n] + 1}{[\lambda n] - n} \|(\sigma_{[\lambda n]}(f, t) - f(t))\| + \frac{n+1}{[\lambda n] - n} \|(\sigma_n(f, t) - f(t))\| \\ &+ C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{k=1}^{[\lambda n]} \left| \left( \frac{c_k}{k} \right) \right|^p \right)^{\frac{1}{p}} \\ &+ C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{k=n}^{[\lambda n]} \left| \left( \frac{c_k - c_{-k}}{k} \right) \right| k \log k \right) \\ &+ C_3 \sum_{k=n}^{[\lambda n]} \left| \Delta \left( \frac{c_k - c_{-k}}{k} \right) \right| k \log k \\ (3.2) \quad &+ C_4 \left( \sum_{k=n}^{[\lambda n]} k^{p-1} \left| \left( \frac{c_k}{k} \right) \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

Since  $\{c_n\}$  is a null sequence and  $\lambda > 1$ , we have

$$\frac{[\lambda n]}{[\lambda n] - n} \sim \frac{\lambda}{\lambda - 1} \text{ as } n \rightarrow \infty, \text{ it follows that } \lim_{n \rightarrow \infty} \frac{[\lambda n]}{[\lambda n] - n} c_n = 0,$$

therefore the 3rd term of (3.2) tends to zero as  $n \rightarrow \infty$ .

Also as  $f \in L^1(T)$ , it follows that

$$\|\sigma_n(f, t) - f(t)\| = o(1), \text{ as } n \rightarrow \infty.$$

Taking *lim sup* of both sides of (3.2) and by condition (1.4),

we have

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \|g_n(f, t) - f(t)\| &\leq C_3 \overline{\lim}_{n \rightarrow \infty} \sum_{k=n}^{[\lambda n]} \left| \Delta \left( \frac{c_k - c_{-k}}{k} \right) \right| k \log k \\ &+ C_4 \overline{\lim}_{n \rightarrow \infty} \left( \sum_{k=1}^{[\lambda n]} k^{p-1} \left| \left( \frac{c_k}{k} \right) \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

Now by taking  $\lim$  as  $\lambda \rightarrow 1$  and by conditions (1.5) and (1.6), we have

$$\overline{\lim}_{n \rightarrow \infty} \|g_n(f, t) - f(t)\| = 0$$

We further notice that

$$\begin{aligned} &\|S_n(f, t) - f(t)\| \\ &\leq \|S_n(f, t) - g_n(f, t)\| + \|g_n(f, t) - f(t)\| \\ &= \|g_n(f, t) - f(t)\| + \left\| \frac{i}{n+1} (c_{n+1}E_n(t) - c_{-(n+1)}E_{-n}(t)) \right\| \\ &\left\| \frac{i}{n+1} (c_{n+1}E_n(t) - c_{-(n+1)}E_{-n}(t)) \right\| = \|g_n(f, t) - S_n(f, t)\| \\ &\leq \|g_n(f, t) - f(t)\| + \|S_n(f, t) - f(t)\| \end{aligned}$$

Since  $\|g_n(f, t) - f(t)\| = o(1)$ , ( $n \rightarrow \infty$ ). by (6.3.1)

$$\text{and by Lemma 3, } \left\| \frac{i}{n+1} (c_{n+1}E_n(t) - c_{-(n+1)}E_{-n}(t)) \right\| = o(1)(n \rightarrow \infty),$$

if and only if  $c_{n+1} \log |n| = o(1)$  ( $|n| \rightarrow \infty$ ).

Therefore  $\|S_n(f, t) - f(t)\| = o(1)$  ( $|n| \rightarrow \infty$ ),

if and only if  $c_{n+1} \log |n| = o(1)$  ( $|n| \rightarrow \infty$ ), the assertion (ii) follows.

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