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CONSTRUCTING A MAXIMAL COFINITARY GROUP

(submitted by M. M. Arslanov)

ABSTRACT. Assuming continuum hypothesis (CH), we construct a maximal cofinitary group step by step. We also outline a way of constructing maximal cofinitary group by assuming the negation of CH and Martin's Axiom (MA).

1. Introduction.

A permutation $g \in \text{Sym}(\mathbb{N})$ is cofinitary iff g has only finitely many fixed points. A group $G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff, for all $g \in G \setminus \{id\}$, g is a cofinitary permutation. Different properties of maximal cofinitary groups has been studied (see, e.g., [A], [T1], [T2], [Z1], [Z2] etc.). In [C], P. Cameron pointed out that Zorn's Lemma (or, Axiom of Choice) implies the existence of maximal cofinitary group. However, questions can still be asked from various directions. For example,

- (1) instead of saying that Zorn's Lemma implies existence of a maximal cofinitary group, is there any concrete example of maximal cofinitary group?
- (2) can we construct a maximal cofinitary group without using Axiom of Choice (or Zorn's Lemma), or say, does there exist a Borel maximal cofinitary group? etc.

In this notes, we give a concrete example of maximal cofinitary group under CH. In fact, we will construct a maximal cofinitary groups under CH step by step. We will also outline a way of constructing maximal cofinitary groups by assuming $\neg CH$ and MA. The set theory notation which we use in this paper are standard, people can find them in a book like [K].

2. Constructing a maximal cofinitaty group under CH.

Let $G \leq \text{Sym}(\mathbb{N})$ be a countable cofinitary group. Let $f \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary. Let $W_G = \{w_i(x) \mid i \in \mathbb{N}\}$ enumerate all words

$$g_1 x^{n_1} g_2 \dots g_t x^{n_t} g_{t+1}$$

which actually involve x such that $g_l \in G \setminus \{id\}$ except that possibly $g_1 = id$ or $g_{t+1} = id$ and $n_i \in \mathbb{Z} \setminus \{0\}$. By induction, we construct a permutation $g \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, g \rangle$ is cofinitary.

At the i -th stage, we do the following. Assume that g_{i-1}^* is a 1-1 finite partial function we defined up to $i-1$ stage.

If $g_{i-1}^*(i)$ is defined, then let $g_i^*(i) = g_{i-1}^*(i)$.

Assume $g_{i-1}^*(i)$ is undefined. Then we do the following.

We want to define $g_i^*(i)$ which satisfies the following condition (α) .

(α) . For every conjugate subword w'_j of w_j , $j < i$, without cancellation, the following holds. Suppose that $w'_j(g_{i-1}^*)(l)$ is undefined, and $w'_j(g_i^*)(l) = l$. Then $w'_j = uzu^{-1}$ without cancellation and $\langle l, n \rangle \in u^{-1}(g_{i-1}^*)$ for some n with $z(g_{i-1}^*)(n) = n$.

Lemma 2.1 *Assume that $w(x) \in W_G$ is a fixed word and $g_{i-1}^*(i)$ is not defined. Then for all but finitely many $m \in \mathbb{N}$ we can define $g_i^*(i) = m$ and the condition (α) holds.*

Proof We argue by induction on the length of the word w . Let $m \in \mathbb{N} \setminus \text{rang}(g_{i-1}^*)$ and consider

$$\bar{g}_i = g_{i-1}^* \cup \{\langle i, m \rangle\}.$$

Suppose that there exists a conjugate subword

$$w_0 = g_1 x^{n_1} g_2 \dots g_t x^{n_t} g_{t+1}$$

of w and an integer l_m such that

$$w_0(g_i^*)(l_m) \text{ is undefined, and } w_0(\bar{g}_i)(l_m) = l_m,$$

and condition (α) fails.

Consider the computation of the second formula in more detail. There exists some $1 \leq j \leq t$ such that

$$r = g_{j+1}(g_{i-1}^*)^{n_{j+1}} g_{j+2} \dots (g_{i-1}^*)^{n_t} g_{t+1}(l_m)$$

is defined, and either

- $n_j > 0$ and there exists $0 \leq k \leq n_j - 1$ such that

$$(g_{i-1}^*)^k(r) \notin \text{dom}(g_{i-1}^*); \text{ or}$$

- $n_j < 0$ and there exists $n_j + 1 \leq k \leq 0$ such that

$$(g_{i-1}^*)^k(r) \notin \text{rang}(g_{i-1}^*).$$

At the next step of the computation, we must use the new data $\langle i, m \rangle$ of \bar{g}_i .

Case (1). We must have that $(g_{i-1}^*)^k(r) = i$ and so

$$(\bar{g}_i)^{k+1}(r) = m.$$

We now consider various subcases.

Case 1.1. Suppose that $k + 1 < n_j$. Then we have to choose that

$$m \in \text{dom}(g_{i-1}^*) \cup \{i\}.$$

Hence there are only finitely many choice for $m \in \mathbb{N}$.

Case 1.2. Suppose that $k + 1 = n_j$. Now we have to consider the rest of the computation.

$$g_1(\bar{g}_i)^{n_1} \dots g_j(m) = l_m.$$

First suppose that $j > 1$. If $n_{j-1} > 0$, then we can continue the computation iff

$$g_j(m) \in \text{dom}(\bar{g}_i) = \text{dom}(g_{i-1}^*) \cup \{n\},$$

and hence there are only finitely many values for m . If $n_{j-1} < 0$, then we must have that

$$g_j(m) \in \text{rang}(\bar{g}_i) = \text{rang}(g_{i-1}^*) \cup \{m\}.$$

Since $g_j \in G$ which is a cofinitary group, there are only finitely many values of m .

Now suppose that $j = 1$. Then it has to be that $g_1(m) = l_m$. There is an apparent difficulty since we do not know how many possibilities there are for l_m . But since we know that

$$g_1(\bar{g}_i)^{n_1} g_2 \dots (\bar{g}_i)^{n_t} g_{t+1}(l_m) = l_m, \text{ and} \\ g_1(g_{i-1}^*)^{n_1} g_2 \dots (g_{i-1}^*)^{n_t} g_{t+1}(l_m) \text{ is undefined,}$$

then this implies that either

$$(1_a). \quad g_{t+1}(l_m) \in \text{rang}(g_{i-1}^*) \cup \text{dom}(g_{i-1}^*), \text{ or}$$

$$(1_b). \quad g_{t+1}(l_m) = i, \text{ and } w_0(x) = g_1 x g_2.$$

For (1_a), it is clear that there are only finitely many possibilities for l_m , and hence only finitely many possibilities for $m = g_1^{-1}(l_m)$.

For (1_b), since $g_1(m) = l_m$ and $g_2(l_m) = i$, then

$$m = g_1^{-1}(g_2^{-1}(i)).$$

Again there are only finitely many values for m .

Case (2). We must have that $g_{i-1}^*(r) = m$ and so

$$(\bar{g}_i)^{k-1}(r) = i.$$

Case 2.1. Suppose that $j < t$. Then

$$(g_{i-1}^*)^k g_{j+1}(g_{i-1}^*)^{n_{i+1}} \dots g_{t+1}(l_m) = m.$$

Since $\text{rang}((g_{i-1}^*)^k g_{j+1} \dots g_{t+1})$ is finite, there are only finitely many values for m .

Case 2.2. Suppose that $j = t$. If $k \neq 0$, then

$$(g_{i-1}^*)^k g_{t+1}(l_m) = m.$$

Since $\text{rang}((g_{i-1}^*)^k g_{t+1})$ is finite, there are only finitely many values for m .

Now assume that $k = 0$. Then $m = g_{t+1}(l_m)$. Once again, there is an apparent difficulty, since we do not know how many possibilities there are for l_m . We have the following two subcases to consider:

(2_a). $n_1 < 0$ in $w_0 = g_1 x^{n_1} \dots g_t x^{n_t} g_{t+1}$.

(2_b). $n_1 > 0$ in $w_0 = g_1 x^{n_1} \dots g_t x^{n_t} g_{t+1}$.

For (2_a), since $n_1 < 0$, then

$$l_m \in g_1(\text{dom}(\bar{g}_i)) = g_1(\text{dom}(g_{i-1}^*) \cup \{i\}),$$

i.e., $g_{t+1}^{-1}(m) \in g_1(\text{dom}(g_{i-1}^*) \cup \{i\})$. Hence,

$$m \in g_{t+1} g_1(\text{dom}(g_{i-1}^*) \cup \{i\}).$$

Thus there are finitely many values for m .

Now we consider the subcase (2_b). Since

$$w_0 = g_1 x^{n_1} \dots g_t x^{n_t} g_{t+1} = g_1 x(x^{n_1-1} g_2 \dots g_t x^{n_t+1}) x^{-1} g_{t+1},$$

we may write that $w_0 = g_1 x z x^{-1} g_{t+1}$. If $z(g_{i-1}^*)(i) = i$ and $g_1 = g_{t+1}^{-1}$, then we are done. Assume otherwise. Then either $z(g_{i-1}^*)(i) \neq i$ or $g_1 \neq g_{t+1}^{-1}$.

If $z(g_{i-1}^*)(i) \neq i$, then there are two subcases:

(I). $z(g_{i-1}^*)(i) = k$, for some $k \in \mathbb{N}$, or

(II). $z(g_{i-1}^*)(i)$ is undefined.

For (I) we may assume that $z(g_{i-1}^*)(i) \in \text{dom}(g_{i-1}^*)$ because otherwise the computation stops. Therefore, we know that

$$g_{i-1}^*(z(g_{i-1}^*)(i)) \neq m, \text{ and } g_{i-1}^*(z(g_{i-1}^*)(i)) \in \text{rang}(g_{i-1}^*).$$

This implies that

$$l_m \in \mathbb{N} \setminus g_1(\text{rang}(g_{i-1}^*));$$

i.e.,

$$g^{t+1}(m) \in \mathbb{N} \setminus g_1(\text{rang}(g_{i-1}^*)).$$

Hence there are finitely many possibilities for m .

For (II), there are two different possibilities:

(II_a). $z(\bar{g}_i)(i) = k$ for some $i \neq k \in \mathbb{N}$,

(II_b). $z(\bar{g}_i)(i) = i$.

For (II_a), there are finitely many possibilities for m , by a similar argument to the subcase (I).

Now suppose that (II_b) holds. By induction hypothesis, if m is a sufficiently large integer, then there exists an expression $z = u z_0 u^{-1}$ and an integer $c \in \mathbb{N}$ such that

(i). $z_0(g_{i-1}^*)(c) = c$, and

(ii). $u^{-1}(\bar{g}_i)(i) = c$.

If there are infinitely many such m , we must have that $g_1 = g_{t+1}$, since G is a cofinitary group. But now (α) holds for all those m such that (i) and (ii) are true.

Finally suppose that $z(g_{i-1}^*)(i) \neq i$ and $g_1 \neq g_{t+1}^{-1}$. Since G is a cofinitary group, there are only finitely many possibilities for m .

We have completed the proof of the lemma. \square

By this lemma, we know that we can define $g_i^*(i)$ which satisfies condition (α) .

A similar argument shows that we can also define $k = (g_i^*)^{-1}(i)$ which satisfies condition (α) .

Let

$$\dot{g}_i = g_{i-1}^* \cup \{\langle i, g_i^*(i) \rangle, \langle k, (g_i^*)^{-1}(i) \rangle\}.$$

Next we want to find a pair $\langle n, f(n) \rangle \in f$ with $n > i$ such that

$$g_i^* = \dot{g}_i \cup \{\langle n, f(n) \rangle\}$$

satisfies condition (α) . The following lemma guarantees that we can find such pair.

Lemma 2.2 *Let G be a cofinitary group and let $f \in \text{Sym}(\mathbb{N}) \setminus G$ be such that $\langle G, f \rangle$ is also a cofinitary group. Assume that $w(x) \in W_G$. Then for all but finitely many $\langle n, f(n) \rangle$, $\dot{g}_i \cup \{\langle n, f(n) \rangle\}$ satisfies condition (α) .*

Proof Since $\langle G, f \rangle$ is a cofinitary group and $f \in \text{Sym}(\mathbb{N}) \setminus G$, we know that f is a cofinitary permutation, i.e., f has finitely many fixed point.

Let

$$w_0 = g_1 x^{n_1} g_2 x^{n_2} \dots g_t x^{n_t} g_{t+1}$$

be a conjugate subword of w , where $n_i \in \mathbb{Z} \setminus \{0\}$ and $g_i \neq id$ except possibly $g_i = id$ or $g_{t+1} = id$. Also let

$$g' = \dot{g}_i \cup \{\langle n, f(n) \rangle\},$$

$$w_0(g')(l_n) = l_n, \text{ and}$$

$$w_0(\dot{g}_i)(l_n) \text{ is undefined.}$$

Consider the point where $\langle n, f(n) \rangle$ is first used. So we have that

$$w_0 = ax^e b, \text{ and } b(\dot{g}_i)(l_n) \in \{n, f(n)\},$$

where $a, b \in W_G \cup G$.

If b involves x , then either

$$n \in \text{rang}(b(\dot{g}_i)), \text{ or}$$

$$f(n) \in \text{rang}(b(\dot{g}_i)), \text{ i.e., } n \in f^{-1}(\text{rang}(b(\dot{g}_i))),$$

and so there are only finitely many possibilities for n . Thus without loss of generality, we may assume that $b = g_{t+1}$ in the following.

Case 1. Suppose that $n_t > 0$.

First suppose that $n_t > 1$. Then

$$f(g_{t+1}(l_n)) = f(n).$$

Hence if $f(n) \notin \text{dom}(\dot{g}_i) \cup \{n\}$, then the computation will stop. Since $\text{dom}(\dot{g}_i)$ is finite and f is a cofinitary permutation, there are only finitely possibilities for n .

Now assume that $n_t = 1$. To make the computation continue, it has to be that

$$g_t(f(n)) \in \text{dom}(\dot{g}_i) \cup \text{rang}(\dot{g}_i) \cup \{n, f(n)\}.$$

Since $g_t \neq id$ and $f \cap g_t$ is finite, there are only finitely many n 's such that

$$g_t(f(n)) = f(n), \text{ or } g_t(f(n)) = n.$$

Hence there are only finitely many n 's to make the computation continue.

So, without loss of generality, we may consider that

$$w_0 = g_1 x g_2.$$

Assume that there are infinitely many l_n such that

$$g_1 \{\langle n, f(n) \rangle\} g_2(l_n) = l_n.$$

This implies that

$$g_1 f g_2 = id.$$

Hence $f = g_1^{-1} g_2^{-1} \in G$ which is a contradiction.

Case 2. Suppose that $n_t < 0$.

First suppose that $n_t < -1$. We know that if

$$n \notin \text{rang}(\dot{g}_i) \cup \{f(n)\}.$$

then the computation stops. There are only finitely many choice for n .

So without loss of generality, we may suppose that $n_t = -1$. Then only

$$g_t(n) \in \text{dom}(\dot{g}_i) \cup \text{rang}(\dot{g}_i) \cup \{n, f(n)\}$$

can make the computation continue. Since there are only finitely many $n \in \mathbb{N}$ such that

$$g_t(n) = n, \text{ or, } g_t(n) = f(n).$$

there are only finitely many possibilities for n .

We consider the last case

$$w_0 = g_1 x^{-1} g_2.$$

If there are infinitely many l_n such that

$$w_0(l_n) = g_1 \{\langle n, f(n) \rangle\} g_2(l_n) = l_n,$$

then $g_1 f^{-1} g_2 = id$. Hence $f^{-1} = g_1^{-1} g_2^{-1}$, i.e., $f = g_2 g_1 \in G$. This is a contradiction as well.

We have proved the lemma. □

Let

$$g_i^* = \dot{g}_i \cup \{\langle n, f(n) \rangle\}.$$

We finished the i -th stage construction.

Now let $g = \bigcup_{i \in \mathbb{N}} g_i^*$.

We have proved the following.

Theorem 2.3 *Let G be a countable cofinitary group and let $f \in \text{Sym}(\mathbb{N}) \setminus G$ be such that $\langle G, f \rangle$ is also a cofinitary group. Then we can recursively construct a permutation $g \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, g \rangle$ is cofinitary and $g \cap f$ is infinite.*

Proof We prove that $\langle G, g \rangle$ is cofinitary by induction on the length of the word $w_i \in W_G (i \in \omega)$ that $|fix(w(g))| < \omega$.

First suppose that we cannot express $w_i = uzu^{-1}$ without cancellation. Then $w(g)(l) = l$ implies that $w_i(g_i^*)(l) = l$. It follows that

$$|fix(w(g))| < \omega.$$

Next suppose that we can express $w = uzu^{-1}$ without cancellation. By induction hypothesis $|fix(z(g))| < \omega$, since

$$|fix(w(g))| = |fix(z(g))|$$

by condition (α) . We know that $|fix(w(g))| < \omega$.

It can be easily seen from Lemma 2.2 that $g \cap f$ is infinite. □

By CH, let $\{f_\alpha \mid \omega \geq \alpha < \omega_1\}$ enumerate all permutations in $Sym(\mathbb{N})$. We construct a maximal cofinitary group G as follows.

Let $g = (01)(234)(5678)\dots$. Let $G_n = \langle g \rangle$ for any $n \in \mathbb{N}$.

At the α -th stage, $\omega \leq \alpha < \omega_1$, we consider the countable cofinitary group $G'_\alpha = \bigcup_{\beta < \alpha} G_\beta$ and f_α .

Assume that $f_\alpha \notin G'_\alpha$ and $\langle G'_\alpha, f_\alpha \rangle$ is cofinitary. Then, by Theorem 2.3, we can construct a g_α such that $G_\alpha = \langle G'_\alpha, g_\alpha \rangle$ is cofinitary, and $g_\alpha \cap f_\alpha$ is infinite.

Let $G = \bigcup_{\alpha < \omega_1} G_\alpha$. Then G is a maximal cofinitary group.

Thus we finished our construction of a maximal cofinitary group under CH.

Remark. By results in section 7 of [M], and Lemma 2.1, Lemma 2.2, you may think that we can prove that there is a Π_1^1 maximal cofinitary group under $V=L$. Unfortunately, it is not the case. Our construction only gives a set of generators of a maximal cofinitary group. Thus the following problem is open and probably hard to solve:

Open Problem 2.4 *Does there exist a Borel (or, analytic, co-analytic, closed) maximal cofinitary group? Or, does there exist a Borel (or, analytic, co-analytic, closed) set of permutations which generate a maximal cofinitary group?*

For more explanation, see [GZ].

3. When CH fails.

In this section, we sketch a construction of a maximal cofinitary group by MA and \neg CH. The idea involved here is similar to the one in section 3. Thus, we will just sketch the construction and point out the p. o. set and some key results we are going to use here.

Definition 3.1 Let $G \leq Sym(\mathbb{N})$ be a cofinitary group. Then the partially ordered set \mathbb{G}_G consists of all conditions of the form $\langle s, F \rangle$ such that

- s is a 1-1 finite partial function from \mathbb{N} to \mathbb{N} ,
- F is a finite subset of W_G ;

where W_G consists of the words $g_1 x^{n_1} \dots g_t x^{n_t} g_{t+1}$ which actually involves x such that $g_i \in G \setminus \{id\}$ except that possibly $g_1 = id$ or $g_{t+1} = id$, and $n_i \in \mathbb{Z} \setminus \{0\}$. If $w(x) \in W_G$ and s is a 1-1 finite partial function from \mathbb{N} to \mathbb{N} , then $W(s)$ denotes the partial function obtained by substituting s for x in $w(x)$. We define $\langle s_2, F_2 \rangle \leq \langle s_1, F_1 \rangle$ iff

- (a). $s_1 \subseteq s_2$ and $F_1 \subseteq F_2$;

(b). For every conjugate subword w_0 of $w \in F$, the following holds. Suppose that $w_0(s_1)(l)$ is undefined and $w_0(s_2)(l) = l$. Then $w_0 = uzu^{-1}$ without cancellation and $\langle l, n \rangle \in u^{-1}(s_2)$ for some n with $z(s_1)(n) = n$.

It is easily seen that the poset \mathbb{G}_G is c.c.c. More explanations about this poset can be found in [Z1] and [Z2]. For example, the following theorem can be proved by some rather complicated density arguments which appeared in [Z1].

Lemma 2.2 (*MA* (κ)). *Let $G \leq \text{Sym}(\mathbb{N})$ be any cofinitary permutation group, where $|G| \leq \kappa$ and $\omega \leq \kappa < 2^\omega$. Then there exists a $g \in \text{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, g \rangle = G * \langle g \rangle$ is a cofinitary permutation group. Here $G * \langle g \rangle$ denotes the free product of G and $\langle g \rangle$. Moreover, if $f \in \text{Sym}(\mathbb{N}) \setminus G$ and $\langle G, f \rangle$ is a cofinitary group, then $|g \cap f| = \omega$, i.e., $\{n \in \mathbb{N} \mid g(n) = f(n)\}$ is infinite.*

Proof See the proofs of Theorem 2.6 and Lemmas 3.2, and 3.3 in [Z1]. \square

Now, let $\{f_\alpha \mid \omega \geq \alpha < 2^\omega\}$ enumerate all permutations in $\text{Sym}(\mathbb{N})$. We construct a maximal cofinitary group G as follows.

Let $g = (01)(234)(5678)\dots$. Let $G_n = \langle g \rangle$ for any $n \in \mathbb{N}$.

At the α -th stage, $\omega \leq \alpha < 2^\omega$, we consider the cofinitary group $G'_\alpha = \bigcup_{\beta < \alpha} G_\beta$ and f_α .

Assume that $f_\alpha \notin G'_\alpha$ and $\langle G'_\alpha, f_\alpha \rangle$ is cofinitary. Then, by Lemma 2.2, we can construct a g_α by MA such that $G_\alpha = \langle G'_\alpha, g_\alpha \rangle$ is cofinitary, and $g_\alpha \cap f_\alpha$ is infinite.

Let $G = \bigcup_{\alpha < 2^\omega} G_\alpha$. We know that G is a maximal cofinitary group of size continuum. We thus finished our construction.

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REFERENCES

- [A] S. A. Adeleke *Embeddings of infinite permutation groups in sharp, highly transitive, and homogeneous groups*, Proceedings of Edinburgh Mathematical Society, **31** (1981), 169–178.
- [C] P. Cameron *Cofinitary permutation groups*, Bulletin of the London Mathematical Society, **28** (1996), 113–140.
- [GZ] S. Gao and Y. Zhang *Definable sets of generators in maximal cofinitary groups* (preprint).
- [K] K. Kunen *Set Theory. An introduction of independence proofs*, 1980, North-Holland.
- [M] A. Miller *Infinite combinatorics and definability*, Annals of Pure and Applied Logic, **41** (1989), 179–203.
- [T1] J. K. Truss *Embeddings of infinite permutation groups*, Proceedings of Groups–St Andrews, 1985, London Math. Soc. Lecture Note Ser. 121 (ed. E. F. Robertson and C. M. Campbell) 1986, 335–351, Cambridge University Press.

- [T2] J. K. Truss *Joint embeddings of infinite permutation groups*, Advances in Algebra and Model Theory (ed. M. Droste and R. Göbel), Algebra, Logic and Applications, vol 9, 1997, 121–134, Gordon and Breach Science Publ.
- [Z1] Y. Zhang *Maximal cofinitary groups*, Archive for Mathematical Logic, **39**, No.1 (2000), 41–52.
- [Z2] Y. Zhang *Permutation groups and covering properties*, Journal of London Mathematical Society **63** No. 2(2001), 1–15.

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