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**PRODUCT PRESERVING BUNDLE FUNCTORS ON  
MULTIFIBERED AND MULTIFOLIATE MANIFOLDS**

(submitted by M. A. Malakhaltsev)

ABSTRACT. We show that the set of the equivalence classes of multifoliate structures is in one-to-one correspondence with the set of equivalence classes of finite complete projective systems of vector space epimorphisms. After that we give the complete description of all product preserving bundle functors on the categories of multifibered and multifoliate manifolds.

In the middle 1980s Eck [2], Kainz and Michor [3], Luciano [8] described all product preserving bundle functors on the category of smooth manifolds in terms of Weil bundles [14] (see also [5]). In 1996 Mikulski [9] classified all product preserving bundle functors on fibered manifolds. In the recent years Weil functors and product preserving functors are of great interest, see e.g. Kolář and Mikulski [6], Kriegl and Michor [7], Muñoz, Rodrigues, and Muriel [11], Mikulski and Tomáš [10, 13].

Kodaira and Spencer in [4] introduced the notion of a multifoliate structure on a smooth manifold. In the present paper, we introduce the category of multifibered manifolds which is a subcategory of the category of multifoliate manifolds and, following the lines of Mikulski [9], describe all product preserving bundle functors on these categories.

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We denote the category of smooth manifolds by  $\mathcal{M}f$  and that of fibered manifolds by  $\mathcal{FM}$  [5]. All manifolds and maps between manifolds under consideration are assumed to be of class  $C^\infty$ .

## 1. PROJECTIVE SYSTEMS OF VECTOR SPACES

Let  $(\Lambda = \{\alpha, \beta, \dots\}, \leq)$  be a partially ordered set. A *projective system* (an *inverse system*) over  $\Lambda$  [1] is a collection  $(S_\alpha, \zeta_\alpha^\beta, \Lambda)$  consisting of sets  $S_\alpha$ ,  $\alpha \in \Lambda$ , and maps  $\zeta_\alpha^\beta : S_\beta \rightarrow S_\alpha$ ,  $\alpha \leq \beta$ , called projections, such that  $\zeta_\alpha^\alpha = \text{id}_{S_\alpha}$  for all  $\alpha \in \Lambda$  and  $\zeta_\alpha^\beta \circ \zeta_\beta^\gamma = \zeta_\alpha^\gamma$  when  $\alpha \leq \beta \leq \gamma$ . The projective limit of a projective system  $(S_\alpha, \zeta_\alpha^\beta, \Lambda)$  is the subset

$$S = \varprojlim S_\alpha \subset \prod_{\alpha \in \Lambda} S_\alpha$$

consisting of all elements  $x = (x_\alpha)$  such that  $\zeta_\alpha^\beta(x_\beta) = x_\alpha$ . If the set  $S$  is not empty, then by  $\zeta_\beta : S \rightarrow S_\beta$  we denote the map which sends  $x = (x_\alpha)$  into  $x_\beta$ . These maps are called *canonical projections*.

It will be convenient to denote projective systems under consideration as follows:  $\zeta = (S_\alpha, \zeta_\alpha^\beta, \Lambda, S)$ .

In this section, we will consider projective systems of vector spaces  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  satisfying the following conditions:

- i)  $L_\alpha$ ,  $\alpha \in \Lambda$ , and  $L$  are finite-dimensional vector spaces over  $\mathbb{R}$ ;
- ii) all the maps  $\xi_\alpha^\beta$  and  $\xi_\alpha$  are linear epimorphisms.

By an isomorphism between two projective systems  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  and  $\xi' = (L'_{\alpha'}, \xi'_{\alpha'}{}^{\beta'}, \Lambda', L')$  we mean a collection  $(\omega, \{\psi_\alpha\}_{\alpha \in \Lambda})$  consisting of an isomorphism  $\omega : \Lambda \rightarrow \Lambda'$  of partially ordered sets and linear isomorphisms  $\psi_\alpha : L_\alpha \rightarrow L'_{\omega(\alpha)}$  such that  $\xi'_{\omega(\alpha)}{}^{\omega(\beta)} \circ \psi_\beta = \psi_\alpha \circ \xi_\alpha^\beta$  for  $\alpha \leq \beta$ . An isomorphism  $(\omega, \{\psi_\alpha\}_{\alpha \in \Lambda})$  gives rise to the isomorphism  $\psi := \varprojlim \psi_\alpha : L \rightarrow L'$  defined by  $\psi((x_\alpha)) = (\psi_\alpha(x_\alpha))$ . The map  $\psi$  is the unique isomorphism between  $L$  and  $L'$  such that  $\xi'_{\omega(\alpha)} \circ \psi = \psi_\alpha \circ \xi_\alpha$ . Projective systems  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  and  $\xi' = (L'_{\alpha'}, \xi'_{\alpha'}{}^{\beta'}, \Lambda', L')$  are said to be *isomorphic* if there exists an isomorphism between them.

An isomorphism from  $\xi$  to itself of the form  $(\text{id}, \{f_\alpha\}_{\alpha \in \Lambda})$  is said to be an *automorphism* of  $\xi$ . Denote by  $GL(\xi)$  the group of all linear automorphisms of  $L$  of the form  $f = \varprojlim f_\alpha$  where  $(\text{id}, \{f_\alpha\})$  is an automorphism of  $\xi$ .

**Definition.** A vector subspace  $K \subset L$  is said to be *invariant* if every  $f \in GL(\xi)$  maps  $K$  into itself.

One can easily see that the sum and the intersection of two invariant subspaces are invariant subspaces. For any  $\alpha \in \Lambda$ , the subspace  $K_\alpha := \ker \xi_\alpha \subset L$  is invariant and  $L_\alpha \cong L/K_\alpha$ . In what follows we will identify  $L_\alpha$  and  $L/K_\alpha$ .

**Definition.** A projective system  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  is said to be *complete* if any finite-codimensional invariant subspace of  $L$  is of the form  $\ker \xi_\alpha$  for some  $\alpha \in \Lambda$ .

Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  be a projective system (not necessarily complete). Consider the set  $\{K_a\}_{a \in \tilde{\Lambda}}$  of all finite-codimensional invariant subspaces  $K_a$  of  $L$ . For any two invariant subspaces  $K_a, K_b$  such that  $K_a \supset K_b$ , denote by  $\xi_a^b : L_b = L/K_b \rightarrow L_a = L/K_a$  the canonical epimorphism. Let us endow  $\tilde{\Lambda}$  with the partial order defined as follows:  $a \leq b$  if and only if  $K_a \supseteq K_b$ . One can easily see that the collection  $\tilde{\xi} = (L_a, \xi_a^b, \tilde{\Lambda}, \tilde{L} = \varprojlim L_a)$  is a complete projective system. We call it *the completion of  $\xi$* . Obviously,  $\tilde{\xi}$  is complete. Since, for any  $\alpha \in \Lambda$ , the subspace  $K_\alpha = \ker \xi_\alpha$  is invariant, one can consider  $\Lambda$  as a subset of  $\tilde{\Lambda}$ .

**Definition.** A projective system  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  is called *finite* if  $\Lambda$  is finite.

Obviously, when  $\xi$  is finite, its limit  $L$  is a finite-dimensional vector space.

**Proposition 1.1.** *If  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  is a finite complete projective system then*

- (1)  $\Lambda$  contains the greatest element  $\varepsilon$ ;
- (2)  $L$  is isomorphic to  $L_\varepsilon$ .

**Proof.** Indeed, the zero subspace is invariant and of finite codimension.

**Definition.** Two projective systems

$$\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L) \text{ and } \xi' = (L'_\alpha, \xi_{\alpha'}^{\beta'}, \Lambda', L')$$

are said to be *equivalent* if there exists an isomorphism  $\varphi : L \rightarrow L'$  such that  $\varphi \circ f \circ \varphi^{-1} \in GL(\xi')$  for any  $f \in GL(\xi)$  and

$$\Phi : f \in GL(\xi) \mapsto \varphi \circ f \circ \varphi^{-1} \in GL(\xi')$$

is a group isomorphism.

One can easily see that isomorphic projective systems are equivalent.

**Proposition 1.2.** *Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  be a projective system and  $\tilde{\xi} = (L_a, \xi_a^b, \tilde{\Lambda}, \tilde{L})$  its completion. Then  $\xi$  and  $\tilde{\xi}$  are equivalent.*

**Proof.** In fact, the maps

$$\varphi = (\varphi_\alpha) : \tilde{L} \rightarrow L, \quad \varphi_\alpha : \tilde{L} \ni x = (x_a)_{a \in \tilde{\Lambda}} \mapsto x_\alpha \in L_\alpha$$

and

$$\psi = (\psi_a) : L \rightarrow \tilde{L}, \quad \psi_a : L \ni x \mapsto x + K_a \in L_a = L/K_a$$

are mutually inverse isomorphisms which induce an isomorphism of the groups  $GL(\xi)$  and  $GL(\tilde{\xi})$ .  $\square$

The proof of the following proposition is immediate.

**Proposition 1.3.** *If complete projective systems  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  and  $\xi' = (L'_\alpha, \xi'^\beta_\alpha, \Lambda', L')$  are equivalent, then  $\xi$  is isomorphic to  $\xi'$ .*

**Definition.** Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  be a projective system. A local diffeomorphism  $\varphi : U \subset L \rightarrow V \subset L$  between two open subsets of  $L$  is called a  $\xi$ -diffeomorphism if for any  $x \in U$  there exist an open subset  $W(x) \subset U$  and a system of diffeomorphisms  $\{\varphi_\alpha : \xi_\alpha(W) \rightarrow \xi_\alpha(\varphi(W))\}_{\alpha \in \Lambda}$  such that  $\varphi_\alpha \circ \xi_\alpha = \xi_\alpha \circ \varphi$  for any  $\alpha \in \Lambda$ .

Denote the pseudogroup of all  $\xi$ -diffeomorphisms by  $\Gamma(\xi)$ . The tangent map  $d\varphi_x$  of any  $\xi$ -diffeomorphism  $\varphi : U \rightarrow V$  at every point  $x \in U$  can be viewed as an element of  $GL(\xi)$ .

**Definition.** A  $\xi$ -structure on an  $n$ -dimensional smooth manifold ( $n = \dim L$ ) is a maximal atlas compatible with the pseudogroup  $\Gamma(\xi)$ . A smooth manifold endowed with a  $\xi$ -structure is called a  $\xi$ -manifold.

**Definition.** Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  and  $\xi' = (L'_\alpha, \xi'^\beta_\alpha, \Lambda, L')$  be two projective systems over the same partially ordered set  $\Lambda$ . A smooth map  $g : U \subset L \rightarrow V \subset L'$  is called a  $\Lambda$ -smooth map if for any  $x \in U$  there exist an open subset  $W(x) \subset U$  and a system of smooth maps  $\{g_\alpha : \xi_\alpha(W) \rightarrow \xi'_\alpha(g(W))\}_{\alpha \in \Lambda}$  such that  $g_\alpha \circ \xi_\alpha = \xi'_\alpha \circ g$  for any  $\alpha \in \Lambda$ .

**Definition.** Let  $M$  be a  $\xi$ -manifold and  $M'$  a  $\xi'$ -manifold. A smooth map  $f : M \rightarrow M'$  between a  $\xi$ -manifold  $M$  and a  $\xi'$ -manifold  $M'$  is called a  $\Lambda$ -smooth map if it is  $\Lambda$ -smooth in terms of the atlases defining  $\xi$ - and  $\xi'$ -structures on these manifolds.

For a fixed finite partially ordered set  $\Lambda$ , all  $\xi$ -manifolds for all projective systems  $\xi$  over  $\Lambda$  together with  $\Lambda$ -smooth maps as morphisms form a subcategory  $\mathcal{M}f_{\text{proj}}(\Lambda)$  of the category  $\mathcal{M}f$ .

**Proposition 1.4.** *The category  $\mathcal{M}f_{\text{proj}}(\Lambda)$  admits products.*

**Proof.** Let  $M$  be a  $\xi$ -manifold and  $M'$  a  $\xi'$ -manifold, where

$$\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L) \text{ and } \xi' = (L'_\alpha, \xi'^\beta_\alpha, \Lambda, L').$$

Then  $M \times M'$  is a  $(\xi \times \xi')$ -manifold, where  $\xi \times \xi'$  denotes the projective system  $(L_\alpha \times L'_\alpha, \xi_\alpha^\beta \times \xi'^\beta_\alpha, \Lambda, L \times L')$ .  $\square$

## 2. MULTIFOLIATE MANIFOLDS

Multifoliate structures on smooth manifolds were introduced by K. Kodaira and D.C. Spencer [4] as follows.

**Definition.** A pair  $(\Lambda, p)$  consisting of a finite partially ordered set  $\Lambda$  and a surjective map

$$p : \{1, \dots, n\} \ni i \mapsto p(i) \in \Lambda$$

is called a *multifoliate structure on the set*  $\{1, \dots, n\}$ .

Denote by  $GL(\Lambda, p)$  the group of all linear isomorphisms

$$f : \mathbb{R}^n \ni (x^i) \mapsto (f^i_j x^j) \in \mathbb{R}^n$$

satisfying the condition

$$f^i_j = 0 \quad \text{if } p(i) \not\geq p(j),$$

and by  $\Gamma(\Lambda, p)$  the pseudogroup of all local diffeomorphisms  $g : U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^n$  such that  $dg_x \in GL(\Lambda, p)$  for all  $x \in U$ .

**Definition.** A  $(\Lambda, p)$ -*multifoliate structure* on an  $n$ -dimensional smooth manifold is a maximal atlas compatible with the pseudogroup  $\Gamma(\Lambda, p)$ . We call the local coordinates determined by a chart of this atlas *adapted coordinates*. A smooth manifold endowed with a  $(\Lambda, p)$ -multifoliate structure is called a  $(\Lambda, p)$ -*multifoliate manifold*.

**Definition.** Let  $M$  be a  $(\Lambda, p)$ -multifoliate manifold and  $N$  be a  $(\Lambda, p')$ -multifoliate manifold. A  $\Lambda$ -*multifoliate map*  $f : M \rightarrow N$  is a smooth map, satisfying the condition

$$\frac{\partial f^a}{\partial x^i} = 0 \quad \text{if } p'(a) \not\geq p(i)$$

in adapted coordinates. Clearly, this definition does not depend on the choice of a local coordinate system.

For a fixed finite partially ordered set  $\Lambda$ , all  $(\Lambda, p)$ -multifoliate manifolds for all surjective maps  $p : \{1, \dots, n\} \rightarrow \Lambda$  and for all natural numbers  $n \geq \text{card } \Lambda$  together with  $\Lambda$ -multifoliate maps as morphisms form a subcategory  $\mathcal{M}f_\Lambda$  of the category  $\mathcal{M}f$ . We call it *the category of multifoliate manifolds over*  $\Lambda$ .

**Proposition 2.1.** *The category  $\mathcal{M}f_\Lambda$  admits products.*

**Proof.** If  $M_a$  is a  $(\Lambda, p_a)$ -multifoliate manifold,  $p_a : \{1, \dots, n_a\} \rightarrow \Lambda$ ,  $a = 1, 2$ , then the product  $M_1 \times M_2$  is a  $(\Lambda, p)$ -multifoliate manifold, where  $p : \{1, \dots, n_1 + n_2\} \rightarrow \Lambda$  is defined by

$$p(i) = \begin{cases} p_1(i), & i \leq n_1; \\ p_2(i - n_1), & i > n_1. \end{cases}$$

□

**Definition.** We say that two multifoliate structures  $(\Lambda, p)$  and  $(\Lambda', p')$  on the same set  $\{1, \dots, n\}$  are *equivalent* if there exists a linear automorphism  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\varphi \circ f \circ \varphi^{-1} \in GL(\Lambda', p')$  for any  $f \in GL(\Lambda, p)$  and

$$\Phi : f \in GL(\Lambda, p) \mapsto \varphi \circ f \circ \varphi^{-1} \in GL(\Lambda', p')$$

is a group isomorphism.

Clearly, if  $(\Lambda, p)$  is a multifoliate structure on  $\{1, \dots, n\}$ , then, for each permutation  $\sigma$  on  $\{1, \dots, n\}$ , the multifoliate structure  $(\Lambda, p \circ \sigma)$  is equivalent to  $(\Lambda, p)$ .

For a multifoliate structure  $(\Lambda, p)$  on  $\{1, \dots, n\}$ , define the sets  $H_\alpha = \{i \mid p(i) \leq \alpha\}$ ,  $\alpha \in \Lambda$ , and let  $k(\alpha) = \text{card } H_\alpha$ . The vector spaces

$$L_\alpha = \{(x^{i_1}, \dots, x^{i_{k(\alpha)}}) \mid x^{i_s} \in \mathbb{R}, i_s \in H_\alpha, s = 1, \dots, k(\alpha)\}$$

and the natural epimorphisms  $\text{pr}_\alpha^\beta : L_\beta \rightarrow L_\alpha$ ,  $\alpha \leq \beta$ , form a projective system whose limit can be naturally identified with  $\mathbb{R}^n$ . Denote this system and its completion, respectively, by  $\xi(\Lambda, p)$  and  $\tilde{\xi}(\Lambda, p)$ .

**Theorem 2.1.** [12] *The correspondence  $(\Lambda, p) \mapsto \tilde{\xi}(\Lambda, p)$  induces a bijection between the equivalence classes of multifoliate structures  $(\Lambda, p)$  and the equivalence classes of finite complete projective systems of vector space epimorphisms.*

**Proof.** We give here a sketch of the proof and refer for details to [12].

Show first that the correspondence  $(\Lambda, p) \mapsto \tilde{\xi}(\Lambda, p)$  induces a map from the set of equivalence classes of multifoliate structures to the set of equivalence classes of finite complete projective systems of vector space epimorphisms. By Propositions 1.2 and 1.3, it suffices to show that the groups  $GL(\Lambda, p)$  and  $GL(\xi(\Lambda, p))$  are isomorphic. In fact, there is a natural isomorphism  $GL(\Lambda, p) \rightarrow GL(\xi(\Lambda, p))$  which assigns to  $g \in GL(\Lambda, p)$  a collection of maps  $\{g_\alpha : L_\alpha \rightarrow L_\alpha\}$  defined as follows:  $g_\alpha(y_\alpha) = \text{pr}_\alpha(g(y))$  where  $y \in \mathbb{R}^n$  is such that  $\text{pr}_\alpha(y) = y_\alpha$ .

To prove that the correspondence indicated in the theorem is one-to-one, we need to pass to the dual inductive system [1].

Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  be a finite complete projective system and let  $\varepsilon \in \Lambda$  be the greatest element. The dual spaces  $L_\alpha^*$  together with the dual maps  $(\xi_\alpha^\beta)^*$  form an inductive system  $\xi^* = (L_\alpha^*, (\xi_\alpha^\beta)^*, \Lambda)$ . The existence of the greatest element implies that the inductive limit of  $\xi^*$  exists and can be identified with the dual space  $L^*$ . Under this identification the dual maps  $\xi_\alpha^* : L_\alpha^* \rightarrow L^*$  are the canonical maps of  $\xi^*$ . Obviously, all the maps  $(\xi_\alpha^\beta)^*$  and  $\xi_\alpha^*$  are monomorphisms. We will call the inductive system  $\xi^* = (L_\alpha^*, (\xi_\alpha^\beta)^*, \Lambda, L^*)$  the *dual of  $\xi$* .

For any  $f \in GL(\xi)$  and for each  $\alpha \in \Lambda$ , we have  $\xi_\alpha^* \circ f_\alpha^* = f^* \circ \xi_\alpha^*$ .

Denote by  $GL(\xi^*)$  the group of all linear automorphisms  $h : L^* \rightarrow L^*$  which are the limits of inductive systems of linear automorphisms  $h_\alpha : L_\alpha^* \rightarrow L_\alpha^*$ . Since the maps  $\xi_\alpha^*$  are monomorphisms, it will be convenient to consider each  $L_\alpha^*$  as a subspace of  $L^*$ . Then  $h_\alpha = h|L_\alpha^*$  or, in other words,  $h$  maps  $L_\alpha^*$  into itself. The correspondence  $f \mapsto f^*$  is an isomorphism of the groups  $GL(\xi)$  and  $GL(\xi^*)$ .

The dual system  $\xi^*$  is *complete* in the sense that it contains all subspaces which are invariant with respect to each  $f^* \in GL(\xi^*)$ .

By a *chain* in  $\xi^*$  we mean a sequence of embeddings

$$L_{\alpha_k}^* \xleftarrow{(\xi_{\alpha_{k-1}}^{\alpha_k})^*} L_{\alpha_{k-1}}^* \xleftarrow{(\xi_{\alpha_{k-2}}^{\alpha_{k-1}})^*} \dots \xleftarrow{(\xi_{\alpha_1}^{\alpha_2})^*} L_{\alpha_1}^*$$

such that  $\alpha_1 < \alpha_2 < \dots < \alpha_k$  and  $\alpha_i$  is the successor of  $\alpha_{i-1}$  in  $\Lambda$ ,  $i = 2, \dots, k$ , (that is,  $\alpha_{i-1} \leq \beta \leq \alpha_i$  implies that either  $\beta = \alpha_{i-1}$  or  $\beta = \alpha_i$ ). The space  $L_{\alpha_k}^*$  is called the *end* of the chain.

$L_\alpha^*$  is said to be a *subspace of the first floor* if  $\alpha$  is a minimal element of  $\Lambda$ .  $L_\alpha^*$  is said to be a *subspace of the  $k$ -th floor* if each chain with end  $L_\alpha^*$  is of length no greater than  $k$  and among all such chains there is at least one of length  $k$ .

If  $L_\alpha^*$  is a subspace of the first floor, we take a basis  $B_\alpha = \{e_\alpha^1, \dots, e_\alpha^{s(\alpha)}\}$  in  $L_\alpha^*$  and call the index  $\alpha$  *distinguished*. Let  $C_1$  be the union of  $B_\alpha$  for all subspaces of the first floor. One can verify that the system  $C_1$  is linearly independent. In fact, the assumption that the system is linearly dependent contradicts the completeness of  $\xi^*$  (see [12] for details).

Let now  $L_\beta^*$  be a space of the second floor. Then either  $L_\beta^* \subset \mathcal{L}\{C_1\}$ , where  $\mathcal{L}\{C_1\}$  is the linear span of the system  $C_1$ , or one can choose a system of linearly independent elements  $B_\beta = \{e_\beta^1, \dots, e_\beta^{s(\beta)}\}$  in  $L_\beta^*$  such that  $L_\beta^* = \mathcal{L}\{e_\beta^1, \dots, e_\beta^{s(\beta)}\} \oplus (L_\beta^* \cap \mathcal{L}\{C_1\})$ . In the latter case the index  $\beta$  is also called *distinguished*. Let  $C_2$  be the union of  $B_\beta$  for all subspaces

of the second floor. The system  $C_1 \cup C_2$  is linearly independent (see [12] for details).

Suppose that we have chosen systems  $C_\ell$  for every  $\ell \leq k$ . If  $L_\gamma^*$  is a space of  $(k+1)$ -th floor, then either  $L_\gamma^* \subset \mathcal{L}_k := \mathcal{L}\{C_1 \cup \dots \cup C_k\}$  or there exists a system of linearly independent elements  $B_\gamma = \{e_\gamma^1, \dots, e_\gamma^{s(\gamma)}\}$  such that  $L_\gamma^* = (L_\gamma^* \cap \mathcal{L}_k) \oplus \mathcal{L}\{e_\gamma^1, \dots, e_\gamma^{s(\gamma)}\}$ . In the latter case the index  $\gamma$  is called *distinguished*. Let  $C_{k+1}$  be the union of  $B_\gamma$  for all subspaces of the  $(k+1)$ -st floor. As above, the system  $C_1 \cup C_2 \cup \dots \cup C_{k+1}$  is linearly independent.

This process stops when we reach  $L^*$ . As a result, we obtain a subset  $\Lambda' \subset \Lambda$  consisting of distinguished elements and the corresponding basis  $\{e^1, \dots, e^n\}$  in  $L^*$ . Let  $p : \{1, \dots, n\} \rightarrow \Lambda'$  be the map defined as follows:  $p(m) = \alpha$  where  $\alpha$  is the minimal distinguished element such that  $e^m \in L_\alpha^*$ . The pair  $(\Lambda', p)$  is a multifoliate structure on  $\{1, \dots, n\}$  and the group  $GL(\xi)$  is isomorphic to  $GL(\Lambda', p)$ .  $\square$

**Corollary 2.1.** *For any finite partially ordered set  $\Lambda$ , the categories  $\mathcal{M}f_{\text{proj}}(\Lambda)$  and  $\mathcal{M}f_\Lambda$  are isomorphic.*

**Corollary 2.2.** *Let  $(\Lambda, p)$  be a multifoliate structure on  $\{1, \dots, n\}$  and  $\xi(\Lambda, p)$  the corresponding projective system. Let  $\tilde{\xi}(\Lambda, p) = (\tilde{L}_a, \tilde{\xi}_a^b, \tilde{\Lambda}, \tilde{L})$  be the completion of  $\xi(\Lambda, p)$  and  $(\Lambda', p')$  the multifoliate structure on  $\{1, \dots, n\}$  determined by  $\tilde{\xi}(\Lambda, p)$ . Then*

- (1) *the partially ordered sets  $\Lambda$  and  $\Lambda'$  are canonically isomorphic;*
- (2) *the multifoliate structures  $(\Lambda, p)$  and  $(\Lambda', p')$  are equivalent.*

**Proof.** (1) Every invariant subspace of  $\tilde{\xi}(\Lambda, p)$  is of the form

$$\tilde{L}_{(\alpha_1, \dots, \alpha_k)} = L / (\ker \xi_{\alpha_1} \cap \dots \cap \ker \xi_{\alpha_k}),$$

where  $\alpha_1, \dots, \alpha_k \in \Lambda$  are pairwise incomparable. Thus,  $\tilde{\Lambda}$  is isomorphic to the set of all finite collections of pairwise incomparable elements  $(\alpha_1, \dots, \alpha_k)$  endowed with the partial order defined as follows:  $(\alpha_1, \dots, \alpha_k) \leq (\beta_1, \dots, \beta_\ell)$  if and only if  $\ker \xi_{\beta_1} \cap \dots \cap \ker \xi_{\beta_\ell} \subseteq \ker \xi_{\alpha_1} \cap \dots \cap \ker \xi_{\alpha_k}$ .

The index  $(\alpha_1, \dots, \alpha_k) \in \tilde{\Lambda}$  is distinguished if and only if  $k = 1$ , and so the set of all distinguished elements is naturally isomorphic to  $\Lambda$ .

(2) From Theorem 2.1 it follows that  $GL(\Lambda, p) \cong GL(\xi(\Lambda, p))$  and  $GL(\tilde{\xi}(\Lambda, p)) \cong GL(\Lambda', p')$ . The rest of the proof follows from Proposition 1.2.  $\square$

**Corollary 2.3.** *If  $(\Lambda, p)$  and  $(\Omega, q)$  are equivalent multifoliate structures on  $\{1, \dots, n\}$ , then*

- (1) *the partially ordered sets  $\Lambda$  and  $\Omega$  are isomorphic;*
- (2) *there exists a permutation  $\sigma$  on  $\{1, \dots, n\}$  such that  $q = p \circ \sigma$ .*

**Proof.** (1) Let  $(\Lambda', p')$  and  $(\Omega', q')$  be the multifoliate structures corresponding to the complete projective systems  $\tilde{\xi}(\Lambda, p)$  and  $\tilde{\xi}(\Omega, q)$  respectively. The systems  $\tilde{\xi}(\Lambda, p)$  and  $\tilde{\xi}(\Omega, q)$  are equivalent. By Proposition 1.3, these systems are isomorphic. Hence the sets  $\Lambda'$  and  $\Omega'$  of their distinguished elements are isomorphic. By Corollary 2.2,  $\Lambda$  and  $\Omega$  are isomorphic.

(2) Let  $\omega : \Lambda \ni \alpha \mapsto \omega(\alpha) \in \Omega$  be an isomorphism. Recall that, for any distinguished index  $\beta \in \Lambda' \cong \Lambda$ ,  $s(\beta)$  denotes the number of linearly independent elements in the system  $B_\beta$  defined in the proof of Theorem 2.1. One can easily see that the cardinality of the subset  $p^{-1}(\beta) \subset \{1, \dots, n\}$  coincides with  $s(\beta)$ . Since the projective systems  $\tilde{\xi}(\Lambda, p)$  and  $\tilde{\xi}(\Omega, q)$  are isomorphic, for every distinguished index  $\beta \in \Lambda' \cong \Lambda$ , the numbers  $s(\beta)$  and  $s(\omega(\beta))$  coincide. This means that  $p^{-1}(\alpha)$  and  $q^{-1}(\omega(\alpha))$  have the same cardinality for any  $\alpha \in \Lambda$ . From this observation it follows that one can find a permutation  $\sigma$  on  $\{1, \dots, n\}$  such that  $q = p \circ \sigma$ . In general, such a permutation is not unique.  $\square$

3. MULTIFIBERED MANIFOLDS. THE CLASSIFICATION THEOREM

**Definition.** Let  $\xi = (L_\alpha, \xi_\alpha^\beta, \Lambda, L)$  be a projective system of vector spaces and let  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  be a projective system such that all  $M_\alpha$  and  $M = \varprojlim M_\alpha$  are smooth manifolds and all maps  $\pi_\alpha^\beta : M_\beta \rightarrow M_\alpha$  and  $\pi_\alpha : M \rightarrow M_\alpha$  are surjective submersions. Let  $\mathcal{A}$  be a  $\xi$ -structure on  $M$ . We call  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  a *multifibered manifold* if the  $\xi$ -structure  $\mathcal{A}$  on  $M$  is compatible with all projections  $\pi_\alpha$  in the following sense: for any point  $x = (x_\alpha) \in M$ , there are charts  $(U, h)$  centered at  $x$  on  $M$  and  $(U_\alpha, h_\alpha)$  centered at  $x_\alpha$  on  $M_\alpha$ ,  $\alpha \in \Lambda$ , such that the following diagram commutes

$$\begin{array}{ccc}
 U & \xrightarrow{h} & L \\
 \pi_\alpha \downarrow & & \downarrow \xi_\alpha \\
 U_\alpha & \xrightarrow{h_\alpha} & L_\alpha .
 \end{array}$$

It follows from Corollary 2.1 that  $M$  carries a structure of multifoliate manifold. For any point  $x = (x_\alpha) \in M$  the projective system of tangent spaces  $\xi_x = (T_{x_\alpha} M_\alpha, (d\pi_\alpha^\beta)_{x_\beta}, \Lambda, T_x M)$  is isomorphic to  $\xi$ .

**Definition.** A multifibered map  $f : \pi \rightarrow \bar{\pi}$  between two multifibered manifolds  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  and  $\bar{\pi} = (\bar{M}_\alpha, \bar{\pi}_\alpha^\beta, \Lambda, \bar{M})$  is a collection of maps  $\{f_\alpha : M_\alpha \rightarrow \bar{M}_\alpha\}_{\alpha \in \Lambda}$  such that for all  $\alpha \leq \beta$  the diagram

$$\begin{array}{ccc} M_\beta & \xrightarrow{f_\beta} & \bar{M}_\beta \\ \pi_\alpha^\beta \downarrow & & \downarrow \bar{\pi}_\alpha^\beta \\ M_\alpha & \xrightarrow{f_\alpha} & \bar{M}_\alpha \end{array}$$

commutes. Each multifibered map determines a unique smooth map  $f : M \rightarrow \bar{M}$ .

Multifibered manifolds over  $\Lambda$  together with multifibered maps form a subcategory of the category  $\mathcal{M}f_\Lambda$  of multifoliate manifolds over  $\Lambda$ . We denote it by  $\mathcal{F}M_\Lambda$ .

**Proposition 3.1.** *The category  $\mathcal{F}M_\Lambda$  admits products.*

**Proof.** If  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  and  $\bar{\pi} = (\bar{M}_\alpha, \bar{\pi}_\alpha^\beta, \Lambda, \bar{M})$  are two multifibered manifolds, then their product is  $\pi \times \bar{\pi} := (M_\alpha \times \bar{M}_\alpha, \pi_\alpha^\beta \times \bar{\pi}_\alpha^\beta, \Lambda, M \times \bar{M})$ .  $\square$

The categories  $\mathcal{F}M_\Lambda$  and  $\mathcal{M}f_\Lambda$  are local categories over manifolds.

**Definition.** An *inductive system of Weil algebra homomorphisms* over  $\Lambda$  is a collection  $\mu = (A_\alpha, \mu_\alpha^\beta, \Lambda)$  consisting of Weil algebras  $A_\alpha$ ,  $\alpha \in \Lambda$ , and Weil algebra homomorphisms  $\mu_\beta^\alpha : A_\alpha \rightarrow A_\beta$ ,  $\alpha \leq \beta$ , such that  $\mu_\alpha^\alpha = \text{id}_{A_\alpha}$  for all  $\alpha \in \Lambda$  and  $\mu_\gamma^\beta \circ \mu_\beta^\alpha = \mu_\gamma^\alpha$  when  $\alpha \leq \beta \leq \gamma$ . Let  $\mu = (A_\alpha, \mu_\alpha^\beta, \Lambda)$  and  $\bar{\mu} = (\bar{A}_\alpha, \bar{\mu}_\alpha^\beta, \Lambda)$  be two inductive systems of Weil algebra homomorphisms. By a *morphism*  $\nu : \mu \rightarrow \bar{\mu}$  we mean a family  $\nu = (\nu_\alpha)_{\alpha \in \Lambda}$  of Weil algebra homomorphisms  $\{\nu_\alpha : A_\alpha \rightarrow \bar{A}_\alpha\}$  such that for all  $\alpha \leq \beta$  the diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\nu_\alpha} & \bar{A}_\alpha \\ \mu_\beta^\alpha \downarrow & & \downarrow \bar{\mu}_\beta^\alpha \\ A_\beta & \xrightarrow{\nu_\beta} & \bar{A}_\beta \end{array}$$

commutes.

**Theorem 3.1.** *Any product preserving bundle functor  $F$  on the category  $\mathcal{F}M_\Lambda$  or  $\mathcal{M}f_\Lambda$  is uniquely determined by the inductive system  $\mu = (\mu_\beta^\alpha : A_\alpha \rightarrow A_\beta)$  of Weil algebra homomorphisms. Any natural transformation  $\eta : F \rightarrow \bar{F}$  is uniquely determined by the morphism  $\nu : \mu \rightarrow \bar{\mu}$  of inductive systems of Weil algebra homomorphisms.*

Since, by Theorem 2.1, any multifoliate manifold is locally a multi-fibered manifold, it is enough to consider the case of a bundle functor  $F : \mathcal{FM}_\Lambda \rightarrow \mathcal{FM}$ .

The proof of the Theorem 3.1 is essentially the same as the Mikulski's proof [9] for the case of a bundle functor  $\mathcal{FM} \rightarrow \mathcal{FM}$ . We will reproduce the main scheme of the proof.

Let  $\mu = (G_\alpha, \mu_\beta^\alpha, \Lambda)$  be an inductive system of natural transformations of bundle functors, i.e., for any  $\alpha \in \Lambda$ , there is given a bundle functor  $G_\alpha : \mathcal{M}f \rightarrow \mathcal{FM}$  and for any  $\alpha, \beta \in \Lambda$  such that  $\alpha \leq \beta$ , there is given a natural transformation  $\mu_\beta^\alpha : G_\alpha \rightarrow G_\beta$  with the properties  $\mu_\beta^\alpha \circ \mu_\gamma^\beta = \mu_\gamma^\alpha$  and  $\mu_\alpha^\alpha = \text{id}$ . We define a bundle functor  $\prod_\mu G_\alpha : \mathcal{FM}_\Lambda \rightarrow \mathcal{FM}$  as follows.

Consider a multifibered manifold  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$ . We let

$$\prod_\mu G_\alpha(\pi) := \{(x_\alpha) \mid G_\beta(\pi_\alpha^\beta)(x_\beta) = \mu_\beta^\alpha(M_\alpha)(x_\alpha)\} \subset \prod_{\alpha \in \Lambda} G_\alpha(M_\alpha).$$

The set  $\prod_\mu G_\alpha(\pi)$  is a submanifold in  $\prod_{\alpha \in \Lambda} G_\alpha(M_\alpha)$ . We define the map

$$p_\mu(\pi) : \prod_\mu G_\alpha(\pi) \rightarrow M$$

as follows. Consider the bundle projection

$$\prod_{\alpha \in \Lambda} G_\alpha(M_\alpha) \rightarrow \prod_{\alpha \in \Lambda} M_\alpha.$$

The image of its restriction to  $\prod_\mu G_\alpha(\pi)$  coincides with  $M = \varprojlim M_\alpha$ , thus defining the map  $p_\mu(\pi) : \prod_\mu G_\alpha(\pi) \rightarrow M$  which is a surjective submersion.

Let  $f = (f_\alpha) : \pi \rightarrow \bar{\pi}$  be a multifibered map. We set

$$\prod_\mu G_\alpha(f) := \text{the restriction of } \prod_{\alpha \in \Lambda} G_\alpha(f_\alpha).$$

The map

$$\prod_\mu G_\alpha(f) : \prod_\mu G_\alpha(\pi) \rightarrow \prod_\mu G_\alpha(\bar{\pi})$$

is well-defined since all  $\mu_\beta^\alpha$  are natural transformations.

The correspondence

$$\prod_\mu G_\alpha : \mathcal{FM}_\Lambda \rightarrow \mathcal{FM}$$

is a bundle functor.

Now let  $\bar{\mu} = (\bar{\mu}_\beta^\alpha : \bar{G}_\alpha \rightarrow \bar{G}_\beta)$  be another inductive system of natural transformations of bundle functors. Suppose that there is given a family

$\nu = (\nu_\alpha : G_\alpha \rightarrow \overline{G}_\alpha)$  of natural transformations such that, for any manifold  $X$  and  $\alpha \leq \beta$ , the diagram

$$\begin{array}{ccc} G_\alpha(X) & \xrightarrow{\nu_\alpha(X)} & \overline{G}_\alpha(X) \\ \mu_\beta^\alpha(X) \downarrow & & \downarrow \overline{\mu}_\beta^\alpha(X) \\ G_\beta(X) & \xrightarrow{\nu_\beta(X)} & \overline{G}_\beta(X) \end{array} \quad (1)$$

commutes. Then we define the natural transformation

$$\prod_{\mu, \overline{\mu}} \nu_\alpha : \prod_\mu G_\alpha \rightarrow \prod_{\overline{\mu}} \overline{G}_\alpha$$

as follows.

For a multifibered manifold  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$ , we define the map

$$\prod_{\mu, \overline{\mu}} \nu_\alpha(\pi) : \prod_\mu G_\alpha(\pi) \rightarrow \prod_{\overline{\mu}} \overline{G}_\alpha(\pi)$$

to be the restriction of  $\prod_{\alpha \in \Lambda} \nu_\alpha(M_\alpha)$ . Since each  $\nu_\alpha$  is a natural transformation, the map  $\prod_{\mu, \overline{\mu}} \nu_\alpha(\pi)$  is well-defined. The family

$$\prod_{\mu, \overline{\mu}} \nu_\alpha = \left\{ \prod_{\mu, \overline{\mu}} \nu_\alpha(\pi) \right\} : \prod_\mu G_\alpha \rightarrow \prod_{\overline{\mu}} \overline{G}_\alpha$$

is a natural transformation.

Let us denote by  $pt$  a one-point manifold. Consider a smooth manifold  $X$ . For any  $\alpha \in \Lambda$ , we construct a multifibered manifold  $i_\alpha(X) = (X_\gamma, r_\delta^\gamma, \Lambda, X)$  in the following way. We let  $X_\gamma = X$  if  $\gamma \geq \alpha$ , and  $X_\gamma = pt$  otherwise. Each projection  $r_\delta^\gamma$  is either the identity map  $\text{id}_X : X \rightarrow X$  if  $\gamma \geq \alpha$ ,  $\delta \geq \alpha$ , or the unique map  $pt_X : X \rightarrow pt$  if  $\gamma \geq \alpha$ ,  $\delta \not\geq \alpha$ , or the unique map  $pt \rightarrow pt$  if  $\gamma \not\geq \alpha$ ,  $\delta \not\geq \alpha$ . Clearly,  $\varprojlim X_\alpha = X$ . We can consider any map  $f : X \rightarrow Y$  as a multifibered map  $f : i_\alpha(X) \rightarrow i_\beta(Y)$  for  $\alpha \leq \beta$ . Thus we obtain the bundle functors

$$i_\alpha : \mathcal{M}f \rightarrow \mathcal{F}M_\Lambda$$

and the natural transformations

$$\text{id}_\beta^\alpha : i_\alpha \rightarrow i_\beta, \quad \alpha \leq \beta,$$

consisting of  $\mathcal{F}M_\Lambda$ -morphisms  $\text{id}_X : i_\alpha(X) \rightarrow i_\beta(X)$ . Obviously, the functors  $i_\alpha$  preserve products.

Let  $F : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be a bundle functor. Consider the bundle functors

$$G_\alpha^F := F \circ i_\alpha : \mathcal{M}f \rightarrow \mathcal{F}M. \quad (2)$$

If  $F$  preserves products, then the functors  $G_\alpha^F$  also preserve products.

We define an inductive system  $\mu^F = ((\mu^F)_\beta^\alpha)$  of natural transformations as follows:

$$(\mu^F)_\beta^\alpha := F(\text{id}_\beta^\alpha) : G_\alpha^F \rightarrow G_\beta^F. \quad (3)$$

Let  $\overline{F} : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be another bundle functor, and let  $\eta = \{\eta_\pi\} : F \rightarrow \overline{F}$  be a natural transformation. We define the family of natural transformations

$$\nu^\eta = (\nu_\alpha^\eta : G_\alpha^F \rightarrow G_\alpha^{\overline{F}})$$

by

$$\nu_\alpha^\eta(X) := \eta_{i_\alpha(X)} : G_\alpha^F(X) \rightarrow G_\alpha^{\overline{F}}(X) \quad (4)$$

for any manifold  $X$ . The diagram

$$\begin{array}{ccc} G_\alpha^F(X) & \xrightarrow{\nu_\alpha^\eta(X)} & G_\alpha^{\overline{F}}(X) \\ (\mu^F)_\beta^\alpha(X) \downarrow & & \downarrow (\mu^{\overline{F}})_\beta^\alpha(X) \\ G_\beta^F(X) & \xrightarrow{\nu_\beta^\eta(X)} & G_\beta^{\overline{F}}(X) \end{array}$$

commutes for any manifold  $X$  and any  $\alpha \leq \beta$ .

Let  $F : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be a bundle functor. Following Mikulski, we construct a natural transformation

$$\Theta = \{\Theta_\pi\} : F \rightarrow \prod_{\mu^F} G_\alpha^F.$$

Let  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  be a multifibered manifold. For any  $\alpha \in \Lambda$  we define a multifibered map  $j_\alpha : \pi \rightarrow i_\alpha(M_\alpha)$  as follows: we let  $(j_\alpha)_\gamma := \pi_\alpha^\gamma$  if  $\alpha \leq \gamma$  and  $(j_\alpha)_\gamma := pt_{M_\gamma}$  otherwise.

The image of the map

$$\prod_{\alpha \in \Lambda} F(j_\alpha) : F(\pi) \rightarrow \prod_{\alpha \in \Lambda} F(i_\alpha(M_\alpha)) = \prod_{\alpha \in \Lambda} G_\alpha^F(M_\alpha)$$

is contained in  $\prod_{\mu^F} G_\alpha^F(\pi)$ . Therefore, the map

$$\Theta_\pi := \prod_{\alpha \in \Lambda} F(j_\alpha) : F(\pi) \rightarrow \prod_{\mu^F} G_\alpha^F(\pi) \subset \prod_{\alpha \in \Lambda} G_\alpha^F(M_\alpha).$$

is well-defined.

The family  $\Theta = \{\Theta_\pi\} : F \rightarrow \prod_{\mu^F} G_\alpha^F$  is a natural transformation.

Let now  $\mu = (\mu_\beta^\alpha : G_\alpha \rightarrow G_\beta)$  be an inductive system of natural transformations of bundle functors  $G_\alpha : \mathcal{M}f \rightarrow \mathcal{F}M$ . Consider the corresponding bundle functor  $F := \prod_\mu G_\alpha : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$ . Denote by

$\overset{\circ}{\mu} = (\overset{\circ}{\mu}_\beta^\alpha : \overset{\circ}{G}_\alpha \rightarrow \overset{\circ}{G}_\beta)$  the corresponding inductive system of natural transformations (3). Then

$$\overset{\circ}{G}_\gamma(X) = \{x_\alpha \in G_\alpha(X_\alpha) \mid (\mu^F)_\beta^\alpha(M_\alpha)(x_\alpha) = G_\beta^F(\pi_\alpha^\beta)(x_\beta)\} \subset \prod_{\alpha \in \Lambda} G_\alpha(X_\alpha),$$

where  $X_\alpha = X$  for  $\alpha \geq \gamma$ , otherwise  $X_\alpha = pt$ .

For any manifold  $X$  and for any  $\alpha \in \Lambda$ , we define the map

$$\mathcal{O}_\alpha(X) : \overset{\circ}{G}_\alpha(X) \rightarrow G_\alpha(X) \quad (5)$$

as the restriction of the standard projection  $\prod_{\beta \in \Lambda} G_\beta(X_\beta) \rightarrow G_\alpha(X)$ .

The families

$$\mathcal{O}_\alpha = \{\mathcal{O}_\alpha(X)\} : \overset{\circ}{G}_\alpha \rightarrow G_\alpha$$

are natural transformations. They all are natural equivalences if and only if every map  $\mu_\beta^\alpha(pt) : G_\alpha(pt) \rightarrow G_\beta(pt)$  is a diffeomorphism. The diagram

$$\begin{array}{ccc} \overset{\circ}{G}_\alpha(X) & \xrightarrow{\mathcal{O}_\alpha(X)} & G_\alpha(X) \\ \overset{\circ}{\mu}_\beta^\alpha(X) \downarrow & & \downarrow \mu_\beta^\alpha(X) \\ \overset{\circ}{G}_\beta(X) & \xrightarrow{\mathcal{O}_\beta(X)} & G_\beta(X) \end{array}$$

is commutative for any manifold  $X$  and any  $\alpha \leq \beta$ .

Suppose now that the inductive system  $\mu = (\mu_\beta^\alpha : G_\alpha \rightarrow G_\beta)$  satisfies the condition that all the maps  $\mu_\beta^\alpha(pt) : G_\alpha(pt) \rightarrow G_\beta(pt)$  are diffeomorphisms.

For any multifibered manifold  $\pi = (M_\alpha, \pi_\alpha^\beta, \Lambda, M)$  we let

$$T^\mu(\pi) = \begin{cases} G_\alpha(X) & \text{if } \pi = i_\alpha(X) \text{ for some } X \in \mathcal{M}f, \alpha \in \Lambda, \\ \prod_\mu G_\alpha(\pi) & \text{otherwise.} \end{cases}$$

Then  $T^\mu(\pi)$  is a fibered manifold over  $M = \varprojlim M_\alpha$ . We also define the map

$$I_\pi : T^\mu(\pi) \rightarrow \prod_\mu G_\alpha(\pi)$$

as follows:

$$I_\pi = \begin{cases} \mathcal{O}_\alpha^{-1}(X) & \text{if } \pi = i_\alpha(X), \\ \text{id}_{\prod_\mu G_\alpha(\pi)} & \text{otherwise,} \end{cases}$$

where  $\mathcal{O}_\alpha(X) : \overset{\circ}{G}_\alpha(X) = \prod_\mu G_\alpha(i_\alpha(X)) \rightarrow G_\alpha(X)$  are defined by (5).

We let

$$T^\mu(f) := I_{\bar{\pi}}^{-1} \circ \prod_{\alpha \in \Lambda} G_\alpha(f_\alpha) \circ I_\pi.$$

The correspondence  $T^\mu$  thus defined is a bundle functor  $\mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$ , and the family

$$I = \{I_\pi\} : T^\mu \rightarrow \prod_\mu G_\alpha$$

is a natural transformation.

If all  $G_\alpha$  preserve products, then  $G_\alpha(pt) = pt$ , hence the maps  $\mu_\beta^\alpha(pt)$  are diffeomorphisms. In this case,  $I$  is a natural equivalence and the functor  $T^\mu$  also preserves products.

Let now  $\bar{\mu} = (\bar{\mu}_\beta^\alpha : \bar{G}_\alpha \rightarrow \bar{G}_\beta)$  be another inductive system of natural transformations such that all  $\bar{\mu}_\beta^\alpha(pt)$  are diffeomorphisms, and let  $\nu = (\nu_\alpha : G_\alpha \rightarrow \bar{G}_\alpha)$  be a family of natural transformations such that the diagram (1) is commutative. Following Mikulski, we define a natural transformation  $\tilde{\nu} = \{\tilde{\nu}_\pi\} : T^\mu \rightarrow T^{\bar{\mu}}$  to be the composition

$$\tilde{\nu}_\pi : T^\mu(\pi) \xrightarrow{I_\pi} \prod_\mu G_\alpha(\pi) \xrightarrow{\Pi_{\mu, \bar{\mu}} \nu_\alpha(\pi)} \prod_{\bar{\mu}} \bar{G}_\alpha(\pi) \xrightarrow{\bar{I}_\pi^{-1}} T^{\bar{\mu}}(\pi)$$

for any multifibered manifold  $\pi$ .

In the case  $F = T^\mu$  the natural transformations  $(\mu^F)_\beta^\alpha : G_\alpha^F \rightarrow G_\beta^F$  coincide with  $\mu_\beta^\alpha$ , i.e.,

$$\mu^F = \mu \quad \text{if} \quad F = T^\mu.$$

Let  $F : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be a bundle functor such that  $(\mu^F)_\beta^\alpha(pt)$ ,  $\alpha \leq \beta$ , are diffeomorphisms.

Then we define a natural transformation  $\kappa = \{\kappa_\pi\} : F \rightarrow T^{\mu^F}$  to be the composition

$$\kappa_\pi : F(\pi) \xrightarrow{\Theta_\pi} \prod_{\mu^F} G_\alpha^F(\pi) \xrightarrow{I_\pi^{-1}} T^{\mu^F}(\pi) \quad (6)$$

for any multifibered manifold  $\pi$ .

The proofs of the following propositions are similar to the proofs of Theorems 2.1 and 2.2 in [9].

**Proposition 3.2.** (1) *Let  $F : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be a product preserving bundle functor. Then the natural transformation  $\kappa : F \rightarrow T^{\mu^F}$  is a natural equivalence.*

(2) *If  $\mu = (\mu_\beta^\alpha : G_\alpha \rightarrow G_\beta)$  is an inductive system of natural transformations between product preserving bundle functors  $G_\alpha : \mathcal{M}f \rightarrow \mathcal{F}M$  and  $\kappa$  is the natural transformation (6) for  $F = T^\mu$ , then  $\kappa : T^\mu \rightarrow T^\mu$  and  $\kappa_\pi = \text{id}_{T^\mu(\pi)}$  for any multifibered manifold  $\pi$ .*

(3) *For  $\mu = (\mu_\beta^\alpha : G_\alpha \rightarrow G_\beta)$  the functor  $T^\mu$  is a product preserving bundle functor on the category  $\mathcal{F}M_\Lambda$  unique up to a natural equivalence*

such that the natural transformation  $\mu^F$  corresponding to  $F = T^\mu$  coincides with  $\mu$ .

**Proposition 3.3.** *Let  $F, \overline{F} : \mathcal{F}M_\Lambda \rightarrow \mathcal{F}M$  be two product preserving bundle functors. Let  $\mu^F = ((\mu^F)_\beta : G_\alpha^F \rightarrow G_\beta^F)$  and  $\mu^{\overline{F}} = ((\mu^{\overline{F}})_\beta : G_\alpha^{\overline{F}} \rightarrow G_\beta^{\overline{F}})$  be the corresponding inductive systems of natural transformations. Let  $\nu = (\nu_\alpha : G_\alpha^F \rightarrow G_\alpha^{\overline{F}})$  be the family of natural transformations such that the diagram*

$$\begin{array}{ccc} G_\alpha^F(X) & \xrightarrow{\nu_\alpha(X)} & G_\alpha^{\overline{F}}(X) \\ (\mu^F)_\beta^\alpha(X) \downarrow & & \downarrow (\mu^{\overline{F}})_\beta^\alpha(X) \\ G_\beta^F(X) & \xrightarrow{\nu_\beta(X)} & G_\beta^{\overline{F}}(X) \end{array}$$

is commutative for any manifold  $X$ . Then the natural transformation  $\eta = \{\eta_\pi\} : F \rightarrow \overline{F}$  given by the compositions

$$\eta_\pi : F(\pi) \xrightarrow{\kappa_\pi} T^{\mu^F}(\pi) \xrightarrow{\tilde{\nu}_\pi} T^{\mu^{\overline{F}}}(\pi) \xrightarrow{\overline{\kappa}_\pi^{-1}} \overline{F}(\pi)$$

is the unique natural transformation  $F \rightarrow \overline{F}$  such that  $\nu_\alpha^\eta = \nu_\alpha$ , where  $\nu_\alpha^\eta$  is defined by (4).

**Definition.** We say that two bundle functors  $F$  and  $\overline{F}$  are *equivalent* if there exists a natural equivalence  $\eta : F \rightarrow \overline{F}$ . We say that two inductive systems of natural transformations  $\mu$  and  $\overline{\mu}$  are *equivalent* if there exists a family  $\nu = (\nu_\alpha)$  of natural transformations such that the diagram (1) is commutative for any manifold  $X$ .

The following proposition completes the proof of Theorem 3.1. It is proved just the same as Corollary 2.3 in [9].

**Proposition 3.4.** *The correspondence  $F \rightarrow \mu^F$  induces a bijection between the equivalence classes of product preserving bundle functors on the category  $\mathcal{F}M_\Lambda$  and the equivalence classes of inductive systems of natural transformations of product preserving bundle functors on the category  $\mathcal{M}f$ . The inverse bijection is induced by the correspondence  $\mu \rightarrow T^\mu$ .*

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