INFINITE-DIMENSIONAL SYMMETRIES: THE KEY TO UNDERSTANDING GRAVITY?

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Colloquium
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Of all fundamental forces, gravity is the most mysterious.

GRAVITY AND INFINITE-DIMENSIONAL SYMMETRIES

Finite-dimensional Lie algebras underlie our understanding of all non-gravitational interactions (electromagnetic, weak and strong nuclear forces) through the Yang-Mills construction.

There are many indications that a deeper understanding of gravity requires infinite-dimensional Lie algebras.

One of these indications comes from the analysis of the dynamics of gravity in the cosmological context, which leads to « cosmological billiards ». These billiards exhibit unexpected connections with tilings of hyperbolic space, and « Coxeter groups ».

This points to the fact that infinite-dimensional Kac-Moody algebras of hyperbolic type are likely to play a central role in the « ultimate » formulation of gravity.

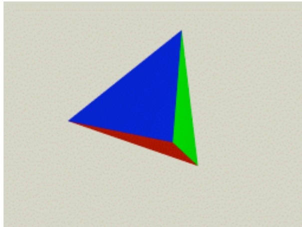
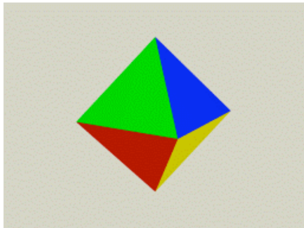
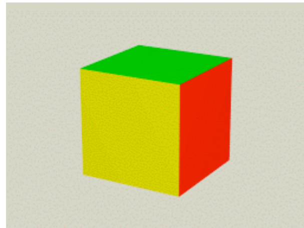
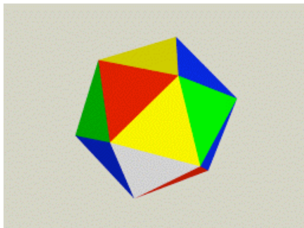
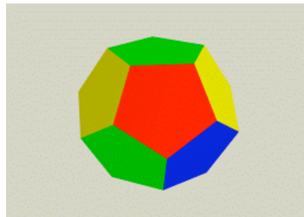
Purpose of colloquium is to explain this last paragraph!

CONTENTS

- Spherical reflection groups
- Affine reflection groups
- Hyperbolic reflection groups
- Infinite-dimensional Lie algebras
- Cosmological billiards
- Exhibiting the symmetry
- Conclusions

Coxeter Groups

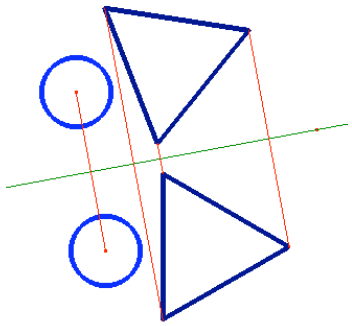
THE FIVE PLATONIC SOLIDS

Tetrahedron {3,3} 	
Octahedron {3,4} 	Cube {4,3} 
Icosahedron {3,5} 	Dodecahedron {5,3} 

http://home.teleport.com/~tpgettys/platonic.shtml

Symmetry groups

Reflection in a line (hyperplane)



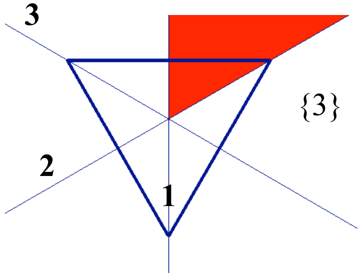
$s^2 = 1$

All Euclidean isometries are products of reflections

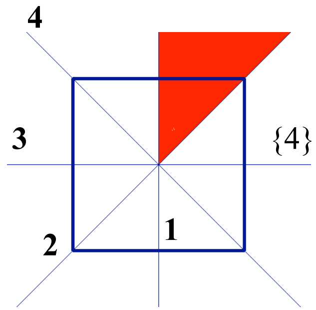
Symmetry groups of regular polytopes are all finite reflection groups
(= groups generated by a finite number of reflections)

Number of generating reflections = dimension of space

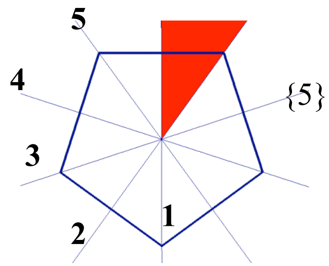
Dihedral groups



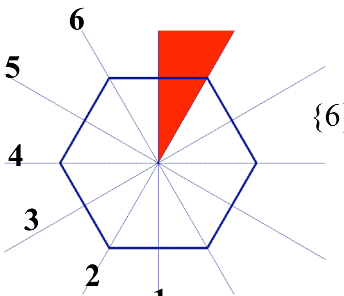
$I_2(3)$, order 6



$I_2(4)$, order 8



$I_2(5)$, order 10



$I_2(6)$, order 12

etc …

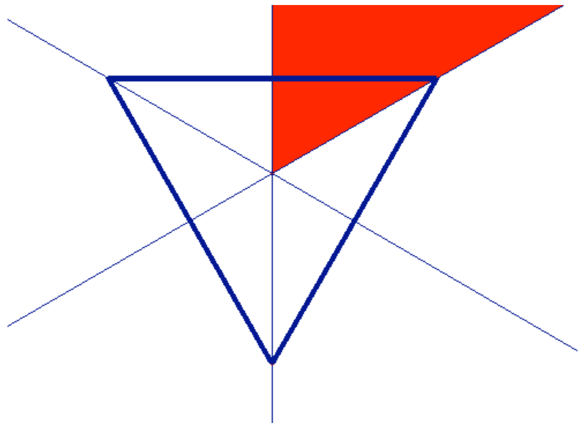
$$(s_1)^2=1,$$

$$(s_2)^2=1,$$

$$(s_1 s_2)^p = 1$$

(fundamental domain in red)

FUNDAMENTAL DOMAIN



Region that intersects each orbit once and only once – drawn in red.

Group generated by reflections in the sides of the domain.

Angles between sides: integer submultiples of π (here $\pi/3$).

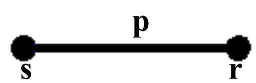
Coxeter Groups

The previous groups are examples of Coxeter groups: these are (by definition) generated by a finite set of reflections s_i obeying the relations:

$$(s_i)^2 = 1;$$
$$(s_i s_j)^{m_{ij}} = 1$$

with $m_{ij} = m_{ji}$ positive integers (=1 for i = j and >1 for different i,j's)

Notation: $(s\,r)^p = 1$



angles between reflection axes: π/p

no line if p = 2

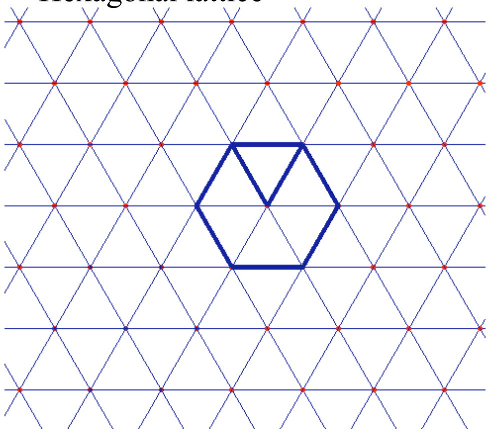
p not written when it is equal to 3

(2 lines if p = 4, 3 lines if p = 6)

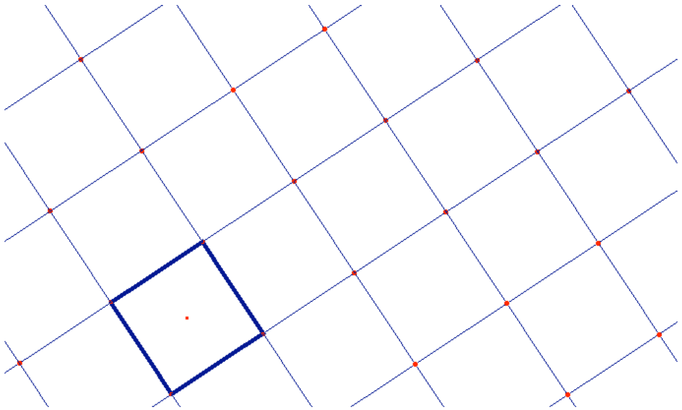
Crystallographic dihedral groups

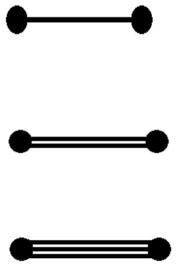
p = 3, 4, 6

Hexagonal lattice



Square lattice





A_2

$B_2 - C_2$

G_2

	A_2	B_2/C_2	G_2
|G|	6	8	12
N	3	4	6

|G| = group order

N = number of reflections

Symmetries of Platonic Solids

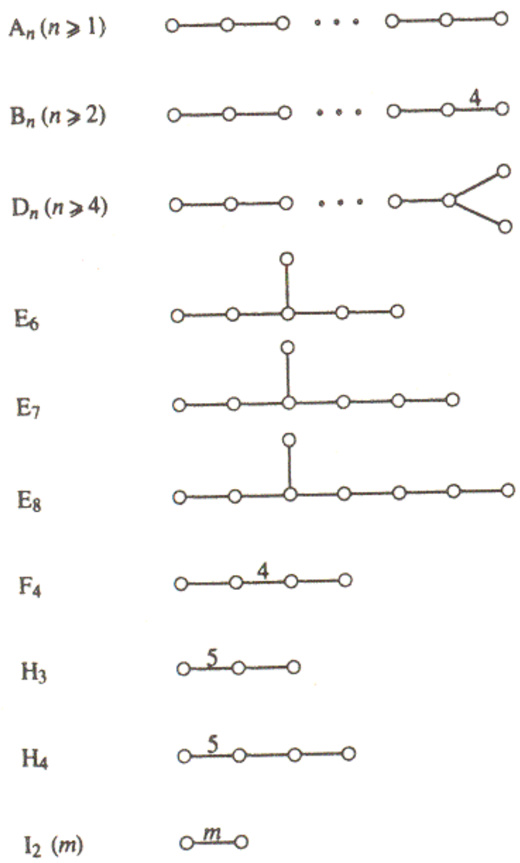
G is in all cases a Coxeter group

$\{s_1, s_2, s_3\}$; $(s_i)^2 = 1$; $(s_i s_j)^{m_{ij}} = 1$; $m_{ij} = 2,3,4,5$ (i different from j)

		$\|G\|$	N
Tetrahedron	 A_3	24	6
Cube and octahedron	 B_3/C_3	48	9
Icosahedron and dodecahedron	 H_3	120	15

H_3 is not crystallographic

List of Finite Reflection Groups (= Finite Coxeter Groups)



Coxeter graphs of finite Coxeter groups
(source: J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press 1990)

	|G|	N
A_n	(n+1)!	n(n+1)/2
B_n/ C_n	2^n n!	n^2
D_n	2^{n-1} n!	n(n-1)
E_6	$2^7\ 3^4\ 5$	36
E_7	$2^{10}\ 3^4\ 5\ 7$	63
E_8	$2^{14}\ 3^5\ 5^2\ 7$	120
F_4	$2^7\ 3^2$	24
G_2	12	6
H_3	120	15
H_4	14400	60

Comments

• In dimensions > 4, there are only 3 regular polytopes: the regular n-simplex (triangle, tetrahedron …), the cross polytope (square, octahedron …) and its dual, the hypercube (square, cube …). The symmetry group of the regular n-simplex is A_n, that of the cross polytope and of the hypercube is B_n (~ C_n).

• In dimension 4, there are 6 (convex) regular polytopes. Besides the three just mentioned, there are:
- the 24-cell {3,4,3} with symmetry group F_4 (24 octahedral faces); and
- the 120-cell {5,3,3} and its dual, the 600-cell {3,3,5} with symmetry group H_4 (120 dodecahedra in one case, 600 tetrahedra in the other).

• H_3 and H_4 are not crystallographic.

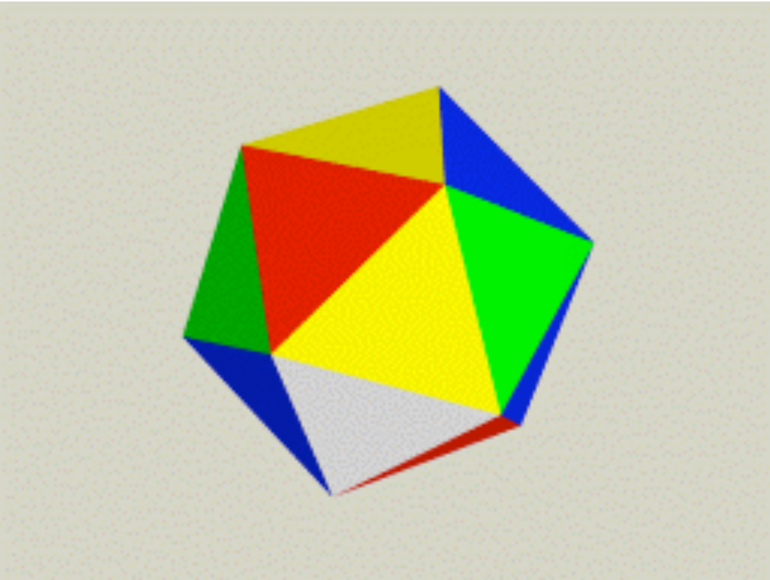
• D_n, E_6, E_7 and E_8 are finite reflection groups but are not symmetry groups of regular polytopes (generalization).

• Fundamental domain is always a (spherical) simplex

• A very nice reference: H.S.M. Coxeter, *Regular polytopes*, Dover 1973

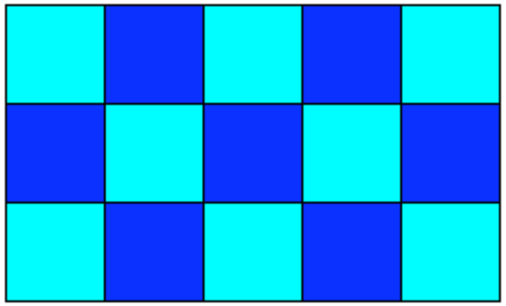
Affine Reflection Groups

In previous cases, the hyperplanes of reflection contain the origin and thus leave the unit sphere invariant (« spherical case »)

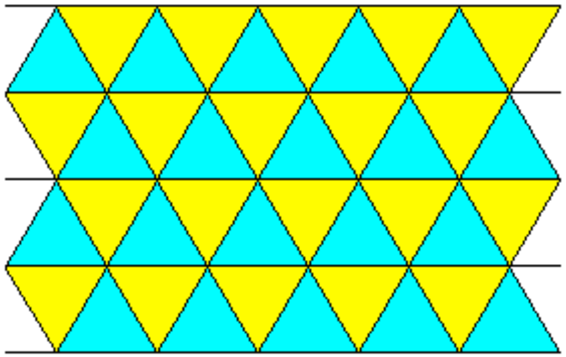


One can relax this condition and consider reflections about arbitrary hyperplanes in Euclidean space (« affine case »).

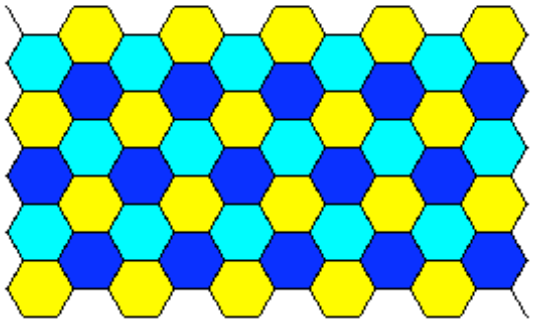
Regular tilings of the plane



{4,4}

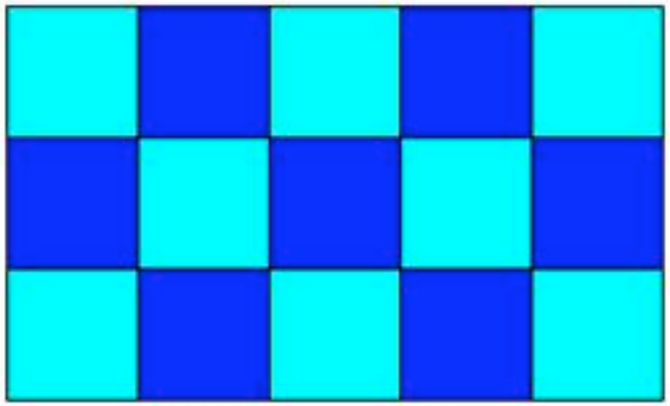


{3,6}

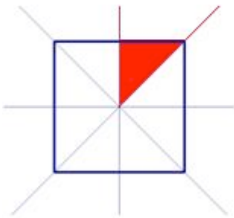


{6,3}

FUNDAMENTAL DOMAIN



{4,4}

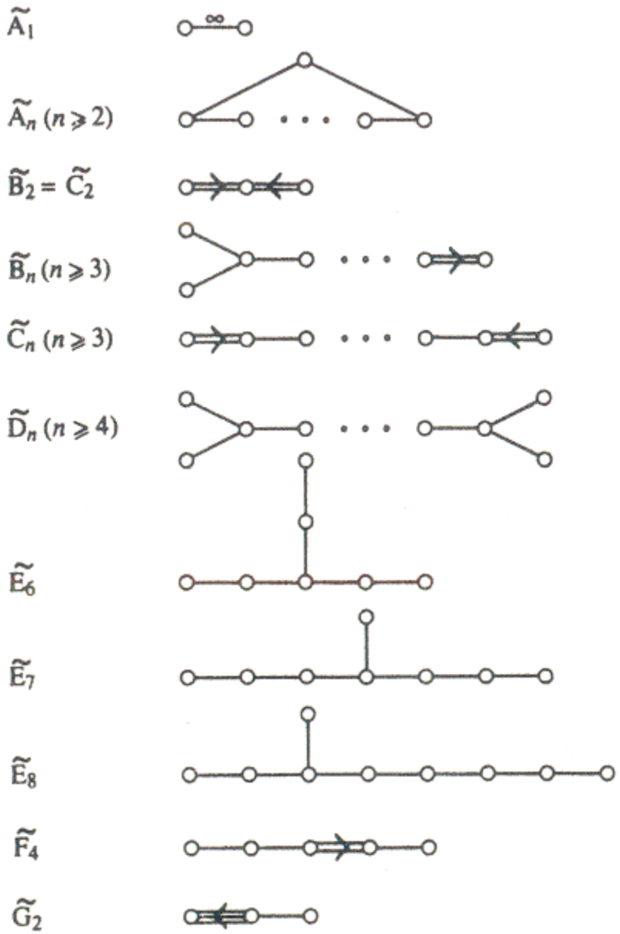


Fundamental domain is a simplex.

Angles between sides: integer submultiples of π (here $\pi/4$ and $\pi/2$).

Group generated by reflections in the sides of the fundamental domain.

Classification of affine Coxeter groups



Remarks

- Affine Coxeter Groups are infinite
- Fundamental region is an Euclidean simplex

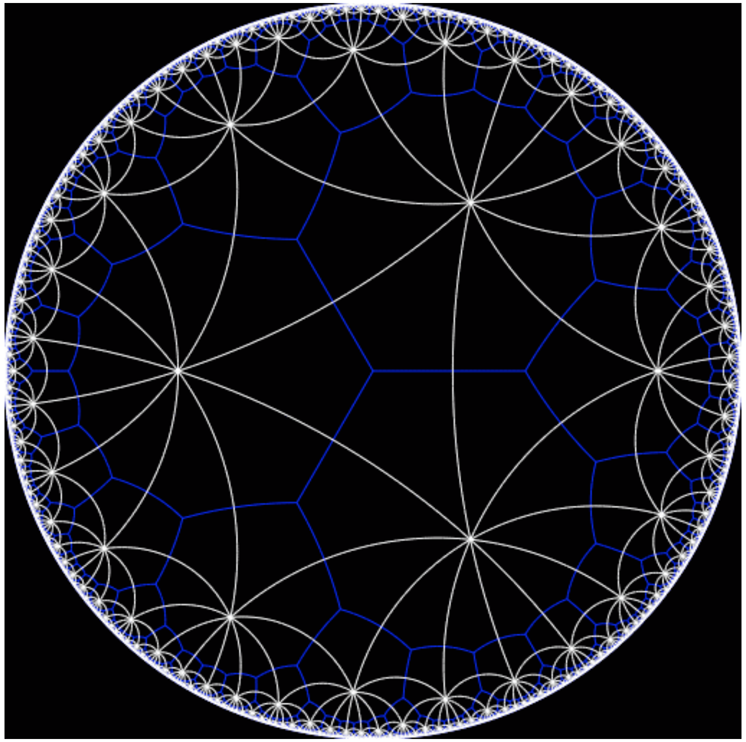
Coxeter graphs of affine Coxeter groups
(source: J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press 1990)

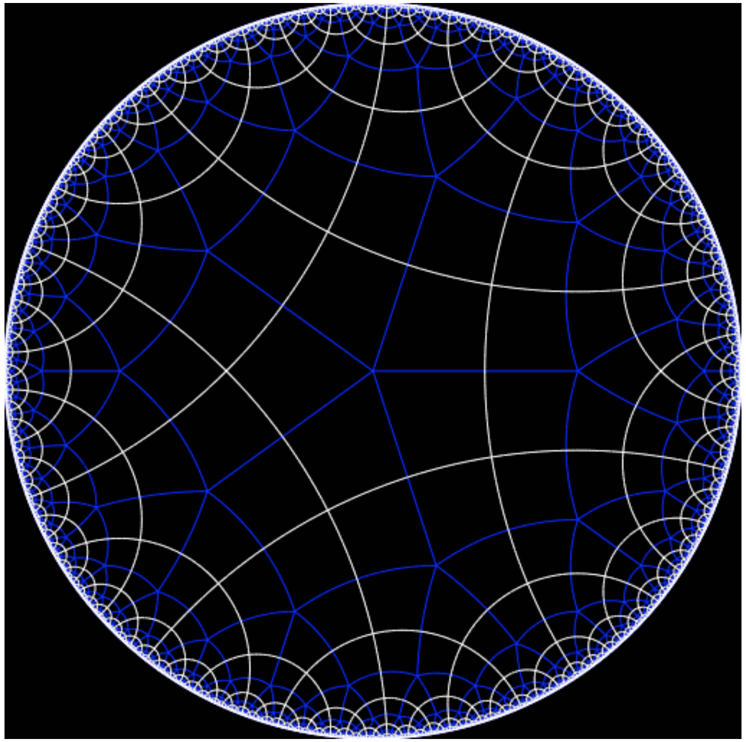
Hyperbolic Reflection Groups

One can also consider reflection groups in hyperbolic space.

These groups are also infinite.

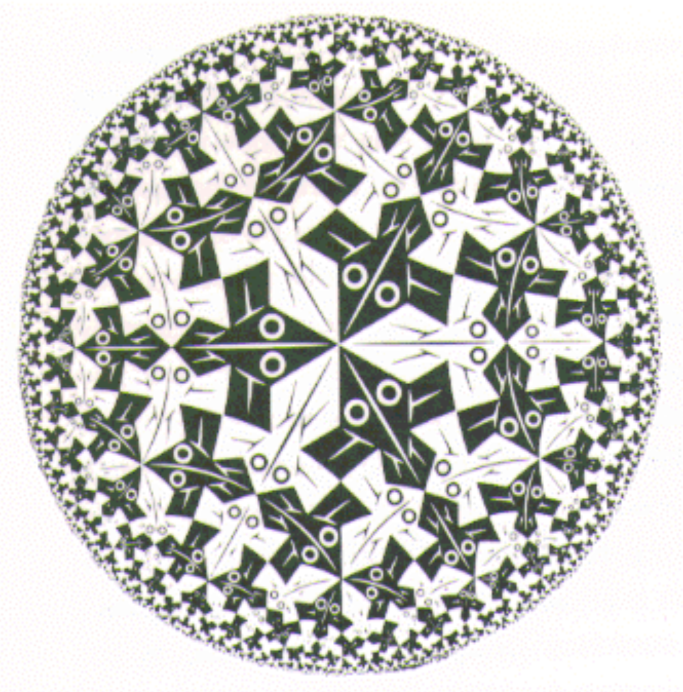
Tilings of the hyperbolic plane





http://www.hadron.org/~hatch/HyperbolicTesselations/

Circle-limits (M.C. Escher)

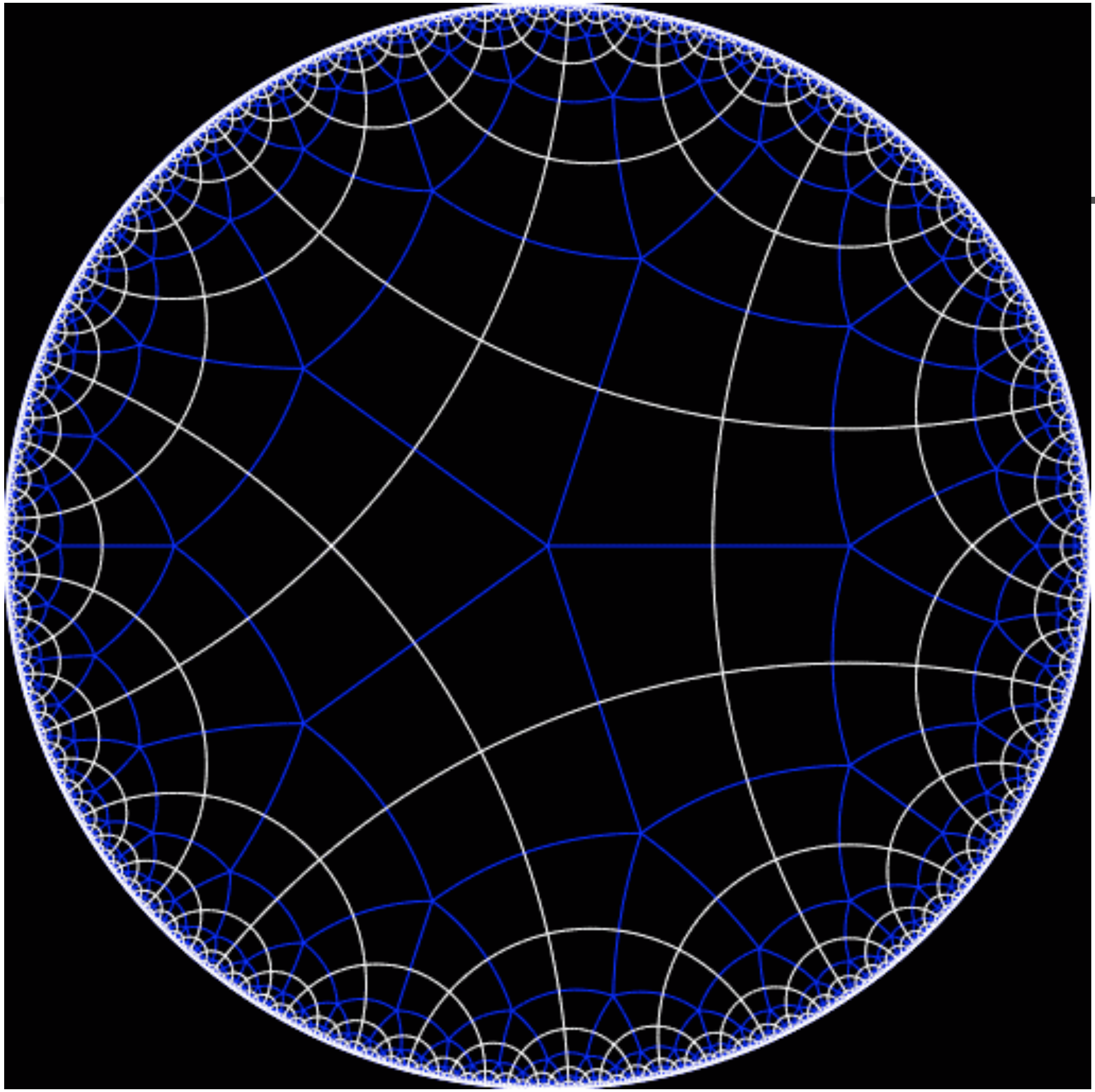


New feature: Fundamental domain need not be a simplex.

It can always be taken to be a Coxeter polyhedron.

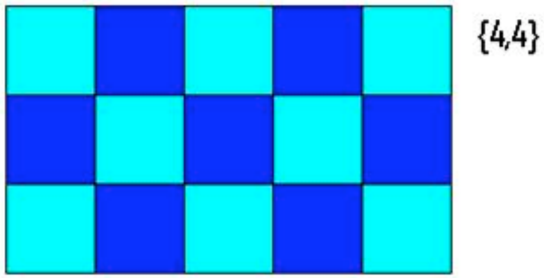
Coxeter polyhedron = (acute-angled) polyhedron with angles that are integer submultiples of π ($\pi/2$, $\pi/3$, $\pi/4$ etc)

Reflections in the sides provide a standard Coxeter presentation of the group



Note : in Euclidean space or on sphere : acute-angled polyhedron is a simplex.

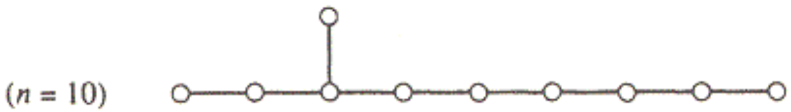
Acute-angled d-gon in plane :
Sum of angles = $\pi(d-2)$
Acuted-angled polygon : $\pi(d-2) \leq d\ (\pi/2)$, which implies $d \leq 4$,
with $d = 4$ (rectangle) leading to a decomposable situation (direct product structure).



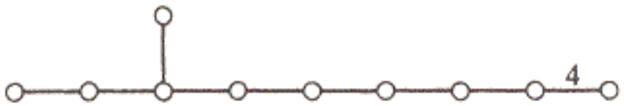
Hence $d = 3$ (triangle) is the only non trivial case

Classification

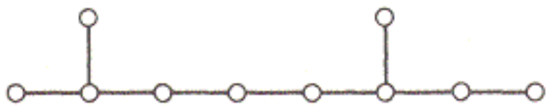
Hyperbolic simplex reflection groups exist only in hyperbolic spaces of dimension < 10. In the maximum dimension 9, the groups are generated by 10 reflections. There are three possibilities, all of which are relevant to M-theory . (See e.g. Humphreys, *Reflection Groups and Coxeter Groups*, for the complete list.)



$\mathbf{E_{10}}$



$\mathbf{BE_{10} - CE_{10}}$



$\mathbf{DE_{10}}$

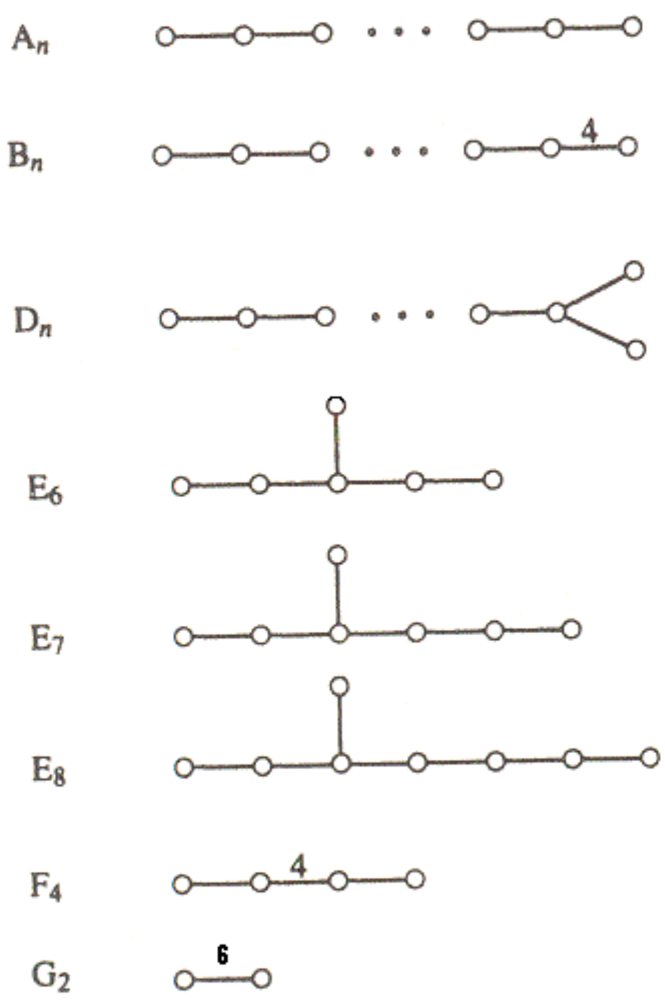
Note: finite-volume Coxeter polyhedra in n-dimensional hyperbolic space exist only for $n \leq 996$.

Infinite-dimensional Symmetry Groups

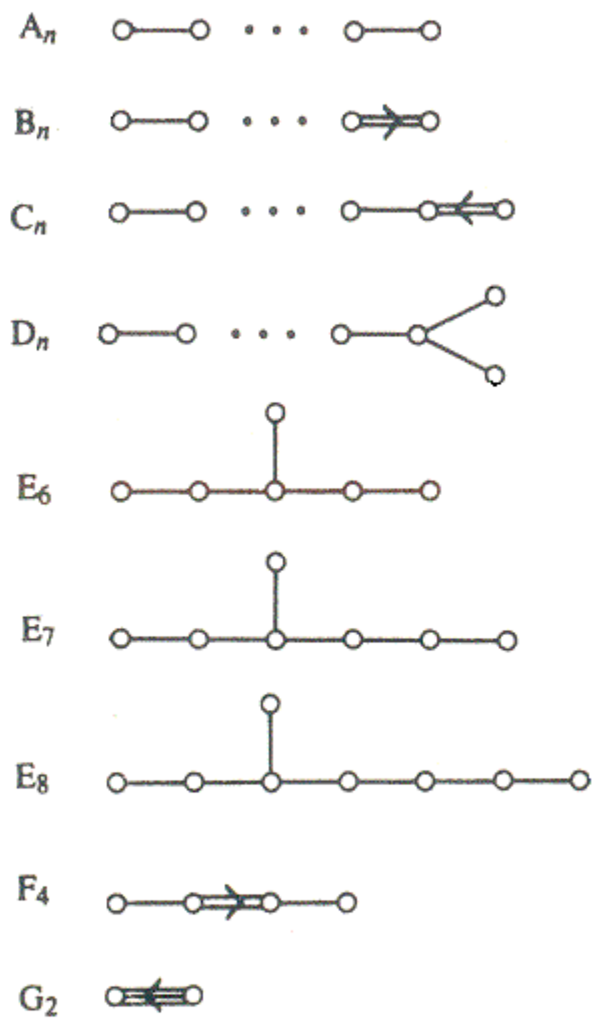
Crystallographic Coxeter Groups and Kac-Moody Algebras

There is an intimate connection between crystallographic Coxeter groups and Lie groups/Lie algebras.

Lie groups are continuous groups (e.g. SO(3)). The ones usually met in physics so far are finite-dimensional (depend on a finite number of continuous parameters). A great mathematical achievement has been the complete classification of all finite-dimensional, simple Lie groups (Lie algebras are the vector spaces of « infinitesimal transformations »).

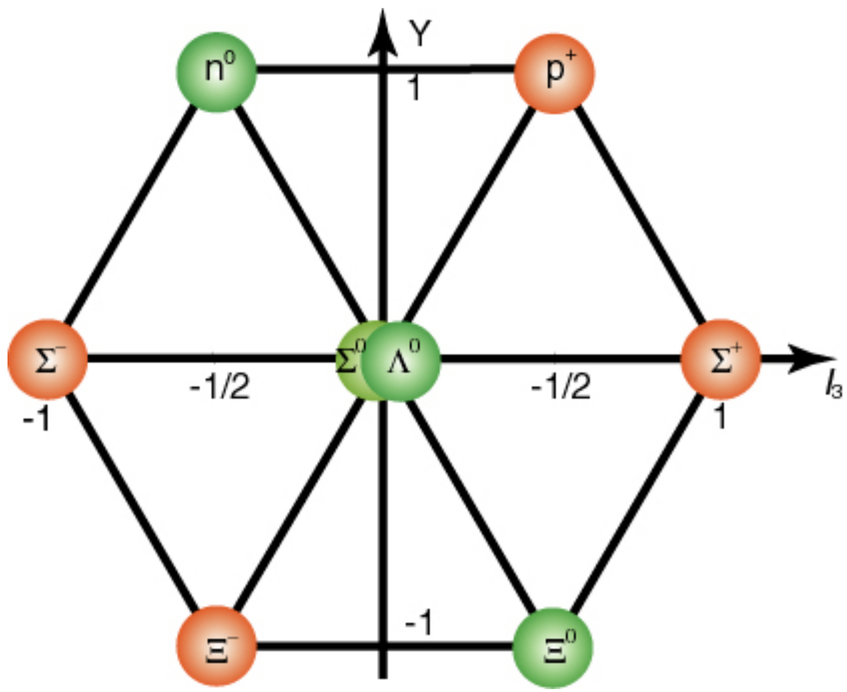


Coxeter graphs of finite Coxeter groups



Dynkin diagrams of finite-dimensional Lie algebras

The connection between crystallographic finite Coxeter groups and finite-dimensional simple Lie algebras is that the Coxeter groups are the « Weyl groups » of the Lie algebras.
Coxeter groups may thus signal a much bigger symmetry.



$I_2(3)$ versus SU(3)

Weyl group of SU(2)

Algebra of angular momentum J^3, J^+, J^-
Angular momentum can always be assumed to be along the third axis.
Fixes the angular momentum up to the sign (+j can be changed into –j by a rotation).
After conjugation to the Cartan subalgebra, there remains a $Z_2 = S_2$ ambiguity, which is the Weyl group of SU(2).
Representations described in terms of eigenvalues of J^3 (Cartan subalgebra) have symmetry $m \rightarrow -m$

Weyl group of SU(n)

Unitary symmetry and permutation group

The Coxeter group A_n is isomorphic to the permutation group S_{n+1} of n+1 objects.

Consider the group SU(n+1) of (n+1)-dimensional unitary matrices (of unit determinant).

SU(n+1) acts on itself:

$$U \rightarrow U' = M^* \, U \, M$$

(unitary change of basis, adjoint action)

By a change of basis, one can diagonalize U (« U is conjugate to an element in the Cartan subalgebra »). The Weyl = Coxeter group A_n is what is left of the original unitary symmetry
once U has been diagonalized since the diagonal form of U is determined up to a permutation of the n+1 eigenvalues.

Infinite Coxeter groups

The same connection holds for infinite Coxeter groups; but in that case the corresponding Lie algebra is infinite-dimensional and of the Kac-Moody type.

Infinite-dimensional Lie algebras (i.e., infinite-dimensional symmetries) are playing an increasingly important role in physics. In the gravitational case, the relevant Kac-Moody algebras are of hyperbolic or Lorentzian type (beyond the affine case).

These algebras are unfortunately still poorly understood.

Cosmological Billiards

Infinite Coxeter groups of hyperbolic type emerge when one investigates the dynamics of gravity in extreme situations. For M-theory, it is E_{10} that is relevant.

Cosmological Billiards

Dynamics of scale factors exhibits interesting features in the strong field regime corresponding to a cosmological singularity (« big bang »), or a black hole singularity (« inside Schwarzschild »).

(Belinskii, Khalatnikov and Lifshitz)

- Dynamics of scale factors is chaotic in the vicinity of a cosmological singularity.

- It is the same dynamics as that of a billiard motion in a region of hyperbolic space

- The billiard region exhibits remarkable properties

The example of pure gravity in 3+1 dimensions

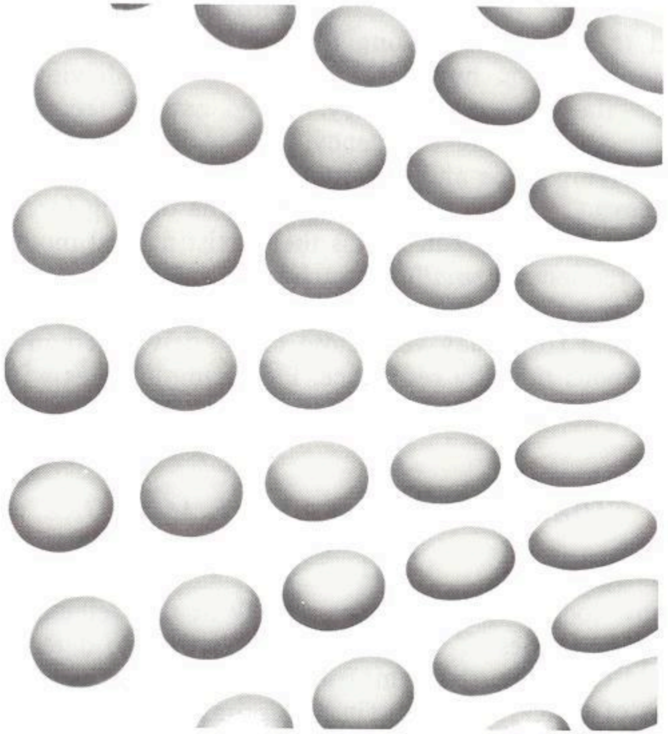
$ds^2 = - dt^2 + a^2(t,\mathbf{x})\, \mathbf{l}^2 + b^2(t,\mathbf{x})\, \mathbf{m}^2 + c^2(t,\mathbf{x})\, \mathbf{n}^2$

l, **m**, **n** are orthogonal spatial frames

a, b, c are the scale factors

Assume singularity at $t = 0$ or $x^0 = - \ln t \to \infty$

Focus on time dependence at a given spatial point **x**



The tidal field of a spherical mass represented by tidal ellipsoids.

Two scale factors are squeezed and one is stretched. This is characteristic of the familiar tidal effects of gravity …
but there is a change with time of the directions of stretching and squeezing

Source: H.C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, Norton 1976

Kasner behaviour

$a(t)$ » t^{p_1}, $b(t)$ » t^{p_2}, $c(t)$ » t^{p_3},

with $p_1 < 0$, $p_2 > 0$, $p_3 > 0$

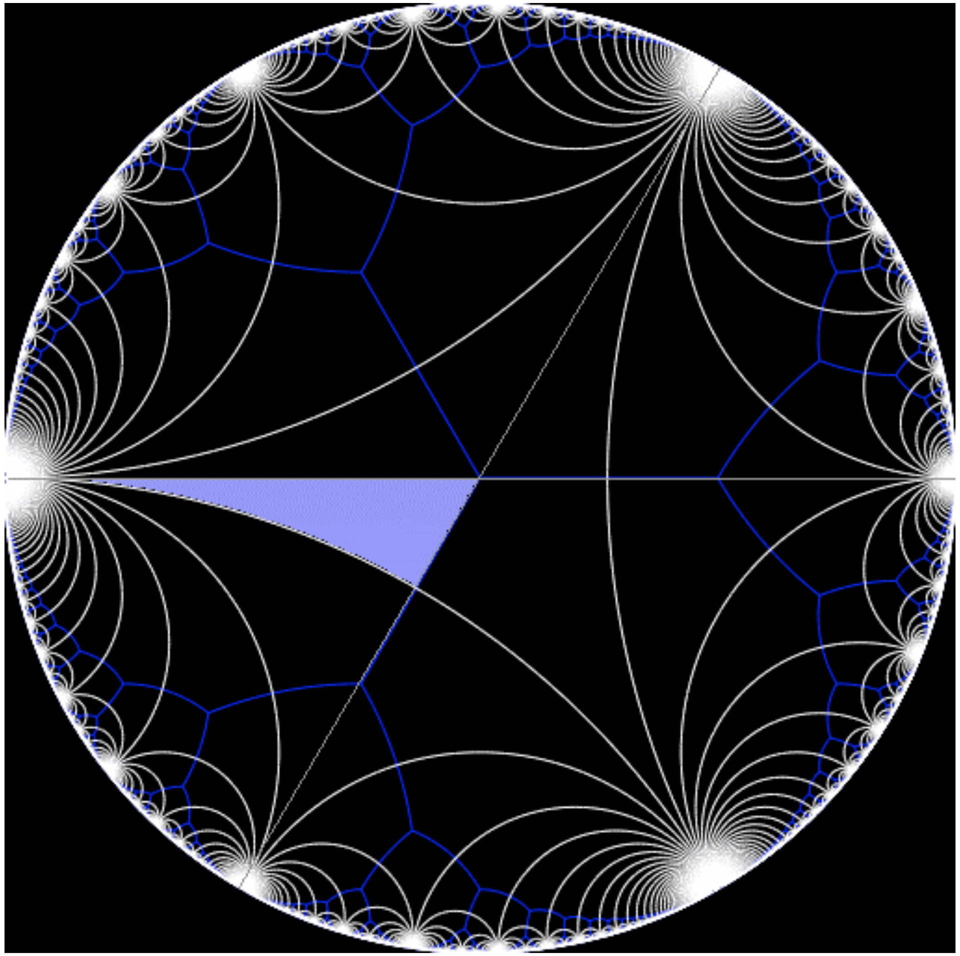
(Infinite) stretching along **l** and (infinite) squeezing along **m** and **n** as $t \rightarrow 0$

Transition to a new Kasner behaviour (« collision »)
before one reaches the singularity

$a(t)$ » t^{q_1}, $b(t)$ » t^{q_2}, $c(t)$ » t^{q_3},
with $q_1 > 0$, $q_2 < 0$, $q_3 > 0$ (say)

And so on (infinite number of Kasner regimes as $t \rightarrow 0$), in a chaotic way

Billiard description



Dynamics can be mapped on billiard dynamics in some region of hyperbolic space.

Free flight = Kasner behaviour

Collision against a wall = change from one Kasner regime to another

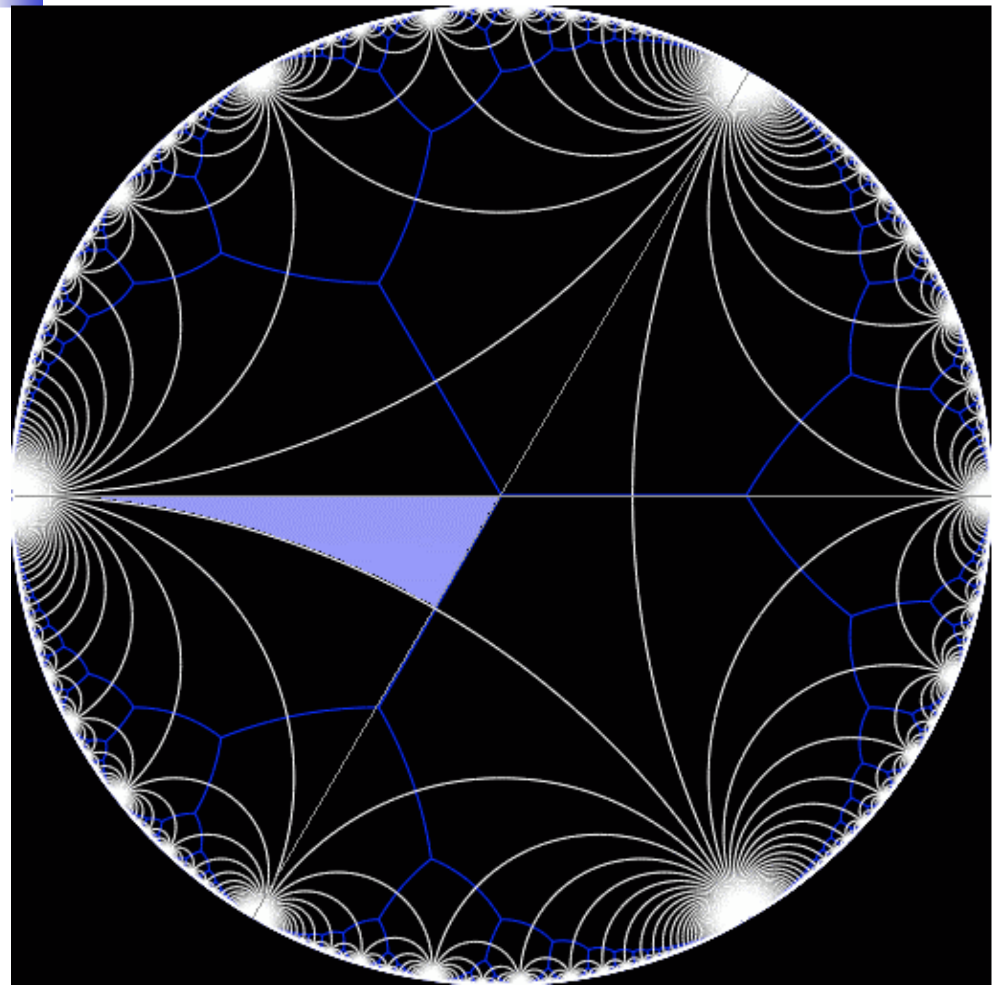
Emergence of hyperbolic Coxeter groups

- The same analysis remains valid for gravity in higher dimensions (but there are then more scale factors)

- It also holds true when one couples antisymmetric tensors to gravity (as requested by string/M-theory)

- Furthermore, the billiard region is the fundamental region of a hyperbolic Coxeter group (the reflections against the walls being the fundamental reflections generating the group).

Examples



Pure gravity in 4 spacetime Dimensions.

The billiard is a triangle with angles $\pi/2$, $\pi/3$ and 0, corresponding to the Coxeter group (2,3, infinity).

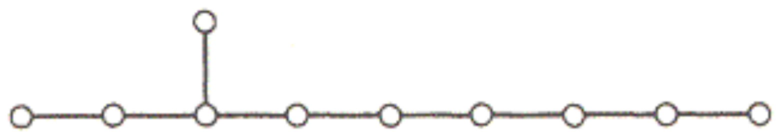
The triangle is the fundamental region of the group PGL(2,Z).

Arithmetical chaos

M-theory and E_{10}

Truncation to 11-dimensional supergravity

Billiard is fundamental Weyl chamber of E_{10}



Heterotic string: BE_{10}
Bosonic string DE_{10}

Is E_{10} the symmetry algebra (or a subalgebra of the symmetry algebra) of M-theory? (perhaps $E_{10}(Z)$, E_{11}, $E_{11}(Z)$)

Similar conclusions come from dimensional reduction to D= 4, 3, 2, 1 (?), 0 (?):

E_7
E_8
E_9
E_{10} ?
E_{11} ?

Exhibiting the Symmetry

- Can one rewrite the Einstein (+ p-form) Lagrangian in a manner that makes the symmetry manifest?

- Promising attemps exist but are so far only partially successful

Non-linear sigma model $E_{10}/K(E_{10})$

- Consider (1+0) non-linear sigma model based on « symmetric space »

$$E_{10}/K(E_{10})$$

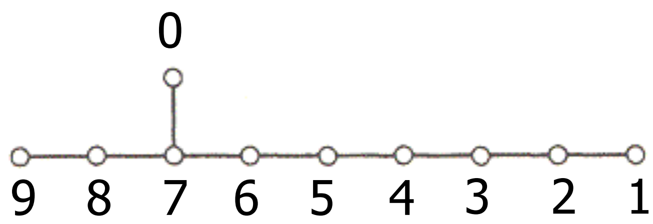
(G/H – compare with SO(3)/SO(2) – here infinite number of fields)

- Write corresponding Lagrangian according to standard rules for coset models

$L \sim Tr\,(P^2)$, $\quad P = (1/2)\left(g^{-1}\,dg + (g^{-1}\,dg)^T\right)$

This Lagrangian is manifestly E_{10} invariant

Expand group element according to « level »



(number of times root α_0 appears)

- Perfect match with the (bosonic) fields of 11-dimensional supergravity at low levels

- Perfect match of the sigma-model equations of motion with the (bosonic) equations of 11-dimensional supergravity

- … but what about higher levels? (recent work on gaugings and level-4 roots)

- Difficulties with « dual graviton »

- Difficulties with higher spin gauge fields (described by Young tableaux of mixed symmetry)

- etc

Conclusions

- Gravity remains the most mysterious of all the fundamental interactions

- There are indications that infinite-dimensional Lie algebras related to hyperbolic structures will be crucial ingredients for a deeper understanding of gravity (characteristic feature of gravity)

- Indications come from the study of the dynamics in extreme regime (cosmological billiards), but also from other approaches (BPS states)

- Fermions fit into the picture (representations of compact subgroup K(E10))

- Indications that quantum corrections are also compatible with conjectured symmetry

- Cosmological deformations and gaugings also seem to fit into the picture

But much more remains to be done!