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FOURIER PROBLEM WITH BOUNDED BAIRE DATA

Miroslav Dont, Praha

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Abstract. The Fourier problem on planar domains with time moving boundary is considered using integral equations. Solvability of those integral equations in the space of bounded Baire functions as well as the convergence of the corresponding Neumann series are proved.

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In [2] the Fourier problem for some regions in the plane $\mathbb{R}^2$ with time moving boundary was solved. The solution was expressed by means of a combination of single and double layer heat potentials (and also of the Weierstrass integral). The problem was considered on regions of the type

$$M = \{\, [x,t] \in \mathbb{R}^2 \bigm| t \in (a,b), x > \varphi(t) \,\}$$

or of the type

$$M = \{\, [x,t] \in \mathbb{R}^2 \bigm| t \in (a,b), \varphi_1(t) < x < \varphi_2(t) \,\},$$

where $\varphi, \varphi_1, \varphi_2$ are continuous functions of bounded variation on a compact interval $\langle a,b \rangle$ [and $\varphi_1(t) < \varphi_2(t)$ on $\langle a,b \rangle$]. In [2] only continuous boundary values were considered. A very simple assertion from functional analysis will enable us to solve relevant integral equations not only in the space of continuous functions but also in the space of bounded Baire functions. This makes it possible to solve the Fourier

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problem for non-continuous boundary conditions (in a sense) and also to prove convergence of a simple numerical method for the above mentioned integral equations (this will be done in a forthcoming paper). At the end of the present paper convergence of the Neumann series of operators corresponding to the integral equations mentioned above is proved.

1. Preliminary

By ${}^*\mathbb{R}^1$ we denote the extended real line (that is ${}^*\mathbb{R}^1 = \mathbb{R}^1 \cup \{+\infty, -\infty\}$). By a function on a set M we mean a numerical function, that is a mapping from M to ${}^*\mathbb{R}^1$; a real function is a mapping from M to $\mathbb{R}^1$. By a continuous function we will always mean a *real* continuous function.

For a real function f on an interval $J \subset \mathbb{R}^1$, $M \subset J$, the variation of f on M will be denoted by $\operatorname{var}[f; M]$. It is well known that $\operatorname{var}[f; \cdot]$ is an outer measure and its restriction to $\operatorname{var}[f; \cdot]$-measurable sets is a measure. The integral of a function $F\colon M \to {}^*\mathbb{R}^1$ with respect to this measure will be denoted by

$$\int_M F \,\mathrm{d} \operatorname{var} f, \qquad \int_M F(\tau) \,\mathrm{d}\big(\operatorname{var} f(\tau)\big) \qquad \text{etc.}$$

(where $M \subset J$ is a $\operatorname{var}[f; \cdot]$-measurable set and F is supposed to be $\operatorname{var}[f; \cdot]$-measurable, of course). If f is of locally finite variation on J (that is, if $\operatorname{var}[f; I] < +\infty$ for any compact interval $I \subset J$), then by

$$\int_M F \,\mathrm{d} f$$

we mean the Lebesgue-Stieltjes integral of F.

Let us recall some basic notation, notions and assertions from [1].

Let $a, b \in \mathbb{R}^1$, $a < b$, be fixed and let φ be a continuous function on $\langle a, b\rangle$, $\operatorname{var}[\varphi; \langle a, b\rangle] < +\infty$. Denote

$$K = \big\{ [\varphi(t), t] \bigm| t \in \langle a, b\rangle \big\}; \tag{1.1}$$

K is a compact set in $\mathbb{R}^2$, of course.

For $[x, t] \in \mathbb{R}^2$, $t > a$, define a (real) function $\alpha_{x,t}$ on the interval $\langle a, \min\{t, b\})$ by

$$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2\sqrt{t - \tau}} \tag{1.2}$$

$[\tau \in \langle a, \min\{t,b\})]$. Recall that from the assumption that φ is of finite variation on $\langle a,b\rangle$ it follows that $\alpha_{x,t}$ has locally finite variation on $\langle a, \min\{t,b\})$ and that

$$\operatorname{var}_\tau \Big[\frac{x-\varphi(\tau)}{2\sqrt{t-\tau}}; \langle a,c)\Big] \leqslant \frac{1}{2\sqrt{t-c}}\Big\{\operatorname{var}\big[\varphi; \langle a,b\rangle\big] + \sup_{\tau\in\langle a,b\rangle}\big|x-\varphi(\tau)\big|\Big\} \tag{1.3}$$

for any $c \in \langle a, \min t, b)$ (the subscript τ in var_τ indicates that the variation is considered with respect to the variable τ).

1.1. Parabolic variation. Let $[x,t] \in \mathbb{R}^2$. For $\alpha, r > 0$, $\alpha < +\infty$ let $n_{x,t}(r,\alpha)$ stand for the number of all points (finite or $+\infty$) of the set

$$K \cap \Big\{[\xi,\tau] \in \mathbb{R}^2 \;\Big|\; t-\tau = \Big(\frac{\xi-x}{2\alpha}\Big)^2, 0 < t-\tau < r\Big\}.$$

It is known that for any $[x,t] \in \mathbb{R}^2$, $r > 0$, the function $n_{x,t}(r,\alpha)$ is a measurable function of the variable $\alpha \in (0,+\infty)$. Denoting

$$V_K(r; x,t) = \int_0^{+\infty} \mathrm{e}^{-\alpha^2} n_{x,t}(r,\alpha)\,\mathrm{d}\alpha \tag{1.4}$$

we have

$$V_K(r; x,t) = \int_{\max\{a,t-r\}}^{\min\{t,b\}} \mathrm{e}^{-\alpha^2_{x,t}(\tau)}\,\mathrm{d}\big(\operatorname{var}\alpha_{x,t}(\tau)\big) \tag{1.5}$$

whenever $\max\{a, t-r\} < \min\{t,b\}$, otherwise $V_K(r;x,t) = 0$ (see [1], Lemma 1.1, Definition 1.1). Further, we write $V_K(+\infty; x,t) = V_K(x,t)$; the function $V_K(\cdot\,,\cdot)$ is called the *parabolic variation* of the set (curve) K.

For any fixed $r > 0$ the function $V_K(r;\cdot)$ is lower-semicontinuous on $\mathbb{R}^2$ and finite on $\mathbb{R}^2 \setminus K$ ([1], Lemma 1.2).

The basic property of the parabolic variation concerns its boundedness. The following assertion holds ([1], Theorem 1.1).

Let $t_0 \in \langle a,b\rangle$ and suppose that

$$\sup\big\{\, V_K(\varphi(t),t) \;\big|\; t \in \langle a,b\rangle, |t-t_0| < \delta \,\big\} < +\infty$$

for some $\delta > 0$. Then there exists a neighbourhood U of $[\varphi(t_0), t_0]$ (in $\mathbb{R}^2$) such that

$$\sup_{[x,t]\in U} V_K(x,t) < +\infty.$$

If

$$\sup_{t\in\langle a,b\rangle} V_K\bigl(\varphi(t),t\bigr) = c < +\infty,$$

then V_K *is bounded on* $\mathbb{R}^2$*—we have that* $V_K(x,t) \leqslant c + \sqrt{\pi}$ *for each* $[x,t]\in\mathbb{R}^2$.

1.2. Operator T. We will always suppose that $\langle a,b\rangle$ is a compact interval in $\mathbb{R}^1$, φ is a continuous function with finite variation on $\langle a,b\rangle$; K is the set in $\mathbb{R}^2$ given by (1.1).

$\mathscr{C}(\langle a,b\rangle)$ stands for the space of all continuous functions on $\langle a,b\rangle$ endowed with the supremum norm, that is

$$\|f\| = \|f\|_{\mathscr{C}} = \sup_{t\in\langle a,b\rangle} \bigl|f(t)\bigr|$$

for $f\in\mathscr{C}(\langle a,b\rangle)$.

Further, let $\mathscr{B}(\langle a,b\rangle)$ denote the space of all bounded Baire functions on $\langle a,b\rangle$ endowed also with the supremum norm which we will denote by $\|\ldots\|_{\mathscr{B}}$ or simply by $\|\ldots\|$. Note that $\mathscr{C}(\langle a,b\rangle)$, $\mathscr{B}(\langle a,b\rangle)$ are Banach spaces and that $\mathscr{C}(\langle a,b\rangle)$ is a closed subspace of $\mathscr{B}(\langle a,b\rangle)$.

For $f\in\mathscr{B}(\langle a,b\rangle)$ the potential $Tf = T_K f$ is defined in the following way. For $[x,t]\in\mathbb{R}^2$ we put $Tf(x,t)=0$ whenever $t\leqslant a$, while

(1.6)

$$Tf(x,t) = T_K f(x,t) = \frac{2}{\sqrt{\pi}}\int_a^{\min\{t,b\}} f(\tau)\exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right)\,\mathrm{d}_\tau\left(\frac{x-\varphi(\tau)}{2\sqrt{t-\tau}}\right)$$

if $t>a$ and the integral on the right hand side exists and is finite (Definition 2.1 in [1]).

It is seen easily that if $V_K(x,t) < +\infty$ then $Tf(x,t)$ is defined and

$$\bigl|Tf(x,t)\bigr| \leqslant \|f\|_{\mathscr{B}}\,\frac{2}{\sqrt{\pi}}V_K(x,t). \tag{1.7}$$

As we have noted $V_K(x,t) < +\infty$ on $\mathbb{R}^2\setminus K$ (assuming $\operatorname{var}[\varphi;\langle a,b\rangle] < +\infty$) and thus for any $f\in\mathscr{B}(\langle a,b\rangle)$, Tf is defined at least on $\mathbb{R}^2\setminus K$. On $\mathbb{R}^2\setminus K$ the function $Tf(x,t)$ is equal to a combination of a double and a single layer heat potentials and solves the heat equation there (see [1], Remark 2.1).

Let $[x,t]\in\mathbb{R}^2$ be fixed and suppose that $V_K(x,t) < +\infty$. For $f\in\mathscr{B}(\langle a,b\rangle)$ put

$$T^{\mathscr{B}}_{x,t}(f) = T^{\mathscr{B}}_{x,t}f = Tf(x,t). \tag{1.8}$$

Then $T^{\mathscr{B}}_{x,t}$ is a linear functional on $\mathscr{B}(\langle a,b\rangle)$, which is continuous due to (1.7). Further, let $T^{\mathscr{C}}_{x,t}$ be the restriction of $T^{\mathscr{B}}_{x,t}$ to $\mathscr{C}(\langle a,b\rangle)$. Let $\|T^{\mathscr{B}}_{x,t}\|$ denote the norm of $T^{\mathscr{B}}_{x,t}$, $\|T^{\mathscr{C}}_{x,t}\|$ the norm of $T^{\mathscr{C}}_{x,t}$, that is

$$\begin{aligned}\|T^{\mathscr{B}}_{x,t}\| &= \sup\big\{\, T^{\mathscr{B}}_{x,t}(f) \;\big|\; f \in \mathscr{B}(\langle a,b\rangle), \|f\|_{\mathscr{B}} \leqslant 1 \,\big\},\\ \|T^{\mathscr{C}}_{x,t}\| &= \sup\big\{\, T^{\mathscr{C}}_{x,t}(f) \;\big|\; f \in \mathscr{C}(\langle a,b\rangle), \|f\|_{\mathscr{C}} \leqslant 1 \,\big\}.\end{aligned}$$

Using (1.7) we have

$$\|T^{\mathscr{C}}_{x,t}\| \leqslant \|T^{\mathscr{B}}_{x,t}\| \leqslant \frac{2}{\sqrt{\pi}} V_K(x,t). \tag{1.9}$$

Further we get (suppose $t > a$)

$$\begin{aligned}\frac{2}{\sqrt{\pi}} V_K(x,t) &= \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} \mathrm{e}^{-\alpha^2_{x,t}(\tau)}\, \mathrm{d}\big(\operatorname{var} \alpha_{x,t}(\tau)\big)\\ &= \sup\left\{ \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau)\mathrm{e}^{-\alpha^2_{x,t}(\tau)}\, \mathrm{d}\big(\alpha_{x,t}(\tau)\big) \;\middle|\; f \in \mathscr{C}(\langle a,b\rangle), |f| \leqslant 1 \right\}\\ &= \sup\big\{\, Tf(x,t) \;\big|\; f \in \mathscr{C}(\langle a,b\rangle), \|f\|_{\mathscr{C}} \leqslant 1 \,\big\} = \|T^{\mathscr{C}}_{x,t}\|.\end{aligned}$$

Together with (1.9) we thus obtain

$$\|T^{\mathscr{C}}_{x,t}\| = \|T^{\mathscr{B}}_{x,t}\| = \frac{2}{\sqrt{\pi}} V_K(x,t). \tag{1.10}$$

Now let us recall an assertion concerning the limits of the potential Tf at points of the curve K. In this connection the point $[\varphi(a), a]$ $(\in K)$ plays a special role. One can see that Tf has no limit at this point if $f(a) \neq 0$ (and f is continuous at a). We shall thus restrict ourselves to the case $f(a) = 0$. Let us denote

$$\begin{aligned}\mathscr{B}_0(\langle a,b\rangle) &= \big\{\, f \in \mathscr{B}(\langle a,b\rangle) \;\big|\; f(a) = 0 \,\big\},\\ \mathscr{C}_0(\langle a,b\rangle) &= \big\{\, f \in \mathscr{C}(\langle a,b\rangle) \;\big|\; f(a) = 0 \,\big\}.\end{aligned} \tag{1.11}$$

Then the following is valid (see [1], Theorem 2.1).

Let $t_0 \in \langle a,b\rangle$, $x_0 = \varphi(t_0)$. Then there exist finite limits

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ t\in\langle a,b\rangle, x>\varphi(t)}} Tf(x,t), \qquad \lim_{\substack{[x,t]\to[x_0,t_0]\\ t\in\langle a,b\rangle, x<\varphi(t)}} Tf(x,t) \tag{1.12}$$

for each $f \in \mathscr{C}_0(\langle a,b\rangle)$ if and only if there is $\delta > 0$ such that

$$\sup\big\{\, V_K(\varphi(t),t) \;\big|\; t \in (t_0-\delta, t_0+\delta) \cap \langle a,b\rangle \,\big\} < +\infty. \tag{1.13}$$

If the condition (1.13) *is fulfilled for some* $\delta > 0$ *then the limits* (1.12) *exist and are finite for each* $f \in \mathscr{B}_0(\langle a, b\rangle)$ *which is continuous at* t_0.

Let $[x,t] \in K$, $t > a$. Recall that if $V_K(x,t) < +\infty$ then there exists a (finite or infinite) limit $\lim\limits_{\tau \to t-} \alpha_{x,t}(\tau)$ [the function $\alpha_{x,t}$ is defined by (1.2)]. In this case we put

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau);$$

$\alpha_{x,t}$ is thus defined on $\langle a, t\rangle$. Further, let G be the function on ${}^*\mathbb{R}^1$ defined by

$$G(t) = \begin{cases} 0, & t = -\infty, \\ \displaystyle\int_{-\infty}^{t} \mathrm{e}^{-x^2}\,\mathrm{d}x, & t > -\infty. \end{cases} \tag{1.14}$$

Using the function G and the value $\alpha_{x,t}(t)$ one can express the values of the limits (1.12). If $[x_0, t_0] \in K$, $t_0 > a$, and the condition (1.13) is fulfilled (for some $\delta > 0$) then for any $f \in \mathscr{B}(\langle a, b\rangle)$ continuous at t_0 we have (see [1], Remark 2.4)

$$\lim_{\substack{[x,t]\to[x_0,t_0] \\ t\in\langle a,b\rangle, x>\varphi(t)}} Tf(x,t) = Tf(x_0,t_0) + f(t_0)\Big[2 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{x_0,t_0}(t_0)\big)\Big], \tag{1.15}$$

$$\lim_{\substack{[x,t]\to[x_0,t_0] \\ t\in\langle a,b\rangle, x<\varphi(t)}} Tf(x,t) = Tf(x_0,t_0) - f(t_0)\frac{2}{\sqrt{\pi}} G\big(\alpha_{x_0,t_0}(t_0)\big). \tag{1.16}$$

Note that in the case $t_0 = a$ the values of those limits are zero if, in addition, $f(a) = 0$.

2. Fredholm radius of the operator $\overline{T}_0$

The operator $\overline{T}_0$ was studied in detail in [2] but it was considered only as an operator on $\mathscr{C}_0(\langle a, b\rangle)$. Here we will deal with an extension of $\overline{T}_0$ from $\mathscr{C}_0(\langle a, b\rangle)$ onto $\mathscr{B}(\langle a, b\rangle)$.

As in the previous section let $\langle a, b\rangle$ be a compact interval in $\mathbb{R}^1$, φ a continuous function with finite variation on $\langle a, b\rangle$, K is defined by (1.1). For $[x,t] \in \mathbb{R}^2$, $t > a$, the function $\alpha_{x,t}$ is defined on $\langle a, \min\{t, b\}\rangle$ by (1.2). Throughout this section we suppose that

$$\sup_{[x,t]\in K} V_K(x,t) < +\infty. \tag{2.1}$$

Then for any $[x,t] \in K$, $t > a$, the limit

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau) \tag{2.2}$$

exists. Thus for each $[x,t] \in K$, $t > a$, the function $\alpha_{x,t}$ is defined on $\langle a,t\rangle$. Note that one can see easily that the set of all $t \in (a,b\rangle$ such that $\alpha_{\varphi(t),t}(t) = 0$ is dense in $\langle a,b\rangle$ [$\alpha_{\varphi(t),t}(t) = 0$ for almost all $t \in (a,b\rangle$].

The symbols $\mathscr{C}(\langle a,b\rangle)$, $\mathscr{C}_0(\langle a,b\rangle)$, $\mathscr{B}(\langle a,b\rangle)$, $\mathscr{B}_0(\langle a,b\rangle)$ and $Tf = T_K f$ will stand for the same as in Section 1.

2.1. Operators $\widetilde{T}$, $\overline{T}$, $\overline{T}_0$. These operators were defined and studied in [2] as operators on $\mathscr{C}_0(\langle a,b\rangle)$; let us recall their definitions.

Assuming (2.1) the limits (1.12) exist and are finite for each $[x_0,t_0] \in K$ and any $f \in \mathscr{C}_0(\langle a,b\rangle)$. One can thus define

$$\widetilde{T}_+ f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' > \varphi(t')}} Tf(x',t'), \tag{2.3}$$

$$\widetilde{T}_- f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' < \varphi(t')}} Tf(x',t') \tag{2.4}$$

for $f \in \mathscr{C}_0(\langle a,b\rangle)$, $t \in \langle a,b\rangle$. It is easy to see that $\widetilde{T}_+ f, \widetilde{T}_- f \in \mathscr{C}_0(\langle a,b\rangle)$ for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ and that $\widetilde{T}_+, \widetilde{T}_-$ are linear operators on $\mathscr{C}_0(\langle a,b\rangle)$ mapping $\mathscr{C}_0(\langle a,b\rangle)$ into itself. It follows from (1.15), (1.16) that

$$\widetilde{T}_+ f(t) = Tf\big(\varphi(t),t\big) + f(t)\Big[2 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\Big], \tag{2.5}$$

$$\widetilde{T}_- f(t) = Tf\big(\varphi(t),t\big) - f(t)\frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big) \tag{2.6}$$

for $f \in \mathscr{C}_0(\langle a,b\rangle)$, $t \in (a,b\rangle$ [$\widetilde{T}_+ f(a) = \widetilde{T}_- f(a) = 0$].

Further, put

$$\overline{\mathcal{T}} f(t) = Tf\big(\varphi(t),t\big) \tag{2.7}$$

for $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in \langle a,b\rangle$. In general, $\overline{\mathcal{T}} f$ is not continuous on $\langle a,b\rangle$ even for $f \in \mathscr{C}_0(\langle a,b\rangle)$. Since $\alpha_{\varphi(t),t}(t) = 0$ for almost all $t \in (a,b\rangle$, one can see from (2.5), (2.6) and the fact that $\widetilde{T}_+, \widetilde{T}_-$ map $\mathscr{C}_0(\langle a,b\rangle)$ into $\mathscr{C}_0(\langle a,b\rangle)$ that $\overline{\mathcal{T}} f \in \mathscr{C}_0(\langle a,b\rangle)$ for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ if and only if $\alpha_{\varphi(t),t}(t) = 0$ for each $t \in (a,b\rangle$. Let I denote the identity operator on $\mathscr{C}_0(\langle a,b\rangle)$, $\mathcal{I}$ the identity operator on $\mathscr{B}(\langle a,b\rangle)$. Let $\overline{T}$ be the restriction of $\overline{\mathcal{T}}$ to $\mathscr{C}_0(\langle a,b\rangle)$. In the case $\alpha_{\varphi(t),t}(t) = 0$ for each $t \in (a,b\rangle$ we then have $\overline{T}\colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$ and

$$\widetilde{T}_+ = \overline{T} + I, \qquad \widetilde{T}_- = \overline{T} - I$$

by (2.5), (2.6). But in general $\alpha_{\varphi(t),t}(t)=0$ does not hold for all $t\in(a,b\rangle$. Then instead of $\overline{T}$ one can consider an operator $\overline{T}_0$ defined on $\mathscr{C}_0(\langle a,b\rangle)$ by

$$\overline{T}_0=\widetilde{T}_+-I \tag{2.8}$$

or [which is the same by (2.5), (2.6)]

$$\overline{T}_0=\widetilde{T}_-+I. \tag{2.9}$$

Since $\widetilde{T}_+$, $\widetilde{T}_-$ are defined on $\mathscr{C}_0(\langle a,b\rangle)$ only, $\overline{T}_0$ can be defined by (2.8) [or by (2.9)] also only on $\mathscr{C}_0(\langle a,b\rangle)$. From (2.5), (2.6) we get

$$\overline{T}_0 f(t)=\overline{T}f(t)+f(t)\Big[1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big] \tag{2.10}$$

for $f\in\mathscr{C}_0(\langle a,b\rangle)$, $t\in(a,b\rangle$ [and $\overline{T}_0f(a)=0$].

But the right hand side of (2.10) has sense for any $\mathscr{B}(\langle a,b\rangle)$ if we write here $\overline{\mathcal{T}}$ instead of $\overline{T}$. For $f\in\mathscr{B}(\langle a,b\rangle)$ define $\overline{\mathcal{T}}_0 f$ by putting $\overline{\mathcal{T}}_0 f(a)=0$ and

$$\overline{\mathcal{T}}_0 f(t)=\overline{\mathcal{T}}f(t)+f(t)\Big[1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\Big] \tag{2.11}$$

for $t\in(a,b\rangle$. Then $\overline{\mathcal{T}}_0$ is a linear extension of $\overline{T}_0$ from $\mathscr{C}_0(\langle a,b\rangle)$ onto $\mathscr{B}(\langle a,b\rangle)$.

Operators $\overline{T}_0$, $\overline{\mathcal{T}}_0$ are linear, and they are also bounded as we shall see later. We know that $\overline{T}_0\colon\mathscr{C}_0(\langle a,b\rangle)\to\mathscr{C}_0(\langle a,b\rangle)$ but it is not clear at the first sight whether analogously $\overline{\mathcal{T}}_0\colon\mathscr{B}(\langle a,b\rangle)\to\mathscr{B}(\langle a,b\rangle)$.

Recall that any linear continuous operator $P\colon\mathscr{C}_0(\langle a,b\rangle)\to\mathscr{C}_0(\langle a,b\rangle)$ can be written in the form

$$(Pf)(t)=\int_a^b f(\tau)\,\mathrm{d}\big(\lambda_t^P(\tau)\big) \tag{2.12}$$

[$f\in\mathscr{C}_0(\langle a,b\rangle)$, $t\in(a,b\rangle$], where for each $t\in(a,b\rangle$, λ_t^P is a function with finite variation on $\langle a,b\rangle$. Then

$$\|P\|=\sup_{t\in(a,b\rangle}\operatorname{var}\big[\lambda_t^P;\langle a,b\rangle\big]. \tag{2.13}$$

Now we want to express the operator $\overline{T}_0$ and also $\overline{\mathcal{T}}_0$ in the form (1.12). For $t\in(a,b\rangle$ put

$$\overline{\lambda}_t(\tau)=\begin{cases}\dfrac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(\tau)\big) & \text{for } \tau\in\langle a,t),\\[2mm] \dfrac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big) & \text{for } \tau\in\langle t,b\rangle\end{cases} \tag{2.14}$$

and further let $\overline{\lambda}_a(\tau) = 1$ for each $\tau \in \langle a, b\rangle$. Let us show that then for any $f \in \mathscr{B}(\langle a, b\rangle)$, $t \in \langle a, b\rangle$, we have

$$\overline{T} f(t) = Tf\bigl(\varphi(t), t\bigr) = \int_a^b f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}_t(\tau)\bigr) \tag{2.15}$$

(the integral on the right hand side is considered as the Lebesgue-Stieltjes integral). For $t = a$ the equality (2.15) is clear. Using Lemma 0.2 from [1] (substitution theorem) we get for $t \in (a, b\rangle$, $c \in (a, b\rangle$ that

$$\begin{aligned}\frac{2}{\sqrt{\pi}} \int_a^{\min\{c,t\}} &\mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)}\,\mathrm{d}\bigl(\alpha_{\varphi(t),t}(\tau)\bigr) \\ &= \frac{2}{\sqrt{\pi}}\Bigl[G\bigl(\alpha_{\varphi(t),t}(\min\{c,t\})\bigr) - G\bigl(\alpha_{\varphi(t),t}(a)\bigr)\Bigr] = \int_a^c \mathrm{d}\bigl(\overline{\lambda}_t(\tau)\bigr).\end{aligned}$$

Since

$$Tf\bigl(\varphi(t), t\bigr) = \frac{2}{\sqrt{\pi}} \int_a^t f(\tau)\mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)}\,\mathrm{d}\bigl(\alpha_{\varphi(t),t}(\tau)\bigr),$$

the equality (2.15) follows.

Now let us define functions $\overline{\lambda}^0_t$. Put $\overline{\lambda}^0_a(\tau) = 1$ for $\tau = \langle a, b\rangle$ and for $t \in (a, b\rangle$ let

$$\overline{\lambda}^0_t(\tau) = \begin{cases} \dfrac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(\tau)\bigr) & \text{for } \tau \in \langle a, t),\\ 1 & \text{for } \tau \in \langle t, b\rangle.\end{cases} \tag{2.16}$$

Note that in general $\overline{\lambda}^0_t$ is not continuous at $\tau = t$ and

$$\overline{\lambda}^0_t(t) - \lim_{\tau\to t-} \overline{\lambda}^0_t(\tau) = 1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr).$$

If μ_t is the Lebesgue-Stieltjes measure [on the interval $\langle a, b\rangle$] corresponding to the function $\overline{\lambda}_t$ and μ^0_t is the Lebesgue-Stieltjes measure corresponding to $\overline{\lambda}^0_t$, then [for $t \in (a, b\rangle$]

$$\mu^0_t = \mu_t + \Bigl[1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\Bigr]\delta_t,$$

where δ_t is the Dirac measure supported by $\{t\}$. It follows from this fact, equality (2.15) and the definition of $\overline{T}_0$ that

$$\overline{T}_0 f(t) = \int_a^b f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_t(\tau)\bigr) \tag{2.17}$$

for any $f \in \mathscr{B}(\langle a, b\rangle)$, $t \in \langle a, b\rangle$.

It is seen from the expression (2.17) of $\overline{\mathcal{T}}_0$ that if $f_n \in \mathscr{B}(\langle a,b\rangle)$, $f_n \to f$ *pointwise* on $\langle a,b\rangle$ and $\|f_n\| \leqslant k$ for some $k \in \mathbb{R}^1$ then $\overline{\mathcal{T}}_0 f_n \to \overline{\mathcal{T}}_0 f$ *pointwise* on $\langle a,b\rangle$. Since $\overline{T}_0 f \in \mathscr{C}_0(\langle a,b\rangle)$ for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ it follows that $\overline{\mathcal{T}}_0 f \in \mathscr{B}_0(\langle a,b\rangle)$ for any $f \in \mathscr{B}_0(\langle a,b\rangle)$. Given $f \in \mathscr{B}(\langle a,b\rangle)$ let $f_0 \in \mathscr{B}_0(\langle a,b\rangle)$ be such that $f_0(t) = f(t)$ for $t \in (a,b\rangle$. Then $\overline{\mathcal{T}}_0 f = \overline{\mathcal{T}}_0 f_0$ and we see that

$$\overline{\mathcal{T}}_0 \colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{B}(\langle a,b\rangle)$$

[even $\overline{\mathcal{T}}_0 \colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{B}_0(\langle a,b\rangle)$].

It is seen easily that

$$\operatorname{var}\bigl[\overline{\lambda}_t; \langle a,b\rangle\bigr] = \frac{2}{\sqrt{\pi}} V_K\bigl(\varphi(t),t\bigr)$$

and

$$\operatorname{var}\bigl[\overline{\lambda}_t^0; \langle a,b\rangle\bigr] = \frac{2}{\sqrt{\pi}} V_K\bigl(\varphi(t),t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|.$$

This last equality enables us to express the norms of the operators $\overline{T}_0$, $\overline{\mathcal{T}}_0$ [norms with respect to $\mathscr{C}_0(\langle a,b\rangle)$ and $\mathscr{B}(\langle a,b\rangle)$, respectively]:

$$\|\overline{T}_0\| = \|\overline{\mathcal{T}}_0\| = \sup_{t \in (a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\bigl(\varphi(t),t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|\right]. \tag{2.18}$$

Since the function G is bounded, it follows from the assumption (2.1) that the operators $\overline{T}_0$, $\overline{\mathcal{T}}_0$ are bounded.

2.2. Operators $\mathcal{H}$. In this section let ψ be a given function continuous on $\langle a,b\rangle$. For $r \geqslant 0$ define an operator ${}^r\mathcal{H}^\psi_\varphi = {}^r\mathcal{H}^\psi$ on $\mathscr{B}(\langle a,b\rangle)$ by

$${}^r\mathcal{H}^\psi_\varphi f(t) = \begin{cases} 0 & \text{if } t \leqslant a+r, \\ \dfrac{2}{\sqrt{\pi}} \displaystyle\int_a^{t-r} f(\tau) \mathrm{e}^{-\alpha_{\psi(t),t}(\tau)} \,\mathrm{d}\bigl(\alpha_{\psi(t),t}(\tau)\bigr) & \text{if } t > a+r \end{cases} \tag{2.19}$$

for $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in \langle a,b\rangle$. Denote further

$$\mathcal{H}^\psi = \mathcal{H}^\psi_\varphi = {}^0\mathcal{H}^\psi_\varphi.$$

Lemma 2.1. *Let $\psi \in \mathscr{C}(\langle a,b\rangle)$, $r > 0$. Then*

$${}^r\mathcal{H}^\psi_\varphi \colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle) \tag{2.20}$$

and ${}^r\mathcal{H}^\psi_\varphi$ is a compact operator on $\mathscr{B}(\langle a,b\rangle)$.

Proof. If $r \geqslant b-a$ then all is clear because then ${}^r\mathcal{H}_\varphi^\psi$ is the zero operator.

Suppose that $0 < r < b-a$ and denote

$$\mathscr{D} = \{ f \mid f \in \mathscr{B}(\langle a,b\rangle), \|f\| \leqslant 1 \}.$$

We have to prove that ${}^r\mathcal{H}_\varphi^\psi(\mathscr{D})$ is a relatively compact subset of $\mathscr{C}_0(\langle a,b\rangle)$. In order to do that it suffices to show that ${}^r\mathcal{H}_\varphi^\psi(\mathscr{D})$ is a set of equicontinuous and uniformly bounded functions on $\langle a,b\rangle$ vanishing at a.

In [2], Lemma 1.2, it was proved that ${}^rH_\varphi^\psi$—the restriction of ${}^r\mathcal{H}_\varphi^\psi$ onto $\mathscr{C}_0(\langle a,b\rangle)$—is compact on the space $\mathscr{C}_0(\langle a,b\rangle)$. In particular, it was proved that if $\mathscr{B}$ is the unit ball in $\mathscr{C}_0(\langle a,b\rangle)$ then ${}^rH_\varphi^\psi(\mathscr{B})$ is a set of equicontinuous and uniformly bounded functions from $\mathscr{C}_0(\langle a,b\rangle)$. In exactly the same way we can prove this for ${}^r\mathcal{H}_\varphi^\psi(\mathscr{D})$ writing everywhere $\mathscr{D}$ instead of $\mathscr{B}$ and ${}^r\mathcal{H}_\varphi^\psi$ instead of ${}^rH_\varphi^\psi$. □

Lemma 2.2. *Given $\psi \in \mathscr{C}(\langle a,b\rangle)$ suppose that $\psi(t) \neq \varphi(t)$ for each $t \in \langle a,b\rangle$. Then*

$$\mathcal{H}_\varphi^\psi \colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle) \tag{2.21}$$

and $\mathcal{H}_\varphi^\psi$ is a compact operator on $\mathscr{B}(\langle a,b\rangle)$.

Proof. We know that for each $r > 0$ the operator ${}^r\mathcal{H}_\varphi^\psi$ is compact and that (2.20) is valid. As the limit (in the norm) of compact operators is compact it suffices to show that

$$\lim_{r\to 0+} \|\mathcal{H}_\varphi^\psi - {}^r\mathcal{H}_\varphi^\psi\| = 0.$$

But this can be done in exactly the same way as $\|H_\varphi^\psi - {}^rH_\varphi^\psi\| \to 0$ (for $r \to 0+$) was proved in the proof of Corollary 1.1 in [2]. □

2.3. The Fredholm radius. Let us note that Lemmas 2.1, 2.2 are valid without the assumption (2.1)—this assumption was not used in the proofs. Throughout this section we will suppose again that the condition (2.1) is fulfilled. Let us recall that then the value $\alpha_{\varphi(t),t}(t)$ is defined for any $t \in (a,b\rangle$. Let us define a function α_K on $(a,b\rangle$ by

$$\alpha_K(t) = \left| 1 - \frac{2}{\pi} G\big(\alpha_{\varphi(t),t}(t)\big) \right|. \tag{2.22}$$

Lemma 2.3. *For each $r > 0$ the function*

$$t \mapsto \frac{2}{\pi} V_K\big(r;\varphi(t),t\big) + \alpha_K(t) \tag{2.23}$$

defined for $t \in (a, b\rangle$ is lower-semicontinuous on $(a, b\rangle$. For $f \in \mathscr{B}(\langle a, b\rangle)$ put

$$\overline{\mathcal{T}}_r f = {}^r\mathcal{H}_\varphi^\varphi f \tag{2.24}$$

and let $\overline{T}_r$ be the restriction of $\overline{\mathcal{T}}_r$ onto $\mathscr{C}_0(\langle a, b\rangle)$, $\mathscr{D}$ the unit ball in $\mathscr{B}(\langle a, b\rangle)$, $\mathscr{D}_0$ the unit ball in $\mathscr{C}_0(\langle a, b\rangle)$. Then for any $t \in (a, b\rangle$ (and $r > 0$)

$$\begin{aligned} \frac{2}{\sqrt{\pi}} V_K\big(r; \varphi(t), t\big) + \alpha_K(t) &= \sup_{f \in \mathscr{D}} \big[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\big] \\ &= \sup_{f \in \mathscr{D}_0} \big[\overline{T}_0 f(t) - \overline{T}_r f(t)\big]. \end{aligned} \tag{2.25}$$

Proof. Let $r > 0$. For $f \in \mathscr{C}_0(\langle a, b\rangle)$ we have $\overline{T}_r f \in \mathscr{C}_0(\langle a, b\rangle)$ by Lemma 2.1 [even $\overline{\mathcal{T}}_r f \in \mathscr{C}_0(\langle a, b\rangle)$ for any $f \in \mathscr{B}(\langle a, b\rangle)$]. Further, $\overline{T}_0 f \in \mathscr{C}_0(\langle a, b\rangle)$ for $f \in \mathscr{C}_0(\langle a, b\rangle)$ and thus

$$(\overline{T}_0 f - \overline{T}_r f) \in \mathscr{C}_0(\langle a, b\rangle).$$

Since the least upper bound of a family of continuous functions is a lower-semicontinuous function it suffices to show that (2.25) is valid.

Given $t \in (a, b\rangle$, $t > a + r$, $f \in \mathscr{B}(\langle a, b\rangle)$ then

$$\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t) = \int_a^b f(\tau)\, \mathrm{d}\big(\overline{\lambda}_t^0(\tau)\big) - \int_a^{t-r} f(\tau)\, \mathrm{d}\big(\overline{\lambda}_t^0(\tau)\big) = \int_{t-r}^b f(\tau)\, \mathrm{d}\big(\overline{\lambda}_t^0(\tau)\big)$$

by (2.17), (2.19). Using the definition of $\overline{\lambda}_t^0$ we see now that

$$\begin{aligned} \sup_{f \in \mathscr{D}} \big[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\big] &= \sup_{f \in \mathscr{D}_0} \big[\overline{T}_0 f(t) - \overline{T}_r f(t)\big] \\ = \operatorname{var}\big[\overline{\lambda}_t^0; \langle t - r, b\rangle\big] &= \frac{2}{\sqrt{\pi}} V_K\big(r; \varphi(t), t\big) + \alpha_K(t). \end{aligned}$$

If $t \in \langle a, a + r\rangle$ (and $t \leqslant b$) then $\overline{\mathcal{T}}_r f(t) = 0$ and thus $\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t) = \overline{\mathcal{T}}_0 f(t)$ and

$$\begin{aligned} \sup_{f \in \mathscr{D}} \big[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\big] &= \sup_{f \in \mathscr{D}_0} \big[\overline{T}_0 f(t) - \overline{T}_r f(t)\big] \\ = \operatorname{var}\big[\overline{\lambda}_t^0; \langle a, b\rangle\big] &= \frac{2}{\sqrt{\pi}} V_K\big(\varphi(t), t\big) + \alpha_K(t). \end{aligned}$$

But in this case $V_K(\varphi(t), t) = V_K(r; \varphi(t), t)$ and the assertion is proved. $\square$

Assuming that the condition (2.1) is fulfilled define for $r > 0$

$$\mathscr{F}_r K = \sup_{t \in \langle a,b \rangle} \frac{2}{\pi} V_K\bigl(r; \varphi(t), t\bigr) \tag{2.26}$$

and

$$\mathscr{F} K = \lim_{r \to 0+} \mathscr{F}_r K \tag{2.27}$$

(this limit clearly exists as $\mathscr{F}_r K$ is non-decreasing with respect to r).

Let B be a Banach space, $P\colon B \to B$ a continuous linear operator. Then by ωP we denote the essential norm of P, that is

$$\omega P = \inf_{A \in \mathfrak{M}} \|P - A\|,$$

where $\mathfrak{M}$ is the set of all compact (linear) operators on B.

Lemma 2.4. *For each* $r > 0$,

$$\mathscr{F}_r K = \sup_{t \in \langle a,b \rangle} \left[\frac{2}{\pi} V_K\bigl(r; \varphi(t), t\bigr) + \alpha_K(t) \right]. \tag{2.28}$$

Further, we have

$$\omega \overline{\mathcal{T}}_0 \leqslant \mathscr{F} K, \tag{2.29}$$

$$\omega \overline{T}_0 \leqslant \mathscr{F} K. \tag{2.30}$$

Proof. Equality (2.28) and inequality (2.30) were proved in [2], Lemma 1.4. But inequality (2.29) can be proved in exactly the same way as (2.30) was proved if we use Lemma 2.1. $\square$

Let us note that by Theorem 1.1 in [2] even $\omega \overline{T}_0 = \mathscr{F} K$. We do not know if the same is valid for $\overline{\mathcal{T}}_0$.

3. Two lemmas

A natural way of solving an integral equation is to consider it as an equation in an appropriate function space. In [2] some integral equations derived from the Fourier problem were investigated in $\mathscr{C}_0(\langle a,b\rangle)$—this corresponds to the Fourier problem with continuous boundary values. In this part we introduce only two auxiliary assertions coming from [7]. These assertions enable us to extend relevant results to the space $\mathscr{B}(\langle a,b\rangle)$ and thus to solve the Fourier problem (in a sense) for non-continuous (bounded) boundary values.

First let us recall one known and simple but useful assertion concerning the expression of the inverse of an operator by the Neumann series. By I we denote the identity operator.

Let L be a Banach space, $A\colon L\to L$ a linear operator and let $\|A\|<1$. Then $(I-A)$ has the inverse operator on L

$$(I-A)^{-1}=\sum_{n=0}^{\infty}A^n \tag{3.1}$$

and

$$\|(I-A)^{-1}\|\leqslant\frac{1}{1-\|A\|}. \tag{3.2}$$

If B is an operator on a normed space L then $\|B\|$ denotes the corresponding operator norm of B. For $x_n,x\in L$ we mean by $x_n\to x$ that $\lim\limits_{n\to\infty}x_n=x$ in the norm.

Lemma 3.1. *Given a normed linear space L let*

$$B_n\colon L\to L,\qquad B\colon L\to L\qquad\qquad (n=1,2,3,\dots)$$

be linear operators. Suppose that B_n^{-1}, B^{-1} exist, B_n^{-1} are bounded and there is $M\in\mathbb{R}^1$ such that $\|B_n^{-1}\|\leqslant M$. Suppose that

$$B_nx\to Bx \tag{3.3}$$

for each $x\in L$. If $x_n,x\in L$, $x_n\to x$, then

$$B_n^{-1}x_n\to B^{-1}x. \tag{3.4}$$

In particular, let L be a Banach space, A_n, $A\colon L\to L$ bounded linear operators and suppose that there is $\lambda\in\mathbb{R}^1$, $\lambda<1$, such that

$$\|A_n\|\leqslant\lambda,\qquad \|A\|\leqslant\lambda. \tag{3.5}$$

Suppose that

$$A_n x \to Ax \tag{3.6}$$

for each $x \in L$. *Then*

$$(I - A_n)^{-1} x_n \to (I - A)^{-1} x \tag{3.7}$$

whenever $x_n, x \in L$, $x_n \to x$.

Proof. It is seen easily that

$$B_n^{-1} - B^{-1} = B_n^{-1}(B - B_n)B^{-1}.$$

Let $x \in L$, $\varepsilon > 0$. By the assumption (3.3) there is n_0 such that

$$\big\|(B - B_n)(B^{-1}x)\big\| < \frac{\varepsilon}{M}$$

for $n > n_0$. Hence for $n > n_0$ we get

$$\|B_n^{-1}x - B^{-1}x\| = \big\|B_n^{-1}(B - B_n)(B^{-1}x)\big\| \leqslant \|B_n^{-1}\|\big\|(B - B_n)(B^{-1}x)\big\| < \varepsilon$$

and thus

$$B_n^{-1}x \to B^{-1}x$$

for any $x \in L$. If now $x_n \to x$ then

$$\begin{aligned} \|B_n^{-1}x_n - B^{-1}x\| &\leqslant \|B_n^{-1}x_n - B_n^{-1}x\| + \|B_n^{-1}x - B^{-1}x\| \\ &\leqslant M\|x_n - x\| + \|B_n^{-1}x - B^{-1}x\| \to 0 \end{aligned}$$

and the first part of the lemma is proved.

The second part follows immediately from the first using the fact that $(I - A_n)^{-1}$, $(I - A)^{-1}$ exist (under the given assumptions) and that

$$\big\|(I - A_n)^{-1}\big\| \leqslant \frac{1}{1 - \lambda}.$$

The assertion is proved. □

Lemma 3.2. *Let* X *be a Banach space,* $X_0 \subset X$ *its complete subspace. Let* Q, B *be bounded linear operators,*

$$Q\colon X \to X, \qquad B\colon X \to X_0,$$

let $\|Q\| < 1$, and suppose that $Q\colon X_0 \to X_0$. Then

$$(I - Q - B)^{-1}(0) \subset X_0. \tag{3.8}$$

Suppose in addition that B is compact. If for each $f \in X_0$ the equation (with unknown g)

$$(I - Q - B)g = f \tag{3.9}$$

has a unique solution in X_0 then for each $f \in X$, (3.9) is uniquely solvable in X.

Proof. Let $x \in X$ be such that

$$(I - Q - B)x = 0. \tag{3.10}$$

Since $\|Q\| < 1$ by assumption and $Q\colon X_0 \to X_0$ we have also

$$(I - Q)^{-1}\colon X_0 \to X_0. \tag{3.11}$$

Equality (3.10) can be written in the form $(I - Q)x = Bx$, that is in the form

$$x = (I - Q)^{-1}Bx.$$

Since by assumption $Bx \in X_0$, it follows from (3.11) that $x \in X_0$ and thus (3.8) is proved.

The second part of the lemma follows from (3.8) and the Riesz-Schauder theory. Since $\|Q\| < 1$ and B is compact, by the Riesz-Schauder theory it suffices to verify that the null space of $(I - Q - B)$ is trivial. But if (3.9) is uniquely solvable over X_0 then we get from (3.8) that

$$\bigl\{\, x \in X \bigm| (I - Q - B)x = 0 \,\bigr\} = \bigl\{ x \in X_0 \bigm| (I - Q - B)x = 0 \,\bigr\} = \{0\}$$

and the assertion is proved. $\square$

In the paper [2] the Fourier problem for continuous boundary values was solved. Using lemma 3.2 we can now extend those results into the case of bounded Baire boundary values. Analogously to [2] we shall distinguish cases of unbounded and bounded regions.

4.1. The case of unbounded region. In this section we will use the following notation. Let $\langle a,b\rangle$ be a compact interval in $\mathbb{R}^1$, $\varphi\colon \langle a,b\rangle \to \mathbb{R}^1$ a continuous function of finite variation on $\langle a,b\rangle$ and denote

$$K = \{\, [\varphi(t),t] \bigm| t \in \langle a,b\rangle \,\}. \tag{4.1}$$

Further put

$$M = \{\, [x,t] \bigm| t \in (a,b), x > \varphi(t) \,\}, \tag{4.2}$$

$$B = K \cup \{\, [x,a] \bigm| x \geqslant \varphi(a) \,\}. \tag{4.3}$$

In Section 2 functions $\widetilde{T}_{+}f$, $\widetilde{T}_{-}f$ were defined for $f \in \mathscr{C}_0(\langle a,b\rangle)$ by

$$\widetilde{T}_{+}f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t]\\ t'\in\langle a,b\rangle, x'>\varphi(t')}} Tf(x',t'), \tag{4.4}$$

$$\widetilde{T}_{-}f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t]\\ t'\in\langle a,b\rangle, x'<\varphi(t')}} Tf(x',t') \tag{4.5}$$

($t \in \langle a,b\rangle$) assuming

$$\sup_{[x,t]\in K} V_K(x,t) < +\infty; \tag{4.6}$$

in this section we will always suppose that (4.6) is fulfilled.

By (4.4), (4.5) $\widetilde{T}_{+}f$, $\widetilde{T}_{-}f$ can be defined only for $f \in \mathscr{C}_0(\langle a,b\rangle)$. As we have seen,

$$\widetilde{T}_{+}f(t) = Tf\bigl(\varphi(t),t\bigr) + f(t)\left[2 - \frac{2}{\sqrt{\pi}}G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right], \tag{4.7}$$

$$\widetilde{T}_{-}f(t) = Tf\bigl(\varphi(t),t\bigr) - \frac{2}{\sqrt{\pi}}f(t)G\bigl(\alpha_{\varphi(t),t}(t)\bigr) \tag{4.8}$$

[recall that for $[x,t] \in K$, $t > a$, the value of $\alpha_{x,t}(t)$ is defined by (2.2)].

Further, we have defined an operator $\overline{\mathcal{T}}$ on $\mathscr{B}(\langle a,b\rangle)$ by

$$\overline{\mathcal{T}}f(t) = Tf\bigl(\varphi(t),t\bigr) \tag{4.9}$$

$[f \in \mathscr{B}(\langle a,b\rangle),\ t \in \langle a,b\rangle]$. The operator $\overline{T}$ is then the restriction of $\overline{\mathcal{T}}$ on $\mathscr{C}_0(\langle a,b\rangle)$.

Let us recall also the definition of the operator $\overline{\mathcal{T}}_0$ on $\mathscr{B}(\langle a,b\rangle)$. If $f \in \mathscr{B}(\langle a,b\rangle)$ then we put $\overline{\mathcal{T}}_0 f(a) = 0$ and

$$\overline{\mathcal{T}}_0 f(t) = \overline{\mathcal{T}} f(t) + f(t)\left[1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right] \tag{4.10}$$

for $t \in (a,b\rangle$. The operator $\overline{T}_0$ is then the restriction of $\overline{\mathcal{T}}_0$ onto $\mathscr{C}_0(\langle a,b\rangle)$ and

$$\overline{T}_0\colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle), \qquad \overline{\mathcal{T}}_0\colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{B}(\langle a,b\rangle)$$

(see Section 2.1), $\overline{T}_0$, $\overline{\mathcal{T}}_0$ are bounded linear operators. Further we have [if I is the identity operator on $\mathscr{C}_0(\langle a,b\rangle)$]

$$\overline{T}_0 = \widetilde{T}_+ - I = \widetilde{T}_- + I,$$

that is

$$\widetilde{T}_+ = \overline{T}_0 + I, \qquad \widetilde{T}_- = \overline{T}_0 - I.$$

Now we define operators $\widetilde{\mathcal{T}}_+$, $\widetilde{\mathcal{T}}_-$ on $\mathscr{B}(\langle a,b\rangle)$ by

$$\widetilde{\mathcal{T}}_+ = \overline{\mathcal{T}}_0 + \mathcal{I}, \qquad \widetilde{\mathcal{T}}_- = \overline{\mathcal{T}}_0 - \mathcal{I} \tag{4.11}$$

[where $\mathcal{I}$ is the identity operator on $\mathscr{B}(\langle a,b\rangle)$]. By (4.10) and (4.7), (4.8), (4.9) [using the fact that $\overline{\mathcal{T}}_0 f(a) = 0$] we obtain the following expression of values $\widetilde{\mathcal{T}}_+ f, \widetilde{\mathcal{T}}_- f$ for $f \in \mathscr{B}(\langle a,b\rangle)$. If $f \in \mathscr{B}(\langle a,b\rangle)$ then

$$\widetilde{\mathcal{T}}_+ f(a) = f(a), \qquad \widetilde{\mathcal{T}}_- f(a) = -f(a)$$

and

$$\widetilde{\mathcal{T}}_+ f(t) = Tf\bigl(\varphi(t),t\bigr) + f(t)\left[2 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right], \tag{4.12}$$

$$\widetilde{\mathcal{T}}_- f(t) = Tf\bigl(\varphi(t),t\bigr) - \frac{2}{\sqrt{\pi}} f(t) G\bigl(\alpha_{\varphi(t),t}(t)\bigr) \tag{4.13}$$

for $t \in (a,b\rangle$. If we put formally

$$\alpha_{\varphi(a),a}(a) = 0$$

then (4.12), (4.13) are valid for any $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in \langle a,b\rangle$.

Let us recall that $\alpha_K(t)$ has been defined for $t \in \langle a,b\rangle$ by

$$\alpha_K(t) = \left|1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\right|. \tag{4.14}$$

Theorem 4.1. *Suppose that*

$$\lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \alpha_K(t)\right] < 1. \tag{4.15}$$

Then for each $g \in \mathscr{B}(\langle a,b\rangle)$ the equation

$$\widetilde{\mathcal{T}}_+ f = g \tag{4.16}$$

(and also the equation $\widetilde{\mathcal{T}}_- f = g$) has a unique solution $f \in \mathscr{B}(\langle a,b\rangle)$.

P r o o f. For $r > 0$ we have defined in Section 2.2 operators $\overline{\mathcal{T}}_r = {}^r\mathcal{H}^{\varphi}_{\varphi}$ by

$$\overline{\mathcal{T}}_r f(t) = {}^r\mathcal{H}^{\varphi}_{\varphi} f(t) = \begin{cases} 0 & \text{if } t \leqslant a + r, \\ \dfrac{2}{\sqrt{\pi}} \displaystyle\int_a^{t-r} f(\tau) \mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)} \,\mathrm{d}\big(\alpha_{\varphi(t),t}(\tau)\big) & \text{if } t > a + r \end{cases} \tag{4.17}$$

for $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in \langle a,b\rangle$. For $r > 0$ the operator $\overline{\mathcal{T}}_r$ is compact on $\mathscr{B}(\langle a,b\rangle)$ and

$$\overline{\mathcal{T}}_r \colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$$

(see Lemma 2.1). By the assumption (4.15) we can choose $r > 0$ such that

$$\sup_{t\in\langle a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \alpha_K(t)\right] < 1. \tag{4.18}$$

Let $\mathscr{D}$ be the unit ball in $\mathscr{B}(\langle a,b\rangle)$. Using (2.25) we get

$$\begin{aligned} \|\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_r\| &= \sup_{f\in\mathscr{D}} \|\overline{\mathcal{T}}_0 f - \overline{\mathcal{T}}_r f\| = \sup_{f\in\mathscr{D}} \Big\{ \sup_{t\in\langle a,b\rangle} \big[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\big] \Big\} \\ &= \sup_{t\in\langle a,b\rangle} \Big\{ \sup_{f\in\mathscr{D}} \big[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\big] \Big\} = \sup_{t\in\langle a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \alpha_K(t)\right]. \end{aligned}$$

Let us denote for a while

$$\begin{aligned} X &= \mathscr{B}(\langle a,b\rangle), & X_0 &= \mathscr{C}_0(\langle a,b\rangle), \\ B &= -\overline{\mathcal{T}}_r, & Q &= -(\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_r). \end{aligned}$$

Since $\widetilde{\mathcal{T}}_+ = \overline{\mathcal{T}}_0 + \mathcal{I}$ the equation (4.16) can be written in the form

$$(\mathcal{I} - Q - B)f = g. \tag{4.19}$$

We know that $Q\colon X \to X$, $Q\colon X_0 \to X_0$ and if $r > 0$ is such that (4.18) is valid then $\|Q\| < 1$. The operator B is compact and $B\colon X \to X_0$. It was shown in [2] that under the assumption (4.15), for each $g \in X_0 = \mathscr{C}_0(\langle a, b\rangle)$ the equation (4.19) [i.e. the equation (4.16)] has a unique solution in X_0. Now it follows immediately from Lemma 3.2 that for each $g \in X$ the equation (4.19) has a unique solution in X, i.e. (4.16) has a unique solution in $\mathscr{B}(\langle a, b\rangle)$. $\square$

Lemma 4.1. *Suppose that* (4.6) *is fulfilled. Given* $f \in \mathscr{B}(\langle a, b\rangle)$, $t_0 \in (a, b\rangle$, *suppose that* f *is continuous at* t_0. *Then* $\overline{\mathcal{T}}_0 f$ *is continuous at* t_0.

Proof. Let us first take notice that if f is continuous on a relatively open interval $J \subset (a, b\rangle$, then $\overline{\mathcal{T}}_0 f$ is continuous on J. Indeed, by Section 1 for each $t \in J$

$$\widetilde{\mathcal{T}}_+ f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ [x',t'] \in M}} Tf(x', t'),$$

which implies that $\widetilde{\mathcal{T}}_+ f$ is continuous on J. Since

$$\overline{\mathcal{T}}_0 f = \widetilde{\mathcal{T}}_+ f + f,$$

also $\overline{\mathcal{T}}_0 f$ is continuous on J.

If $f(t_0) \neq 0$ then

$$\overline{\mathcal{T}}_0 f = \overline{\mathcal{T}}_0\bigl(f - f(t_0)\bigr) + f(t_0)\overline{\mathcal{T}}_0\mathbf{1}$$

(where $\mathbf{1}$ denotes the function which equals 1 on $\langle a, b\rangle$). By the above consideration $\overline{\mathcal{T}}_0\mathbf{1}$ is continuous on $(a, b\rangle$ and it is seen that it suffices to prove the assertion for the case $f(t_0) = 0$. Let us thus suppose that $f(t_0) = 0$.

Denote

$$c = \frac{2}{\sqrt{\pi}} \sup_{[x,t] \in K} V_K(x, t) + 1;$$

$c < +\infty$ by the assumption.

Recall that for $t \in (a, b\rangle$ the function $\overline{\lambda}_t^0$ was defined by

$$\overline{\lambda}_t^0(\tau) = \begin{cases} \dfrac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(\tau)\bigr) & \text{for } \tau \in \langle a, t),\\ 1 & \text{for } \tau \in \langle t, b\rangle \end{cases}$$

[see (2.16)], and $\overline{\lambda}^0_a(\tau) = 1$ for $\tau \in \langle a, b\rangle$. For $t \in (a, b\rangle$ we then have

$$\operatorname{var}\bigl[\overline{\lambda}^0_t; \langle a, b\rangle\bigr] = \frac{2}{\sqrt{\pi}} V_K\bigl(\varphi(t), t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|.$$

Since

$$\left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right| \leqslant 1,$$

we see that

$$\operatorname{var}\bigl[\overline{\lambda}^0_t; \langle a, b\rangle\bigr] \leqslant c \tag{4.20}$$

for each $t \in (a, b\rangle$.

Let $\varepsilon > 0$. Since f is continuous at t_0 [and $f(t_0) = 0$] there is $r > 0$ such that $t_0 - r > a$ and

$$\bigl|f(\tau)\bigr| \leqslant \frac{\varepsilon}{3c} \tag{4.21}$$

for each $\tau \in \langle t_0 - r, t_0 + r\rangle \cap (a, b\rangle$. Then [see (4.20)]

$$\left|\int_{t_0-r}^{t} f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_t(\tau)\bigr)\right| \leqslant \frac{\varepsilon}{3c} \operatorname{var}\bigl[\overline{\lambda}^0_t; \langle a, b\rangle\bigr] \leqslant \frac{\varepsilon}{3} \tag{4.22}$$

for each $t \in (t_0 - r, t_0 + r) \cap (a, b\rangle$. Put

$$f_1(t) = \begin{cases} f(t) & \text{for } t \in \langle a, t_0 - r),\\ 0 & \text{for } t \in \langle t_0 - r, b\rangle. \end{cases}$$

By the above consideration $\overline{\mathcal{T}}_0 f_1$ is continuous on $(t_0 - r, b\rangle$. Thus there is $\delta > 0$, $\delta < r$, such that

$$\bigl|\overline{\mathcal{T}}_0 f_1(t) - \overline{\mathcal{T}}_0 f_1(t_0)\bigr| < \frac{\varepsilon}{3}$$

for $t \in (t_0 - \delta, t_0 + \delta) \cap (a, b\rangle$, that is

$$\left|\int_a^{t_0-r} f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_t(\tau)\bigr) - \int_a^{t_0-r} f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_{t_0}(\tau)\bigr)\right| < \frac{\varepsilon}{3}. \tag{4.23}$$

Consider $t \in (t_0 - \delta, t_0 + \delta) \cap (a, b\rangle$. Since (for such t)

$$\overline{\mathcal{T}}_0 f(t) = \int_a^{t_0-r} f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_t(\tau)\bigr) + \int_{t_0-r}^{t} f(\tau)\,\mathrm{d}\bigl(\overline{\lambda}^0_t(\tau)\bigr),$$

it follows from (4.23) and (4.22) that

$$\begin{aligned}|\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_0 f(t_0)| \leqslant {} & \left| \int_a^{t_0-r} f(\tau)\,\mathrm{d}\big(\overline{\lambda}_t^0(\tau)\big) - \int_a^{t_0-r} f(\tau)\,\mathrm{d}\big(\overline{\lambda}_{t_0}^0(\tau)\big) \right| \\ & + \left| \int_{t_0-r}^{t} f(\tau)\,\mathrm{d}\big(\overline{\lambda}_t^0(\tau)\big) \right| + \left| \int_{t_0-r}^{t_0} f(\tau)\,\mathrm{d}\big(\overline{\lambda}_{t_0}^0(\tau)\big) \right| \\ & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus we see that $\overline{\mathcal{T}}_0 f$ is continuous at t_0. □

Lemma 4.2. *Suppose the condition* (4.15) *is fulfilled. Let* $f \in \mathscr{B}(\langle a,b\rangle)$ *be the solution of the equation*

$$\widetilde{\mathcal{T}}_+ f = g$$

for a given $g \in \mathscr{B}(\langle a,b\rangle)$*. If* g *is continuous at* $t_0 \in (a,b\rangle$ *then also* f *is continuous at* t_0*.*

Proof. Choose $r > 0$ such that

$$\sup_{t\in\langle a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\big(r;\varphi(t),t\big) + \alpha_K(t) \right] < 1.$$

Using the notation from the proof of Theorem 4.1 the equation $\widetilde{\mathcal{T}}_+ f = g$ can be written in the form

$$(\mathcal{I} - Q - B) f = g, \tag{4.24}$$

where $B = -\overline{\mathcal{T}}_r$, $Q = -(\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_r)$. We know that

$$Q\colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{B}(\langle a,b\rangle), \qquad Q\colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle), \qquad \|Q\| < 1,$$
$$B\colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$$

and the operator B is compact. The equality (4.24) can be written in the form

$$(\mathcal{I} - Q) f = g + Bf,$$

i.e.

$$f = (\mathcal{I} - Q)^{-1}[g + Bf].$$

Since $\|Q\| < 1$, we have

$$(\mathcal{I} - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$$

and thus

$$f = \sum_{n=0}^{\infty} Q^n [g + Bf]. \tag{4.25}$$

Since $B\colon \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$, the function Bf is continuous on $\langle a,b\rangle$. The function g is continuous at t_0 by assumption and thus $g + Bf$ is continuous at t_0. The series in (4.25) converges in the norm in $\mathscr{B}(\langle a,b\rangle)$, that is, as a function series, it converges uniformly on $\langle a,b\rangle$. To prove that f is continuous at t_0 it suffices to show that for each $n \in \mathbb{N}$ the function $Q^n[g+Bf]$ is continuous at t_0 and to see this it suffices to show that if $h \in \mathscr{B}(\langle a,b\rangle)$ is continuous at t_0 then Qh is continuous at t_0 as well. But this follows from Lemma 4.1 since

$$Qh = -(\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_r)h = -(\widetilde{\mathcal{T}}_+ - \mathcal{I} - \overline{\mathcal{T}}_r)h = -\widetilde{\mathcal{T}}_+ h + h + \overline{\mathcal{T}}_r h,$$

$\overline{\mathcal{T}}_r h \in \mathscr{C}_0(\langle a,b\rangle)$ and $\overline{\mathcal{T}}_0 h$ is continuous at t_0 by Lemma 4.1. □

Corollary 4.1. *Let $f \in \mathscr{B}(\langle a,b\rangle)$ be the solution of the equation*

$$\widetilde{\mathcal{T}}_+ f = g$$

for a given $g \in \mathscr{B}(\langle a,b\rangle)$. Then the potential Tf solves on M [M is defined by (4.2)] the first boundary value problem with zero initial condition and with the boundary condition g on K in the sense that

$$\lim_{\substack{[x,t]\to[\varphi(t_0),t_0]\\ [x,t]\in M}} Tf(x,t) = g(t_0) \tag{4.26}$$

for each point $t_0 \in (a,b\rangle$ at which g is continuous.

Proof. It is clear that

$$\lim_{\substack{[x,t]\to[x_0,a]\\ [x,t]\in M}} Tf(x,t) = 0$$

for each $x_0 > \varphi(a)$.

Let $t_0 \in (a,b\rangle$ and suppose that g is continuous at t_0. Then by Lemma 4.2 also f is continuous at t_0 and by Section 1 [see (1.15)] the limit

$$\lim_{\substack{[x,t]\to[\varphi(t_0),t_0]\\ [x,t]\in M}} Tf(x,t) = Tf\bigl(\varphi(t_0),t_0\bigr) + f(t_0)\left[2 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t_0),t_0}(t_0)\bigr)\right] \tag{4.27}$$

exists. But by (4.12) the value of the right hand side in (4.27) is equal to $\widetilde{\mathcal{T}}_+ f(t_0)$ and since $\widetilde{\mathcal{T}}_+ f(t_0) = g(t_0)$ we see that (4.26) is valid. □

R e m a r k 4.1. In the case of the first boundary value problem with non-zero initial condition one can use the Weierstrass integral similarly to [2]. Let F be a bounded Baire function on B. A solution of the boundary value problem on M with the boundary condition F on B can be found in the form

$$u(x,t) = Tf(x,t) + \frac{1}{2\sqrt{\pi}} \int_a^{+\infty} \frac{F(\tau,a)}{\sqrt{t-a}} \exp\left(-\frac{(x-\tau)^2}{4(t-a)}\right) \,\mathrm{d}\tau, \tag{4.28}$$

where $f \in \mathscr{B}(\langle a,b\rangle)$ is the solution of the equation

$$\widetilde{\mathcal{T}}_+ f = g,$$

$g(a) = F(\varphi(a),a)$ and

$$g(t) = F\big(\varphi(t),t\big) - \frac{1}{2\sqrt{\pi}} \int_a^{+\infty} \frac{F(\tau,a)}{\sqrt{t-a}} \exp\left(-\frac{(\varphi(t)-\tau)^2}{4(t-a)}\right) \,\mathrm{d}\tau$$

for $t \in (a,b\rangle$. It follows from the well known properties of the Weierstrass integral that

$$\lim_{\substack{[x,t]\to[x_0,a]\\ [x,t]\in M}} u(x,t) = F(x_0,a) \tag{4.29}$$

for almost all $x_0 \in (\varphi(a),+\infty)$ (in the sense of linear measure). Corollary 4.1 yields

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} u(x,t) = F(x_0,t_0) \tag{4.30}$$

for those $[x_0,t_0] \in K$, $t_0 > a$, at which F is continuous.

We do not know if (4.30) is valid in general for almost all $[x_0,t_0] \in K$ (for example in the sense of linear measure on K). In [6] I. Netuka has proved an analogous assertion in the case of the Dirichlet problem in $\mathbb{R}^n$ (for the Laplace equation); the solution was expressed by means of the double layer potential. Instead of limits with respect to the given region the so-called non-tangential (angular) limits were considered (in the proof the non-tangential limits were investigated in detail for the case of discontinuous densities of the double layer potential). Analogous results for the heat potential Tf are not known yet and the question concerning the existence of limits of the form (4.30) or some analogues of angular limits *almost everywhere* on K is still open.

4.2. The case of bounded region. Now let $\langle a,b\rangle$ be a compact interval in $\mathbb{R}^1$ and let φ_1,φ_2 be two continuous functions with finite variation on $\langle a,b\rangle$ such that

$$\varphi_1(t)<\varphi_2(t)\quad\text{for each } t\in\langle a,b\rangle.$$

Let us denote

$$\begin{aligned}K_i&=\{\,[\varphi_i(t),t]\mid t\in\langle a,b\rangle\,\}\qquad\text{for } i=1,2,\\ M&=\{\,[x,t]\mid t\in(a,b),\varphi_1(t)<x<\varphi_2(t)\,\},\\ B&=K_1\cup K_2\cup\{\,[x,a]\mid \varphi_1(a)\leqslant x\leqslant\varphi_2(a)\,\}.\end{aligned}$$

For $i=1,2$, $t\in(a,b\rangle$, $x\in\mathbb{R}^1$ define

$$ {}_i\alpha_{x,t}(\tau)=\frac{x-\varphi_i(\tau)}{2\sqrt{t-\tau}} \tag{4.31}$$

for $\tau\in\langle a,t)$. Parabolic variations corresponding to the functions φ_1, φ_2 (that is to the curves K_1,K_2) will be denoted by V_{K_1}, V_{K_2}, respectively. We will suppose that

$$\sup_{t\in\langle a,b\rangle} V_{K_i}\big(\varphi_i(t),t\big)<+\infty\qquad(i=1,2) \tag{4.32}$$

and for $t\in(a,b\rangle$ define

$$ {}_i\alpha_{\varphi_i(t),t}(t)=\lim_{\tau\to t-}\frac{\varphi_i(t)-\varphi_i(\tau)}{2\sqrt{t-\tau}}\qquad(i=1,2),$$

and further

$$\alpha_{K_i}(t)=\left|1-\frac{2}{\sqrt{\pi}}G\big({}_i\alpha_{\varphi_i(t),t}(t)\big)\right|\qquad(i=1,2).$$

For $f\in\mathscr{B}(\langle a,b\rangle)$ let $T_{K_i}f$ be the heat potential corresponding to the density f considered on K_i $(i=1,2)$, that is

$$T_{K_i}f(x,t)=\frac{2}{\sqrt{\pi}}\int_a^{\min\{t,b\}} f(\tau)\mathrm{e}^{-{}_i\alpha^2_{x,t}(\tau)}\,\mathrm{d}\big({}_i\alpha_{x,t}(\tau)\big)$$

$(i=1,2)$ for $[x,t]\in\mathbb{R}^2$, $t>a$.

Further, for $f\in\mathscr{B}(\langle a,b\rangle)$ let us define functions $\widetilde{\mathcal{T}}_1 f,\widetilde{\mathcal{T}}_2 f$ such that we put $\widetilde{\mathcal{T}}_1 f(a)=\widetilde{\mathcal{T}}_2 f(a)=f(a)$ and

$$\begin{aligned}\widetilde{\mathcal{T}}_1 f(t)&=T_{K_1}f\big(\varphi_1(t),t\big)+f(t)\left[2-\frac{2}{\sqrt{\pi}}G\big({}_1\alpha_{\varphi_1(t),t}(t)\big)\right],\\ \widetilde{\mathcal{T}}_2 f(t)&=-T_{K_2}f\big(\varphi_2(t),t\big)+f(t)\frac{2}{\sqrt{\pi}}G\big({}_2\alpha_{\varphi_2(t),t}(t)\big)\end{aligned}$$

for $t \in (a,b\rangle$. We know that if $t \in (a,b\rangle$ and f is continuous at t then

$$\widetilde{\mathcal{T}}_1 f(t) = \lim_{\substack{[x',t'] \to [\varphi_1(t),t] \\ [x',t'] \in M}} T_{K_1} f(x',t'), \tag{4.33}$$

$$\widetilde{\mathcal{T}}_2 f(t) = -\lim_{\substack{[x',t'] \to [\varphi_2(t),t] \\ [x',t'] \in M}} T_{K_2} f(x',t'). \tag{4.34}$$

Let us denote

$$\mathfrak{B} = \{ [f_1,f_2] \mid f_1, f_1 \in \mathscr{B}(\langle a,b\rangle) \},$$
$$\mathfrak{C}_0 = \{ [f_1,f_2] \mid f_1, f_2 \in \mathscr{C}_0(\langle a,b\rangle) \}.$$

In the space $\mathfrak{B}$ let us consider the norm

$$\|[f_1,f_2]\| = \|[f_1,f_2]\|_{\mathfrak{B}} = \|f_1\|_{\mathscr{B}} + \|f_2\|_{\mathscr{B}}$$

($[f_1,f_2] \in \mathfrak{B}$) and analogously in $\mathfrak{C}_0$

$$\|[f_1,f_2]\| = \|[f_1,f_2]\|_{\mathfrak{C}_0} = \|f_1\|_{\mathscr{C}_0} + \|f_2\|_{\mathscr{C}_0}.$$

One can easily verify that any linear operator $P\colon \mathfrak{C}_0 \to \mathfrak{C}_0$ can be written in the form ($[f_1,f_2] \in \mathfrak{C}_0$)

$$P(f_1,f_2) = \big[P_1 f_1 + P_2 f_2, P_3 f_1 + P_4 f_2\big],$$

where P_i ($i = 1,2,3,4$) are linear operators acting on $\mathscr{C}_0(\langle a,b\rangle)$, $P_i\colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$. The operator P is bounded on $\mathfrak{C}_0$ if and only if all the operators P_i are bounded on $\mathscr{C}_0(\langle a,b\rangle)$; P is compact if and only if all the operators P_i are. Analogously for operators on $\mathfrak{B}$.

On $\mathfrak{B}$ we define an operator $\mathcal{R}$,

$$\mathcal{R}(f_1,f_2)(t) = \big[\widetilde{\mathcal{T}}_1 f_1(t) - T_{K_2} f_2(\varphi_1(t),t), \widetilde{\mathcal{T}}_2 f_2(t) + T_{K_1} f_1(\varphi_2(t),t)\big]$$

($[f_1,f_2] \in \mathfrak{B}$, $t \in \langle a,b\rangle$). Further put

$$\mathcal{R}_0 = \mathcal{R} - \mathcal{I},$$

where $\mathcal{I}$ is the identity operator on $\mathfrak{B}$.

Theorem 4.2. *Suppose that*

$$\max_{i=1,2} \left\{ \lim_{r\to 0+} \sup_{t \in (a,b\rangle} \left[\frac{2}{\pi} V_{K_i}\big(r;\varphi_i(t),t\big) + \alpha_{K_i}(t) \right] \right\} < 1. \tag{4.35}$$

Then for each $[g_1, g_2] \in \mathfrak{B}$ the equation

$$\mathcal{R}(f_1, f_2) = [g_1, g_2] \tag{4.36}$$

has in $\mathfrak{B}$ a unique solution $[f_1, f_2]$.

Proof. The proof is analogous to that of Theorem 4.1. For $r > 0$, $i = 1, 2$ put

$$^r\mathcal{H}_{\varphi_i}^{\varphi_i} f(t) = \begin{cases} 0 & \text{if } t \leqslant a + r, \\ \dfrac{2}{\sqrt{\pi}} \displaystyle\int_a^{t-r} f(\tau) \mathrm{e}^{-{}_i\alpha^2_{\varphi_i(t),t}(\tau)} \, \mathrm{d}\big({}_i\alpha_{\varphi_i(t),t}(\tau)\big) & \text{if } t > a + r \end{cases}$$

for $f \in \mathscr{B}(\langle a, b \rangle)$, $t \in \langle a, b \rangle$. For $r > 0$, $[f_1, f_2] \in \mathfrak{B}$, $t \in \langle a, b \rangle$ then put

$$\mathcal{H}^r(f_1, f_2)(t) = \Big[{}^r\mathcal{H}_{\varphi_1}^{\varphi_1} f_1(t) - T_{K_2} f_2\big(\varphi_1(t), t\big), -{}^r\mathcal{H}_{\varphi_2}^{\varphi_2} f_2(t) + T_{K_1} f_1\big(\varphi_2(t), t\big)\Big].$$

Using the notation from Section 2.2 we can write $\mathcal{H}^r$ in the form

$$\mathcal{H}^r(f_1, f_2) = \big[{}^r\mathcal{H}_{\varphi_1}^{\varphi_1} f_1 - \mathcal{H}_{\varphi_2}^{\varphi_1} f_2, -{}^r\mathcal{H}_{\varphi_2}^{\varphi_2} f_2 + \mathcal{H}_{\varphi_1}^{\varphi_2} f_1\big].$$

The operators ${}^r\mathcal{H}_{\varphi_1}^{\varphi_1}$, ${}^r\mathcal{H}_{\varphi_2}^{\varphi_2}$ [considered as operators on the space $\mathscr{B}(\langle a, b \rangle)$] are compact by Lemma 2.1 while the operators $\mathcal{H}_{\varphi_2}^{\varphi_1}, \mathcal{H}_{\varphi_1}^{\varphi_2}$ are compact by Lemma 2.2 [as $\varphi_1(t) \neq \varphi_2(t)$ for each $t \in \langle a, b \rangle$]; each of those four operators maps $\mathscr{B}(\langle a, b \rangle)$ into $\mathscr{C}_0(\langle a, b \rangle)$. It means that

$$\mathcal{H}^r \colon \mathfrak{B} \to \mathfrak{C}_0$$

and $\mathcal{H}^r$ is a compact operator on $\mathfrak{B}$.

Denote further

$$\mathcal{K}^r = \mathcal{R}_0 - \mathcal{H}^r.$$

Choose $r > 0$ such that

$$\max_{i=1,2} \left\{ \sup_{t \in \langle a, b \rangle} \left[\frac{2}{\sqrt{\pi}} V_{K_i}\big(r; \varphi_i(t), t\big) + \alpha_{K_i}(t) \right] \right\} = c < 1. \tag{4.37}$$

Then

$$\|\mathcal{K}^r\| < 1.$$

For $[f_1, f_2] \in \mathfrak{B}$ we have

$$\mathcal{K}^r(f_1, f_2) = \big[\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}_{\varphi_1}^{\varphi_1} f_1, \widetilde{\mathcal{T}}_2 f_2 - f_2 + {}^r\mathcal{H}_{\varphi_2}^{\varphi_2} f_2\big].$$

For $f_1 \in \mathscr{B}(\langle a,b\rangle)$, $t \in (a,b\rangle$

$$\big(\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1\big)(t) = \frac{2}{\sqrt{\pi}} \int_{\max\{a,t-r\}}^{t} f_1(\tau) \mathrm{e}^{-{}_1\alpha^2_{\varphi_1(t),t}(\tau)} \,\mathrm{d}\big({}_1\alpha_{\varphi_1(t),t}(\tau)\big)$$
$$+ f_1(t)\left[1 - \frac{2}{\sqrt{\pi}} G\big({}_1\alpha_{\varphi_1(t),t}(t)\big)\right]$$

and for $t = a$

$$\big(\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1\big)(a) = 0.$$

Hence we see that for $f_1 \in \mathscr{B}(\langle a,b\rangle)$

$$\|\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1\| \leqslant \|f_1\| \sup_{t\in(a,b\rangle} \left|\frac{2}{\sqrt{\pi}} V_{K_1}\big(\varphi_1(t),t\big) + \alpha_{K_1}(t)\right|. \tag{4.38}$$

In the same way we get for $f_2 \in \mathscr{B}(\langle a,b\rangle)$

$$\|\widetilde{\mathcal{T}}_2 f_2 - f_2 + {}^r\mathcal{H}^{\varphi_2}_{\varphi_2} f_2\| \leqslant \|f_2\| \sup_{t\in(a,b\rangle} \left|\frac{2}{\sqrt{\pi}} V_{K_2}\big(\varphi_2(t),t\big) + \alpha_{K_2}(t)\right|. \tag{4.39}$$

Consider now $[f_1,f_2] \in \mathfrak{B}$. If (4.37) is fulfilled then (4.38), (4.39) imply

$$\begin{aligned}
\big\|\mathcal{K}^r(f_1,f_2)\big\| &= \|\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1\| + \|\widetilde{\mathcal{T}}_2 f_2 - f_2 + {}^r\mathcal{H}^{\varphi_2}_{\varphi_2} f_2\| \\
&\leqslant \|f_1\| \sup_{t\in(a,b\rangle} \left|\frac{2}{\sqrt{\pi}} V_{K_1}\big(\varphi_1(t),t\big) + \alpha_{K_1}(t)\right| \\
&\quad + \|f_2\| \sup_{t\in(a,b\rangle} \left|\frac{2}{\sqrt{\pi}} V_{K_2}\big(\varphi_2(t),t\big) + \alpha_{K_2}(t)\right| \\
&\leqslant c\big(\|f_1\| + \|f_2\|\big) = c\big\|[f_1,f_2]\big\|.
\end{aligned}$$

Now we see that

$$\|\mathcal{K}^r\| \leqslant c < 1$$

(it is not difficult to show that even $\|\mathcal{K}^r\| = c$).

The equation (4.36) can be written in the form

$$(\mathcal{I} + \mathcal{K}^r + \mathcal{H}^r)[f_1,f_2] = [g_1,g_2]. \tag{4.40}$$

It was proved in [2] that for each $[g_1,g_2] \in \mathfrak{C}_0$ the equation (4.36) and hence (4.40) are in $\mathfrak{C}_0$ uniquely solvable. The assertion follows now from Lemma 3.2. $\square$

Lemma 4.3. *Suppose that the condition* (4.35) *is fulfilled. Given* $[g_1, g_2] \in \mathfrak{B}$ *let* $[f_1, f_2] \in \mathfrak{B}$ *be the solution of the equation*

$$\mathcal{R}(f_1, f_2) = [g_1, g_2].$$

Let $t_0 \in (a, b\rangle$. *If* g_1 *or* g_2 *is continuous at* t_0 *then* f_1 *or* f_2, *respectively, is continuous at* t_0.

P r o o f. Choose $r > 0$ such that

$$\max_{i=1,2} \left\{ \sup_{t \in (a,b\rangle} \Big[V_{K_i}\big(r; \varphi_i(t), t\big) + \alpha_{K_i}(t) \Big] \right\} < 1.$$

Like in the proof of Theorem 4.2 we will write the equation $\mathcal{R}(f_1, f_2) = [g_1, g_2]$ in the form

$$(\mathcal{I} + \mathcal{K}^r + \mathcal{H}^r)[f_1, f_2] = [g_1, g_2],$$

where

$$\mathcal{K}^r \colon \mathfrak{B} \to \mathfrak{B}, \qquad \mathcal{K}^r \colon \mathfrak{C}_0 \to \mathfrak{C}_0, \qquad \|\mathcal{K}^r\| < 1, \qquad \mathcal{H}^r \colon \mathfrak{B} \to \mathfrak{C}_0$$

and $\mathcal{H}^r$ is a compact operator. We then have

$$(\mathcal{I} + \mathcal{K}^r)[f_1, f_2] = [g_1, g_2] - \mathcal{H}^r[f_1, f_2],$$

that is

$$[f_1, f_2] = (\mathcal{I} + \mathcal{K}^r)^{-1} \big\{ [g_1, g_2] - \mathcal{H}^r[f_1, f_2] \big\}.$$

Since $\|\mathcal{K}^r\| < 1$ we have

$$(\mathcal{I} + \mathcal{K}^r)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\mathcal{K}^r)^n$$

and hence

$$[f_1, f_2] = \sum_{n=0}^{\infty} (-1)^n (\mathcal{K}^r)^n \big\{ [g_1, g_2] - \mathcal{H}^r[f_1, f_2] \big\}. \tag{4.41}$$

Since $\mathcal{H}^r \colon \mathfrak{B} \to \mathfrak{C}_0$ both components of $\mathcal{H}^r[f_1, f_2]$ are continuous on $\langle a, b\rangle$. If g_i is continuous at t_0 then the i-th component of

$$\big\{ [g_1, g_2] - \mathcal{H}^r[f_1, f_2] \big\}$$

is continuous at t_0. Analogously to the proof of Lemma 4.2 it now suffices to show that if $[h_1, h_2] \in \mathfrak{B}$ and h_1 (or h_2) is continuous at t_0 $[t_0 \in (a, b\rangle]$ then the first (the second, respectively) component of $\mathcal{K}^r[h_1, h_2]$ is continuous at t_0. Then the assertion follows from the fact that the series (4.41) converges uniformly (in each component).

Let $[h_1, h_2] \in \mathfrak{B}$ and suppose, for example, that h_1 is continuous at t_0. The first component of $\mathcal{K}^r[h_1, h_2]$ is of the form (see the proof of Theorem 4.2)

$$\widetilde{\mathcal{T}}_1 h_1 - h_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} h_1.$$

Here ${}^r\mathcal{H}^{\varphi_1}_{\varphi_1} h_1$ is continuous even on $\langle a, b\rangle$ and the continuity of $\widetilde{\mathcal{T}}_1 h_1$ at t_0 follows from Lemma 4.1. $\square$

R e m a r k 4.2. Given $[g_1, g_2] \in \mathfrak{B}$ let $[f_1, f_2] \in \mathfrak{B}$ be the solution of the equation

$$\mathcal{R}(f_1, f_2) = [g_1, g_2].$$

For $[x, t] \in M$ put

$$u(x, t) = T_{K_1} f_1(x, t) - T_{K_2} f_2(x, t).$$

For $x \in (\varphi_1(a), \varphi_2(a))$ surely

$$\lim_{\substack{[x', t'] \to [x, a] \\ [x', t'] \in M}} u(x', t') = 0.$$

It is seen from the definition of $\mathcal{R}$, Lemma 4.3 and Section 1, that if g_1 is continuous at $t_0 \in (a, b\rangle$ then

$$\lim_{\substack{[x, t] \to [\varphi_1(t_0), t_0] \\ [x, t] \in M}} u(x, t) = g_1(t_0),$$

and analogously

$$\lim_{\substack{[x, t] \to [\varphi_2(t_0), t_0] \\ [x, t] \in M}} u(x, t) = g_2(t_0)$$

if g_2 is continuous at t_0. In this sense u can be considered a solution of the first boundary value problem of the heat equation on M with zero initial condition and the boundary condition g_1 on K_1 and g_2 on K_2. In the case of non-zero initial condition one can use the Weierstrass integral analogously to Section 4.1.

4.3. Convergence of the Neumann series. In Sections 4.1, 4.2 solvability of the equation (under appropriate assumptions)

$$(\mathcal{I} + \overline{\mathcal{T}}_0) f = g$$

and of the equation

$$(\mathcal{I}+\mathcal{R}_0)[f_1,f_2]=[g_1,g_2]$$

was proved. If X is a Banach space, $B\colon X\to X$ is a bounded linear operator, $\|B\|<1$, then $(I-B)^{-1}$ exists and

$$(I-B)^{-1}=\sum_{n=0}^{\infty}B^n.$$

In particular, the series on the right hand side converges. If $\|B\|>1$ and $(I-B)^{-1}$ exists, a question arises whether the series

$$\sum_{n=0}^{\infty}B^n$$

or at least for $x\in X$ the series

$$\sum_{n=0}^{\infty}B^n x$$

converges. In the case of our equations we would like to know whether the series

$$\sum_{n=0}^{\infty}(-1)^n\overline{\mathcal{T}_0}^n g$$

converges for $g\in\mathscr{B}(\langle a,b\rangle)$ or whether the series

$$\sum_{n=0}^{\infty}(-1)^n\mathcal{R}_0^n[g_1,g_2]$$

converges for $[g_1,g_2]\in\mathfrak{B}$.

In the following we will use the notation from Section 4.2. Let us prove the following assertion.

Theorem 4.3. *Suppose that*

$$\max_{i=1,2}\left\{\lim_{r\to 0+}\sup_{t\in\langle a,b\rangle}\left[\frac{2}{\sqrt{\pi}}V_{K_i}\bigl(r;\varphi_i(t),t\bigr)+\alpha_{K_i}(t)\right]\right\}<1. \tag{4.42}$$

Then for each $[g_1,g_2]\in\mathfrak{B}$ *the series*

$$\sum_{n=0}^{\infty}(-1)^n\mathcal{R}_0^n[g_1,g_2] \tag{4.43}$$

converges (in the norm in $\mathfrak{B}$). If $[g_1, g_2] \in \mathfrak{B}$,

$$[f_1, f_2] = \sum_{n=0}^{\infty} (-1)^n \mathcal{R}_0^n [g_1, g_2],$$

then

$$\mathcal{R}[f_1, f_2] = [g_1, g_2].$$

P r o o f. Given $[g_1, g_2] \in \mathfrak{B}$ let us prove that

$$\mathcal{R}_0^n [g_1, g_2] \to [0, 0] \qquad \text{for } n \to \infty \tag{4.44}$$

in the sense of the norm in $\mathfrak{B}$, that is, the components of $\mathcal{R}_0^n[g_1, g_2]$ converge to zero uniformly on $\langle a, b\rangle$.

In the proof of Lemma 2.2 we have noted that if $\psi \in \mathscr{C}(\langle a, b\rangle)$, $\psi(t) \neq \varphi(t)$ on $\langle a, b\rangle$, then

$$\lim_{r \to 0+} \|\mathcal{H}_\varphi^\psi - {}^r\mathcal{H}_\varphi^\psi\| = 0.$$

Using the notation from Section 4.2 we thus have

$$\lim_{r \to 0+} \|\mathcal{H}_{\varphi_2}^{\varphi_1} - {}^r\mathcal{H}_{\varphi_2}^{\varphi_1}\| = 0, \qquad \lim_{r \to 0+} \|\mathcal{H}_{\varphi_1}^{\varphi_2} - {}^r\mathcal{H}_{\varphi_1}^{\varphi_2}\| = 0.$$

Since (4.42) is valid there is $r > 0$ such that

$$\max_{i=1,2} \left\{ \sup_{t \in \langle a, b\rangle} \left[\frac{2}{\sqrt{\pi}} V_{K_i}\bigl(r; \varphi_i(t), t\bigr) + \alpha_{K_i}(t) \right] \right\} = \lambda_1, \tag{4.45}$$

$$\max\bigl\{\|\mathcal{H}_{\varphi_2}^{\varphi_1} - {}^r\mathcal{H}_{\varphi_2}^{\varphi_1}\|, \|\mathcal{H}_{\varphi_1}^{\varphi_2} - {}^r\mathcal{H}_{\varphi_1}^{\varphi_2}\|\bigr\} = \lambda_2 \tag{4.46}$$

and

$$\lambda_1 + \lambda_2 = \lambda < 1. \tag{4.47}$$

Define an operator $\mathcal{S} \colon \mathfrak{B} \to \mathfrak{B}$ by

$$\mathcal{S}[f_1, f_2] = \bigl[{}^r\mathcal{H}_{\varphi_1}^{\varphi_1} f_1 - {}^r\mathcal{H}_{\varphi_2}^{\varphi_1} f_2, -{}^r\mathcal{H}_{\varphi_2}^{\varphi_2} f_2 + {}^r\mathcal{H}_{\varphi_1}^{\varphi_2} f_1\bigr]$$

for $[f_1, f_2] \in \mathfrak{B}$ and put

$$\mathcal{U} = \mathcal{R}_0 - \mathcal{S}.$$

For $[f_1, f_2] \in \mathfrak{B}$ we then have

$$\mathcal{U}[f_1, f_2] = \bigl[\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1 - \mathcal{H}^{\varphi_1}_{\varphi_2} f_2 + {}^r\mathcal{H}^{\varphi_1}_{\varphi_2} f_2,$$
$$\widetilde{\mathcal{T}}_2 f_2 - f_2 + {}^r\mathcal{H}^{\varphi_2}_{\varphi_2} f_2 + \mathcal{H}^{\varphi_2}_{\varphi_1} f_1 - {}^r\mathcal{H}^{\varphi_2}_{\varphi_1} f_1\bigr].$$

We have shown in the proof of Theorem 4.2 [see (4.38)] that

$$\|\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1\| \leqslant \|f_1\| \sup_{t\in\langle a,b\rangle} \left| \frac{2}{\sqrt{\pi}} V_{K_1}\bigl(\varphi_1(t), t\bigr) + \alpha_{K_1}(t)\right|;$$

hence

$$\|\widetilde{\mathcal{T}}_1 f_1 - f_1 - {}^r\mathcal{H}^{\varphi_1}_{\varphi_1} f_1 - \mathcal{H}^{\varphi_1}_{\varphi_2} f_2 + {}^r\mathcal{H}^{\varphi_1}_{\varphi_2} f_2\| \leqslant \lambda_1\|f_1\| + \lambda_2\|f_2\|.$$

Using (4.39) we analogously get

$$\|\widetilde{\mathcal{T}}_2 f_2 - f_2 + {}^r\mathcal{H}^{\varphi_2}_{\varphi_2} f_2 + \mathcal{H}^{\varphi_2}_{\varphi_1} f_1 - {}^r\mathcal{H}^{\varphi_2}_{\varphi_1} f_1\| \leqslant \lambda_1\|f_2\| + \lambda_2\|f_1\|.$$

Altogether we thus have

$$\bigl\|\mathcal{U}[f_1, f_2]\bigr\| \leqslant \lambda_1\|f_1\| + \lambda_2\|f_2\| + \lambda_1\|f_2\| + \lambda_2\|f_1\| = \lambda\bigl\|[f_1, f_2]\bigr\|$$

and hence

$$\|\mathcal{U}\| \leqslant \lambda < 1. \tag{4.48}$$

Let us denote

$$\mu = \max\bigl\{1, \|\mathcal{S}\|\bigr\}. \tag{4.49}$$

It is seen from the definition of $\mathcal{S}$, $\mathcal{U}$ that for $[f_1, f_2] \in \mathfrak{B}$, $t \in \langle a, a+r)$ (suppose that $r < b - a$) we have $\mathcal{S}[f_1, f_2](t) = [0, 0]$ and thus

$$\mathcal{R}_0[g_1, g_2](t) = \mathcal{U}[g_1, g_2](t).$$

Now it is seen easily that for $t \in \langle a, a+r)$,

$$\mathcal{R}_0^n[g_1, g_2](t) = \mathcal{U}^n[g_1, g_2](t)$$

for any $n \in \mathbb{N}$ and it follows from (4.48) that the components of $\mathcal{R}_0^n[g_1, g_2]$ converge on $\langle a, a+r)$ uniformly to zero.

Let t_0 be the supremum of such $t \in \langle a, b\rangle$ for which the components of $\mathcal{R}_0^n[g_1, g_2]$ converge on $\langle a, t)$ uniformly to zero—clearly $t_0 \geqslant a + r$.

Choose $t_1 < t_0$ such that $t_0 - t_1 < r$ and let us show that the components of $\mathcal{R}_0^n[g_1, g_2]$ converge on $\langle a, t_1 + r\rangle \cap \langle a, b\rangle$ uniformly to zero; it will follow then that $t_0 = b$ and (4.44) is valid.

Let $\varepsilon > 0$. Since the components of $\mathcal{R}_0^n[g_1, g_2]$ converge on $\langle a, t_1\rangle$ uniformly to zero there is n_0 such that if

$$[f_1^n, f_2^n] = \mathcal{R}_0^n[g_1, g_2]$$

then

$$|f_1^n(t)| + |f_2^n(t)| \leqslant \frac{1-\lambda}{\mu}\varepsilon \tag{4.50}$$

for $n \geqslant n_0$, $t \in \langle a, t_1\rangle$. Denote

$$\kappa = \big\|[f_1^{n_0}, f_2^{n_0}]\big\|. \tag{4.51}$$

Put

$$[h_1, h_2] = \mathcal{R}_0^{n_0+1}[g_1, g_2] = \mathcal{R}_0[f_1^{n_0}, f_2^{n_0}].$$

Since $[h_1, h_2] = [f_1^{n_0+1}, f_2^{n_0+1}]$, it follows from (4.50) that

$$|h_1(t)| + |h_2(t)| \leqslant \frac{1-\lambda}{\mu}\varepsilon \tag{4.52}$$

for $t \in \langle a, t_1\rangle$.

If we denote

$$[h_1^1, h_2^1] = \mathcal{U}[f_1^{n_0}, f_2^{n_0}], \qquad [h_1^2, h_2^2] = \mathcal{S}[f_1^{n_0}, f_2^{n_0}]$$

then

$$[h_1, h_2] = [h_1^1, h_2^1] + [h_1^2, h_2^2].$$

It follows from (4.48), (4.51) that

$$\big\|[h_1^1, h_2^1]\big\| \leqslant \lambda\kappa. \tag{4.53}$$

One can obtain from the definitions of $\mathcal{S}$ and ${}^r\mathcal{H}_\varphi^\psi$ that for $[f_1, f_2] \in \mathfrak{B}$ and $t \in \langle a, b\rangle$, $t > a + r$

$$\mathcal{S}[f_1, f_2](t) = \mathcal{S}[\overline{f}_1, \overline{f}_2], \tag{4.54}$$

where

$$[\overline{f}_1, \overline{f}_2](\tau) = \begin{cases} [f_1, f_2](\tau) & \text{if } a \leqslant \tau \leqslant t - r, \\ [0, 0] & \text{if } t - r < \tau \leqslant b. \end{cases}$$

Hence and from (4.50), (4.49) we get for $t \in \langle t_1, t_1 + r\rangle$

$$\left|h_1^2(t)\right| + \left|h_2^2(t)\right| \leqslant \mu \frac{1-\lambda}{\mu}\varepsilon = (1-\lambda)\varepsilon.$$

Together with (4.53) we now get that

$$\left|h_1(t)\right| + \left|h_2(t)\right| \leqslant \lambda\kappa + (1-\lambda)\varepsilon. \tag{4.55}$$

Let us take notice of the fact that here (4.53) is not necessary but suffices to ensure that

$$\left|h_1^1(t)\right| + \left|h_2^1(t)\right| \leqslant \lambda\kappa$$

for $t \in \langle t_1, t_1 + r\rangle \cap \langle a, b\rangle$. For this (4.51) is not necessary, it suffices to suppose that

$$\left|f_1^{n_0}(t)\right| + \left|f_2^{n_0}(t)\right| \leqslant \kappa \tag{4.56}$$

for $t \in \langle a, t_1 + r\rangle \cap \langle a, b\rangle$.

Let us recapitulate that we have shown that if

$$\left|f_1^{n_0}(t)\right| + \left|f_2^{n_0}(t)\right| \leqslant \frac{1-\lambda}{\mu}\varepsilon$$

for $\langle a, t_1\rangle$ and if (4.56) is valid for $t \in \langle a, t_1 + r\rangle \cap \langle a, b\rangle$ then

$$\left|f_1^{n_0+1}(t)\right| + \left|f_2^{n_0+1}(t)\right| \leqslant \lambda\kappa + (1-\lambda)\varepsilon$$

for $t \in \langle t_1, t_1 + r\rangle \cap \langle a, b\rangle$. Since (4.50) is valid for any $n \geqslant n_0$ we get by induction that for $m \in \mathbb{N}$,

$$\left|f_1^{n_0+m}(t)\right| + \left|f_2^{n_0+m}(t)\right| \leqslant \lambda^m\kappa + (1-\lambda)\varepsilon(\lambda^{m-1} + \lambda^{m-2} + \ldots + 1) \leqslant \lambda^m\kappa + \varepsilon$$

for $t \in \langle t_1, t_1 + r\rangle \cap \langle a, b\rangle$. If we choose m such that $\lambda^m\kappa < \varepsilon$ then

$$\left|f_1^{n_0+m}(t)\right| + \left|f_2^{n_0+m}(t)\right| < 2\varepsilon.$$

Now we see that (4.44) is valid.

D. Medková proved in [5] that if T is a bounded linear operator on a Banach space X, the Fredholm radius of T is greater than 1, then for $x \in X$ the series $\sum\limits_{n=0}^{\infty} T^n x$

converges if and only if $T^n x \to 0$. The Fredholm radius of the operator $\mathcal{R}_0$ is greater than 1 by the assumption (4.42). We have just proved that (4.44) is valid (for each $[g_1, g_2] \in \mathfrak{B}$) and thus the series (4.43) converges.

The second part of the assertion is clear. Indeed, if

$$[f_1, f_2] = \sum_{n=0}^{\infty} (-1)^n \mathcal{R}_0^n [g_1, g_2]$$

then

$$\begin{aligned}
\mathcal{R}[f_1, f_2] = (\mathcal{I} + \mathcal{R}_0)[f_1, f_2] &= \sum_{n=0}^{\infty} (-1)^n \mathcal{R}_0^n [g_1, g_2] + \mathcal{R}_0 \left\{ \sum_{n=0}^{\infty} (-1)^n \mathcal{R}_0^n [g_1, g_2] \right\} \\
&= [g_1, g_2] - \mathcal{R}_0 [g_1, g_2] + \mathcal{R}_0^2 [g_1, g_2] - \cdots \\
&\qquad + \mathcal{R}_0 [g_1, g_2] - \mathcal{R}_0^2 [g_1, g_2] + \mathcal{R}_0^3 [g_1, g_2] - \cdots \\
&= [g_1, g_2].
\end{aligned}$$

The assertion is proved. □

Note that in a similar way one can prove an analogous assertion for operators $\widetilde{\mathcal{T}}_+, \widetilde{\mathcal{T}}_-$ in the case of an unbounded region. We will not repeat here the proof (which is more lucid in this case) but only formulate the assertion.

Theorem 4.4. *Suppose that*

$$\lim_{r \to 0+} \sup_{t \in \langle a, b \rangle} \left[\frac{2}{\pi} V_K \big(r; \varphi(t), t\big) + \alpha_K(t) \right] < 1.$$

Then for each $g \in \mathscr{B}(\langle a, b \rangle)$ the series

$$\sum_{n=0}^{\infty} (-1)^n \overline{\mathcal{T}_0}^n g$$

converges. If

$$f = \sum_{n=0}^{\infty} (-1)^n \overline{\mathcal{T}_0}^n g$$

then $\widetilde{\mathcal{T}}_+ f = g$.

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Author's address: *Miroslav Dont,* Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, Technická 2, 166 27 Praha, Czech Republic, e-mail: `dont@math.feld.cvut.cz`.