

ON THE RATE OF STRONG SUMMABILITY OF DOUBLE
FOURIER SERIES

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Abstract. Estimates of the strong means of Marcinkiewicz type with the Cesàro means of negative order in one of the variables instead of square partial sums are obtained by characteristics constructed on the basis of moduli of continuity.

Keywords: double Fourier series, strong approximation

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1. PRELIMINARIES

Let L^{qq} ($1 \leq q \leq \infty$) [$L^{\infty} = C$] be the class of all 2π -periodic real-valued functions of two variables, integrable in the Lebesgue sense with q -th power (continuous) in the square $Q = [-\pi, \pi; -\pi, \pi]$. Let us define the norm of $f \in L^{qq}$ as

$$\|f\|_{L^{qq}} = \|f(x, y)\|_{L^{qq}} = \begin{cases} \left(\iint_Q |f(x, y)|^q dx dy \right)^{1/q} & \text{when } 1 \leq q < \infty, \\ \sup_{(x, y) \in Q} |f(x, y)| & \text{when } q = \infty. \end{cases}$$

Consider the trigonometric double Fourier series

$$S[f](x, y) = \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} A_{n\nu}(x, y; f)$$

of $f \in L^{qq}$ with partial sums $S_{jk}(x, y; f)$. Denote by $\sigma_{jk}^{(\gamma, \delta)}(x, y) = \sigma_{jk}^{(\gamma, \delta)}(x, y; f)$ the Cesàro means (C, γ, δ) of $S[f]$.

Then

$$\sigma_{jk}^{(\gamma, \delta)}(x, y; f) = \frac{1}{A_j^\gamma A_k^\delta} \sum_{\mu=0}^j \sum_{\nu=0}^k A_{j-\mu}^\gamma A_{k-\nu}^\delta A_{\mu\nu}(x, y; f),$$

where $j, k = 0, 1, 2, \dots$, $\gamma, \delta > -1$ and $A_i^k = \binom{i+k}{i}$.

Double Fourier series of a complex-valued function $f \in L^{qq}$ is defined as

$$S[f](x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn} e^{-i(mx+ny)},$$

where

$$c_{mn} = (2\pi)^{-2} \iint_Q f(x, y) e^{-i(mx+ny)} dx dy.$$

If $f(x, y)$ is real-valued, the Fourier coefficients c_{mn} fulfil the obvious additional conditions. Hence, in our arrangement of $S[f](x, y)$, the expressions $A_{\mu, \nu}(x, y; f)$ are of the form

$$A_{\mu, \nu}(x, y; f) = \alpha_{\mu\nu} \cos \mu x \cos \nu y + \beta_{\mu\nu} \sin \mu x \cos \nu y \\ + \gamma_{\mu\nu} \cos \mu x \sin \nu y + \delta_{\mu\nu} \sin \mu x \sin \nu y,$$

where $\alpha_{\mu\nu}, \beta_{\mu\nu}, \gamma_{\mu\nu}, \delta_{\mu\nu}$ are specific linear combinations of c_{mn} .

For the one dimensional case we will also use the notations $L^q, \|\times\|_{L^q}, S[f], (C, \delta), \sigma_k^\delta(x; f)$ which can be understood analogously to the above.

Let us introduce the integral moduli of continuity of $f \in L^{qq}$

$$\omega_1(\varrho)_{L^{qq}} = \omega_1(\varrho, f)_{L^{qq}} = \sup_{h \in \langle 0, \varrho \rangle} \|f(x+h, y) - f(x, y)\|_{L^{qq}}, \\ \omega_2(\varrho)_{L^{qq}} = \omega_2(\varrho, f)_{L^{qq}} = \sup_{h \in \langle 0, \varrho \rangle} \|f(x, y+h) - f(x, y)\|_{L^{qq}},$$

and the characteristics

$$\varphi_{i,n} = \varphi_{i,n}(q) = \begin{cases} \omega_i\left(\frac{1}{n+1}\right)_{L^{qq}} & \text{when } 1 < q < \infty, \\ \frac{1}{n+1} \sum_{\nu=0}^n \omega_i\left(\frac{1}{\nu+1}\right)_{L^{qq}} & \text{when } q = 1 \text{ or } q = \infty, \end{cases}$$

for $i = 1, 2$,

$$\Phi_{m,n} = \Phi_{m,n}(q) = \varphi_{1,m}(q) + \varphi_{2,n}(q),$$

$$\psi_{i,n} = \psi_{i,n}(p, q) = \begin{cases} \omega_i\left(\frac{1}{n+1}\right)_{L^{qq}} & \text{when } q \leq \max(p, 2), \\ (n+1)^{(1/p)-(1/q)} \omega_i\left(\frac{1}{n+1}\right)_{L^{qq}} & \text{when } \max(p, 2) < q < \infty, \end{cases}$$

for $i = 1, 2$,

$$\Psi_{m,n} = \Psi_{m,n}(p, q) = \psi_{1,m}(p, q) + \psi_{2,n}(p, q),$$

$$X_n = X_n(p, q) = \begin{cases} 1 & \text{when } q \leq \max(p, 2), \\ (n+1)^{1/p-1/q} & \text{when } \max(p, 2) < q < \infty; \end{cases}$$

$$L_n = L_n(\gamma) = L_n(\gamma, p, q) = \begin{cases} 1 & \text{when } \gamma > 0 \text{ or } \gamma = 0, p = 2, q \neq 1, \infty, \\ \log(n+2) & \text{when } \gamma = 0 \text{ and if } p = 2 \text{ then } q = 1, \infty. \end{cases}$$

We shall deal with a regular summability method, determined by an infinite functional sequence $\{\alpha_k(r)/A(r)\}$, $\alpha_k(r) \geq 0$ and $A(r) = \sum_{k=0}^{\infty} \alpha_k(r)$ for $0 < r < 1$, in such a way that the condition $w_k \rightarrow w$ when $k \rightarrow \infty$ implies

$$\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) w_k \rightarrow w \quad \text{when } r \rightarrow 1, 0 < r < 1.$$

The aim of the present paper is to estimate the following strong deviations of Marcinkiewicz type ($1 \leq q \leq \infty$):

$$P^p(r; f)_{L^{q,q}} = \left(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left\| \sigma_{kk}^{(\gamma, \delta-1)} - f \right\|_{L^{q,q}}^p \right)^{1/p} \quad (q \neq \infty),$$

$$Q^p(r; f)_{L^{q,q}} = \left\| \left(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - f(x, y) \right|^p \right)^{1/p} \right\|_{L^{q,q}}$$

$$R^{p,s}(r; f)_{L^{q,q}} = \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left(\sum_{k=2^l-1}^{2^{l+1}-2} \alpha_k(r) \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - f(x, y) \right|^p \right)^{1/p} \right\|_{L^{q,q}}^s \right)^{1/s}$$

for some $p, s > 0$, $\gamma \geq 0$, $\delta > \frac{1}{2}$ and all $r \in (0, 1)$ by means of the characteristics defined before.

Our theorems, presented in Section 2, correspond to the results announced in [3] and [4] for functions of one variable and generalize the results of [5].

The proofs of the theorems are based on a few lemmas which unfortunately cannot be deduced from the one dimensional results by iteration.

For convenience, let $N = [\frac{1}{1-r}]$, $p' = \min(p, q)$, $q' = \max(p, q)$, $\alpha_l = 2^l - 1$, $\beta_l = 2^{l+1} - 2$, $\beta'_l = \min(2^{l+1} - 2, N)$, and let C_j ($j = 1, 2, 3, \dots$) be suitable positive constants independent of f and r . Let $\tilde{\varepsilon}$ denote the real number conjugate to ε , i. e. $1/\varepsilon + 1/\tilde{\varepsilon} = 1$ ($\varepsilon, \tilde{\varepsilon} > 1$).

2. STATEMENT OF RESULTS

Considering a function $f \in L^{q\alpha}$ and numbers r running through the interval $(0, 1)$, we formulate the following seven approximation theorems:

Theorem 1. *Suppose that*

$$(1) \quad \sum_{l=0}^{\infty} \left(\sum_{k=\alpha l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \varphi_{i,k})^{\lambda p}}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \leq C_1 A(r) (1-r) \sum_{k=0}^{\infty} r^k (L_k \varphi_{i,k})^p \quad (i = 1, 2)$$

for some finite $\lambda > 1, \delta$ and p satisfying the condition $(1 - \delta)p < 1 - 1/\lambda$. Then

(i) if $(1 - \delta)q < 1 - 1/\lambda$, we have

$$(2) \quad P^p(r; f)_{L^{q\alpha}} \leq C_2 \left((1-r) \sum_{k=0}^{\infty} r^k (\Phi_{k,k} L_k)^p \right)^{1/p};$$

(ii) if the estimate (1) holds with $\psi_{i,k}$ instead of $\varphi_{i,k}$ then

$$(3) \quad P^p(r; f)_{L^{q\alpha}} \leq C_3 \left((1-r) \sum_{k=0}^{\infty} r^k (\Psi_{k,k} L_k)^p \right)^{1/p}.$$

Theorem 2. *Suppose that*

$$(4) \quad \sum_{l=0}^j \left(\sum_{k=\alpha l}^{\beta'_l} \frac{\alpha_k^\lambda(r) \varphi_{i,k}^{\lambda p}}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \leq C_4 A(r) (1-r) \sum_{k=0}^N \varphi_{i,k}^p \quad (i = 1, 2)$$

where $2^j \leq N + 1 < 2^{j+1}$ and that

$$(5) \quad \left((1-r) \sum_{k=N+1}^{\infty} \alpha_k^\lambda(r) r^{k(1-\lambda)} \right)^{1/\lambda} \leq C_5 A(r) (1-r)$$

with $\lambda > 1$ and p such that $(1 - \delta)p < 1 - 1/\lambda$. Then

(i) we have

$$(6) \quad P^p(r; f)_{L^{q\alpha}} \leq C_6 L_{N-2}(\gamma, p\tilde{\lambda}, q) X_N(\tilde{\lambda}p, q) \left((1-r) \sum_{k=0}^N \Phi_{k,k}^N \right)^{1/p};$$

(ii) if $(1 - \delta)q \leq 1 - 1/\lambda$, we have

$$(7) \quad P^p(r, f)_{L^{q\alpha}} \leq C_7 L_{N-2} \left((1-r) \sum_{k=0}^N \Phi_{k,k}^p \right)^{1/p};$$

(iii) if the estimate (4) holds with $\psi_{i,k}$ instead of $\varphi_{i,k}$, we have

$$(8) \quad P^p(r; f)_{L^{q\alpha}} \leq C_8 L_{N-2} \left((1-r) \sum_{k=0}^N \Psi_{k,k}^p(\tilde{\lambda}p, q) + \Psi_{N,N}^p(\tilde{\lambda}p, q) \right)^{1/p}.$$

Theorem 3. Suppose that

$$(9) \quad \left((1-r) \sum_{k=0}^N \alpha_k^\lambda(r) (k+1)^{-p\delta\lambda} \right)^{1/\lambda} \leq C_9 A(r) (1-r)^{\delta p+1},$$

$$(10) \quad \sum_{k=0}^N \alpha_k(r) \varphi_{i,k}^p \leq C_{10} A(r) (1-r) \sum_{k=0}^N \varphi_{i,k}^p \quad (i = 1, 2),$$

and that the condition (5) holds with $\lambda > 1$, $\delta > \frac{1}{2}$ and $p > 0$ such that $(1-\delta)p < 1 - 1/\lambda$. Then

- (i) condition (6) holds;
- (ii) if $(1-\delta)q < 1 - 1/\lambda$, we have (7).

The relation between p and q shows that sometimes it is more natural to consider the expression $Q^p(r; f)_{L^{q,q}}$, the more so that it is possible to obtain an estimate for a larger range of $1 \leq q \leq \infty$.

Theorem 4. Let the conditions (4) and (5) or (5), (9) and (10) hold with λ, δ, p as before. Then

$$(11) \quad Q^p(r; f)_{L^{q,q}} \leq C_{11} \max(1, (1-r)^{(1/p\lambda)-(1/q)}) L_{N-2} \left((1-r) \sum_{k=0}^N \Phi_{k,k}^p \right)^{1/p}.$$

Theorem 5. Let $f \in L^{q',q'}$ and let the conditions (1) or (4) and (5) or (5), (9) and (10) hold with $\varphi_{i,k} = \varphi_{i,k}(q')$ and λ, δ, p as before. Then

$$(12) \quad Q^p(r; f)_{L^{q',q'}} \leq C_{12} L_{N-2} \left((1-r) \sum_{k=0}^N \Phi_{k,k}^p(q') \right)^{1/p}.$$

Theorem 6. Let us assume (for $i = 1, 2$)

$$(13) \quad \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \varphi_{i,k})}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'} \leq C_{13} A(r) (1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k (L_k \varphi_{i,k})^{p'} \right)^{s/p'}$$

for some finite $\lambda > 1$, δ and p' such that $(1-\delta)p' < 1 - 1/\lambda$. Then

$$(14) \quad R^{p',s}(r; f)_{L^{q,q}} \leq C_{14} \left((1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k L_k^{p'} \Phi_{k,k}^{p'} \right)^{s/p'} \right)^{1/s}.$$

Theorem 7. *If the condition (13) holds with $p' = p$, $\varphi_{i,k} = \varphi_{i,k}(q')$ and with λ , δ , p as in Theorem 1, then*

$$(15) \quad R^{ps}(r; f)_{L^{q'q'}} \leq C_{15} \left((1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k (L_k \Phi_{k,k}(q'))^p \right)^{s/p} \right)^{1/s}.$$

Let us postpone the proofs of our theorems to the next section. Here we will present a few lemmas, which will be needed in these proofs.

3. FUNDAMENTAL LEMMAS

First we formulate an evident

Lemma 1. *Suppose that $g_k \in L^{qq}$ ($k = 0, 1, \dots$) and $p > 0$. Then*

$$(16) \quad \left(\sum_{k=0}^n \|g_k\|_{L^{qq}}^p \right)^{1/p} \leq \max(1, (n+1)^{1/p-1/q}) \left\| \left(\sum_{k=0}^n |g_k(x, y)|^p \right)^{1/p} \right\|_{L^{qq}},$$

$$(17) \quad \left\| \left(\sum_{k=0}^n |g_k(x, y)|^p \right)^{1/p} \right\|_{L^{qq}} \leq \max(1, (n+1)^{1/q-1/p}) \left(\sum_{k=0}^n \|g_k\|_{L^{qq}}^p \right)^{1/p},$$

$$(18) \quad \left\| \left(\sum_{k=0}^{\infty} r^{kp} |g_k(x, y)|^p \right)^{1/p} \right\|_{L^{qq}} \leq \max(1, (1-r^q)^{1/p-1/q}) \left(\sum_{k=0}^{\infty} r^{k \min(p,q)} \|g_k\|_{L^{qq}}^p \right)^{1/p}$$

for $n = 0, 1, 2, \dots$ and $r \in (0, 1)$ (cf. [6] p. 285).

Lemma 2. *Let u_k and v_k be the Poisson and the Poisson conjugate integrals of $f_k \in L^q$ and let*

$$(19) \quad F_k(z) = F(z; f_k) = u_k(r, x) + iv_k(r, x) = \frac{a_0(f_k)}{2} + \sum_{\nu=1}^{\infty} (a_{\nu}(f_k) - ib_{\nu}(f_k)) z^{\nu},$$

where $z = re^{ix}$, $0 < r < 1$ and $a_{\nu}(f_k)$, $b_{\nu}(f_k)$ are the Fourier coefficients of f_k . Then for $r \in (0, 1)$ and $p \geq 1$ we have

$$(20) \quad \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |F'_k(re^{ix})|^p \right)^{1/p} \right\|_{L^q} \leq C_{16} \frac{1}{1-r} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x)|^p \right)^{1/p} \right\|_{L^q},$$

but if $f_k \in L^{q_1}$ and $f_k^{(1)}(x) = f_k(x, y)$, then

$$(20') \quad \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |F'(re^{ix}; f_k^{(1)})|^p \right)^{1/p} \right\|_{L^{q_1}} \leq C_{17} \frac{1}{1-r} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x, y)|^p \right)^{1/p} \right\|_{L^{q_1}},$$

where $n = 0, 1, 2, \dots$ and $F'_k(z) = \frac{dF_k(z)}{dz}$.

Proof. We will prove inequality (20) only, because the proof of the other, (20'), runs similarly.

Analogously to the proof of Lemma 1 in [3] we estimate every term on the right-hand side of the inequality

$$\begin{aligned} & \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{1}{r} \frac{\partial F_k(z)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} \\ & \leq \frac{1}{r} \left(\left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial u_k(r, x)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} + \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial v_k(r, x)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} \right). \end{aligned}$$

On the basis of the Tôyama inequality ([6] p. 285) we have

$$\begin{aligned} & \left\| \frac{1}{r} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial u_k(r, x)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} \\ & = \left\| \frac{1}{r} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{1-r^2}{\pi} \int_{-\pi}^{\pi} (f_k(x) - f_k(x+t)) \frac{r \sin t \, dt}{(1-2r \cos t + r^2)^2} \right|^p \right)^{1/p} \right\|_{L^q} \\ & \leq \left\| \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x) - f_k(x+t)|^p \right)^{1/p} \frac{|t| \, dt}{(1-2r \cos t + r^2)^2} \right\|_{L^q} \\ & \leq \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x) - f_k(x+t)|^p \right)^{1/p} \right\|_{L^q} \frac{|t| \, dt}{(1-2r \cos t + r^2)^2} = I. \end{aligned}$$

If $0 < r \leq \frac{1}{2}$, then

$$I \leq \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} W_n^p(t)_{L^q} \frac{|t| \, dt}{(1-2r \cos t + r^2)^2}$$

where

$$W_n^p(\varrho)_{L^q} = \sup_{|h| \leq \varrho} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x) - f_k(x+h)|^p \right)^{1/p} \right\|_{L^q}.$$

In view of the relation

$$(21) \quad W_n^p(\delta \varrho)_{L^q} \leq (\delta + 1) W_n^p(\varrho)_{L^q} \quad (\delta > 0)$$

we obtain

$$\begin{aligned} I &\leq C_{18} W_n^p(1-r)_{L^q} \int_0^\pi \left(\frac{t}{1-r} + 1 \right) t \, dt \leq C_{19} \frac{1}{1-r} W_n^p(1-r)_{L^q} \\ &\leq \frac{2C_{19}}{1-r} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x)|^p \right)^{1/p} \right\|_{L^q}. \end{aligned}$$

In the case $\frac{1}{2} \leq r < 1$, the substitution $y = \frac{2\sqrt{r}t}{\pi(1-r)}$ leads to

$$\begin{aligned} I &\leq \frac{\pi(1+r)}{2r(1-r)} \int_0^{\frac{2\sqrt{r}}{1-r}} \frac{y}{(1+y^2)^2} W_n^p\left(\frac{\pi y(1-r)}{2\sqrt{r}}\right)_{L^q} \, dy \leq C_{20} \frac{1}{1-r} W_n^p(1-r)_{L^q} \\ &\leq \frac{2C_{20}}{1-r} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x)|^p \right)^{1/p} \right\|_{L^q} \end{aligned}$$

(cf. [2] p. 892).

The estimate of the second term is based on Lemma 2.31 from [7], p. 258, in which the modulus of the derivative $\Phi'(z)$ of a function $\Phi(z)$ regular in a disc is estimated by the Poisson integral of the modulus of $\operatorname{Re} \Phi$ (in our case $\Phi(z) = \frac{\partial}{\partial x} F_k(\varrho e^{ix})$ and the disc is $|z| < 1 - \frac{1-\varrho}{2}$). We obtain

$$\begin{aligned} &\left\| \frac{1}{r} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial v_k(r, x)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} \\ &\leq \left\| \frac{1}{r} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \int_0^r \frac{\partial}{\partial \varrho} \left(\frac{\partial}{\partial x} F_k(\varrho e^{ix}) \right) \, d\varrho \right|^p \right)^{1/p} \right\|_{L^q} \\ &\leq \frac{1}{r} \int_0^r \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \left(\frac{\partial}{\partial x} F_k(\varrho e^{ix}) \right)' \right|^p \right)^{1/p} \right\|_{L^q} \, d\varrho \\ &\leq \frac{1}{r} \int_0^r \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{4}{2\pi(1-\varrho)} \int_{-\pi}^\pi \left| \frac{\partial}{\partial x} u_k \left(1 - \frac{1-\varrho}{2}, t-x \right) \right| \right. \right. \right. \\ &\quad \left. \left. \left. \times \frac{(1 - \frac{1-\varrho}{2})^2 - (1 - (1-\varrho))^2}{2((1 - \frac{1-\varrho}{2})^2 + 2\varrho(1 - \frac{1-\varrho}{2}) \cos t + \varrho^2)} \, dt \right|^p \right)^{1/p} \right\|_{L^q} \, d\varrho \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{r} \int_0^r \frac{4}{2\pi(1-\varrho)} \int_{-\pi}^{\pi} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial}{\partial x} u_k \left(\frac{1+\varrho}{2}, t-x \right) \right|^p \right)^{1/p} \right\|_{L^q} \\
&\quad \times \frac{\left(\frac{1+\varrho}{2} \right)^2 - \varrho^2}{2 \left(\left(\frac{1+\varrho}{2} \right)^2 - \varrho(1+\varrho) \cos t + \varrho^2 \right)} dt d\varrho \\
&= \frac{1}{r} \int_0^r \frac{4}{2\pi(1-\varrho)} \frac{1+\varrho}{2} \int_{-\pi}^{\pi} \left\| \frac{2}{1+\varrho} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial}{\partial x} u_k \left(\frac{1+\varrho}{2}, x \right) \right|^p \right)^{1/p} \right\|_{L^q} \\
&\quad \times \frac{\left(\frac{1+\varrho}{2} \right)^2 - \varrho^2}{2 \left(\left(\frac{1+\varrho}{2} \right)^2 - \varrho(1+\varrho) \cos t + \varrho^2 \right)} dt d\varrho.
\end{aligned}$$

By the previous computations we have

$$\left\| \frac{2}{1+\varrho} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial}{\partial x} u_k \left(\frac{1+\varrho}{2}, x \right) \right|^p \right)^{1/p} \right\|_{L^q} \leq (C_{19} + C_{20}) \frac{2}{1-\varrho} W_n^p \left(\frac{1-\varrho}{2} \right)$$

and so we finally obtain

$$\begin{aligned}
\left\| \frac{1}{r} \left(\frac{1}{n+1} \sum_{k=0}^n \left| \frac{\partial v_k(r, x)}{\partial x} \right|^p \right)^{1/p} \right\|_{L^q} &\leq \frac{1}{r} (C_{19} + C_{20}) \int_0^r \frac{4}{1-\varrho} \frac{1}{1-\varrho} W_n^p \left(\frac{1-\varrho}{2} \right) d\varrho \\
&\leq \frac{1}{r} C_{21} \frac{r}{1-r} W_n^p(1-r) \leq \frac{2}{1-r} C_{21} \left\| \left(\frac{1}{n+1} \sum_{k=0}^n |f_k(x)|^p \right)^{1/p} \right\|_{L^q}.
\end{aligned}$$

Collecting the results, we get (20). \square

Lemma 3. Suppose that $f_k \in L^q$ or $f_k \in L^{q_q}$, $1 \leq q \leq \infty$ and $k = 0, 1, 2, \dots$. If $\delta \geq 0$ and $p \geq 1$, then

$$(22) \quad \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_{\nu}^{\delta}(x; f_{\nu}) \right|^p \right)^{1/p} \right\|_{L^q} \leq C_{22} \left\| \left(\frac{L_n^p(\delta)}{n+1} \sum_{k=n}^{2n} |f_k(x)|^p \right)^{1/p} \right\|_{L^q}$$

or

$$(22') \quad \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_{\nu}^{\delta}(x; f_{\nu}^{(1)}(x)) \right|^p \right)^{1/p} \right\|_{L^{q_q}} \leq C_{23} \left\| \left(\frac{L_n^p(\delta)}{n+1} \sum_{k=n}^{2n} |f_k(x, y)|^p \right)^{1/p} \right\|_{L^{q_q}}$$

for $n = 0, 1, 2, \dots$

Proof. In the proof of (22) we apply the integral representation of the Cesàro means (C, δ) and the Minkowski inequality. Then

$$\begin{aligned} & \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\nu}(x+t) K_{\nu}^{\delta}(t) dt \right|^p \right)^{1/p} \right\|_{L^q} \\ & \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |f_{\nu}(x+t)|^p |K_{\nu}^{\delta}(t)|^p \right)^{1/p} \right\|_{L^q} dt \\ & \leq \left\| \left(\frac{1}{n+1} \sum_{k=n}^{2n} |f_k(x)|^p \right)^{1/p} \right\|_{L^q} \frac{1}{\pi} \int_{-\pi}^{\pi} \max_{n \leq \nu \leq 2n} |K_{\nu}^{\delta}(t)| dt, \end{aligned}$$

where

$$\begin{aligned} K_k^0(t) &= \frac{\sin(k + \frac{1}{2})t}{2 \sin \frac{t}{2}}, \\ K_k^{\delta}(t) &= \frac{1}{A_k^{\delta}} \sum_{\nu=0}^k A_{k-\nu}^{\delta-1} K_k^0(t) \end{aligned}$$

for $\delta > -1$ and $k = 0, 1, 2, \dots$

Since the kernel K_{ν}^{δ} is quasi-positive for $\delta > 0$ ([7] pp. 151, 157), we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \max_{n \leq \nu \leq 2n} |K_{\nu}^{\delta}(t)| dt \leq \begin{cases} C_{22} & \text{for } \delta > 0 \\ C_{22} \log(n+2) & \text{for } \delta = 0. \end{cases}$$

Thus, for $\delta > 0$, (22) is proved.

Our estimates can be essentially improved only for $1 < q < \infty$, $\delta = 0$ and $p = 2$. In this case the inequality (see [8] formula 2.16)

$$(23) \quad \int_{-\pi}^{\pi} \left(\sum_{\nu=0}^n |S_{k_{\nu}}(x; f_{\nu})|^2 \right)^{\frac{q}{2}} dx \leq C_{24} \int_{-\pi}^{\pi} \left(\sum_{\nu=0}^n |f_{\nu}(x)|^2 \right)^{\frac{q}{2}} dx \quad (1 < q < \infty),$$

where k_{ν} is any natural-valued function of ν , for $k_{\nu} = \nu$ gives (22). In a similar way we can obtain (22') and thus the proof of Lemma 3 is complete. \square

Lemma 4. *If $\delta > \frac{1}{2}$ and $p > 0$ such that $(1 - \delta)p < 1$, then, for any q, s such that either $s = q$ or*

$$s \geq \begin{cases} \frac{p}{p-1} & \text{when } p \geq 2, \\ 2 & \text{when } 0 < p \leq 2, \end{cases}$$

$$(24) \quad \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta-1}(x; f_k)|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \leq C_{25} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q},$$

$$(24') \quad \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta-1}(x; f_k^{(1)}(x))|^p \right)^{s/p} \right)^{1/s} \right\|_{L^{qa}} \leq C_{26} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x, y)|^s \right)^{1/s} \right\|_{L^{qa}},$$

where $m, n = 0, 1, 2, \dots$

Proof. We will prove only (24), since (24') can be obtained analogously: using estimate (20') instead of (20).

It can be easily observed that

$$\begin{aligned} & \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta-1}(x; f_k)|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \\ & \leq C_{27} \left(\left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta}(x; f_k)|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \right. \\ & \quad \left. + \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta-1}(x; f_k) - \sigma_{\nu}^{\delta}(x; f_k)|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \right) \\ & = C_{27}(A^I + B^I). \end{aligned}$$

In order to estimate A^I we shall reason as in the proof of Lemma 3. Namely, for $\delta > \frac{1}{2}$, $p, s \geq 1$,

$$\begin{aligned} A^I & \leq \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f_k(x+t) K_{\nu}^{\delta}(t) dt \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \\ & \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x+t)|^s \right)^{1/s} \right\|_{L^q} \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |K_{\nu}^{\delta}(t)|^p \right)^{1/p} dt \\ & \leq \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q} \frac{1}{\pi} \int_{-\pi}^{\pi} \max_{n \leq \nu \leq 2n} |K_{\nu}^{\delta}(t)| dt \\ & \leq C_{22} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q}. \end{aligned}$$

Without loss of generality we can assume $p \geq 1$ because A^I is a non decreasing function of p and the right-hand side of (24) is independent of p .

Now, let us consider the expression

$$B(r) = \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\sum_{\nu=n}^{2n} \nu^{\delta p} r^{\nu p} |\sigma_{\nu}^{\delta-1}(x; f_k) - \sigma_{\nu}^{\delta}(x; f_k)|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q}.$$

Let $p \geq 2$ and $\tilde{p} = \frac{p}{p-2}$. Then $\tilde{p}\delta > 1$, $s \geq \tilde{p}$. Using the inequality

$$\left(\sum_{k=0}^m \|g_k\|_{L^t}^s \right)^{1/s} \leq \max(1, (m+1)^{1/s-1/t}) \left\| \left(\sum_{k=0}^m |g_k(x)|^s \right)^{1/s} \right\|_{L^t}$$

for $g_k \in L^t$, $s > 0$, $t \in \langle 1, +\infty \rangle$, $m = 0, 1, 2, \dots$ ([6], p. 285) and proceeding analogously as in [4] (see the proof of Lemma 3) we obtain

$$\begin{aligned} B(r) &\leq r C_{28} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\int_{-\pi}^{\pi} \frac{|F'_k(re^{i(x+\varphi)})|^{\tilde{p}}}{|1-re^{i\varphi}|^{\delta\tilde{p}}} d\varphi \right)^{s/\tilde{p}} \right)^{1/s} \right\|_{L^q} \\ &\leq r C_{28} \left\| \left(\int_{-\pi}^{\pi} \frac{\left(\frac{1}{m+1} \sum_{k=0}^m |F'_k(re^{i(x+\varphi)})|^s \right)^{\tilde{p}/s}}{|1-re^{i\varphi}|^{\delta\tilde{p}}} d\varphi \right)^{1/\tilde{p}} \right\|_{L^q} \quad (s \neq q). \end{aligned}$$

Applying the Minkowski inequality for $q > \tilde{p}$ and the mean value theorem for integrals for the other q , we get

$$B(r) \leq r C_{28} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |F'_k(re^{ix})|^s \right)^{1/s} \right\|_{L^q} \left\| (1-re^{ix})^{-\delta} \right\|_{L^p}.$$

Lemma 2 leads to

$$B(r) \leq C_{16} C_{28} C_{29} (1-r)^{(1/\tilde{p})-\delta-1} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q},$$

and if $r = 1 - \frac{1}{n+1}$ ($n \geq 1$), then

$$B^I \leq e \left(\frac{2}{n+1} \right)^{\delta} (n+1)^{-1/p} B \left(1 - \frac{1}{n+1} \right) < e 2^{\delta} C_{16} C_{28} C_{29} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q}.$$

In the case $p \in (0, 2)$, we can deduce from the inequality

$$(25) \quad \left(\frac{1}{\sum_{\nu=0}^{\infty} \zeta_{\nu}} \sum_{\nu=0}^{\infty} \zeta_{\nu} \xi_{\nu}^{\gamma_1} \right)^{1/\gamma_1} \leq \left(\frac{1}{\sum_{\nu=0}^{\infty} \zeta_{\nu}} \sum_{\nu=0}^{\infty} \zeta_{\nu} \xi_{\nu}^{\gamma_2} \right)^{1/\gamma_2} \quad \begin{array}{l} (0 < \gamma_1 \leq \gamma_2), \\ (\zeta_{\nu}, \xi_{\nu} > 0) \end{array}$$

that the expression B^I does not exceed

$$e^{2\delta} C_{16} C_{28} C_{29} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |f_k(x)|^s \right)^{1/s} \right\|_{L^q} \quad \text{when } s \neq q.$$

The mean value theorem implies that, for $s = q < \tilde{p}$,

$$\begin{aligned} B(r) &\leq r C_{28} \left(\frac{1}{m+1} \sum_{k=0}^m \left\| \left(\int_{-\pi}^{\pi} \frac{|F'_k(re^{i(x+\varphi)})|^{\tilde{p}}}{|1-re^{i\varphi}|^{\delta\tilde{p}}} d\varphi \right)^{1/\tilde{p}} \right\|_{L^q}^s \right)^{1/s} \\ &= r C_{28} \left(\frac{1}{m+1} \sum_{k=0}^m \|F'_k(re^{i(x+\varphi)})\|_{L^q}^s \right)^{1/s} \|(1-re^{ix})^{-\delta}\|_{L^{\tilde{p}}} \\ &= r C_{28} \left(\frac{1}{m+1} \sum_{k=0}^m \|F'_k(re^{ix})\|_{L^q}^s \right)^{1/s} \|(1-re^{ix})^{-\delta}\|_{L^{\tilde{p}}} \\ &= r C_{28} \left\| \left(\frac{1}{m+1} \sum_{k=0}^m |F'_k(re^{ix})|^s \right)^{1/s} \right\|_{L^q} \|(1-re^{ix})^{-\delta}\|_{L^{\tilde{p}}}. \end{aligned}$$

Collecting the results we obtain (24). □

Lemma 5. For any $\gamma \geq 0$ and $\delta > \frac{1}{2}$, $p > 0$ such that $(1-\delta)p < 1$ we have

$$(26) \quad \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)|^p \right)^{1/p} \right\|_{L^{qq}} \leq C_{30} L_n \varphi_{2,n},$$

$$(27) \quad \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu\nu}^{(\delta,\gamma)}(x,y) - \sigma_{\nu\nu}^{(\delta-1,\gamma)}(x,y)|^p \right)^{1/p} \right\|_{L^{qq}} \leq C_{31} L_n \varphi_{1,n}$$

when $n = 0, 1, 2, \dots$

P r o o f. First we will prove (26) and therefore we will denote by G the expression on the left-hand side of (26). Then

$$G = \left(\int_{-\pi}^{\pi} \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} |\sigma_{\nu}^{\delta}(x; [\sigma_{\nu}^{\delta}(y; f^{(2)}(y)) - \sigma_{\nu}^{\delta-1}(y; f^{(2)}(y))])|^p \right)^{1/p} \right\|_{L^q}^q dy \right)^{1/q}.$$

By virtue of Lemma 3 we obtain

$$G \leq \begin{cases} C_7 H & \text{when } \gamma > 0 \text{ or } \gamma = 0, p = 2, 1 < q < \infty, \\ C_7 \log(n+2) H & \text{when } \gamma = 0, p \neq 2 \text{ or } \gamma = 0, p = 2, q = 1, \infty. \end{cases}$$

where

$$H = \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_{\nu}^{\delta}(y; f^{(2)}(y)) - \sigma_{\nu}^{\delta-1}(y; f^{(2)}(y)) \right|^p \right)^{1/p} \right\|_{L^{q\alpha}}.$$

From the proof of Lemma 2 and by Theorem 2.30 (II) of [7], p. 258, we can obtain the following generalization of the inequality (20):

$$\left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| F'_k(re^{ix}) \right|^p \right)^{1/p} \right\|_{L^q} \leq \begin{cases} C_{32} \frac{W_n^p(1-r)_{L^q}}{1-r} & \text{when } 1 < q < \infty, \\ C_{33} \left(\frac{W_n^p(1-r)_{L^q}}{1-r} + \int_{1-r}^1 \frac{W_n^p(t)_{L^q}}{t^2} dt \right) & \text{when } q = 1, \infty. \end{cases}$$

Hence, analogously to the proof of Lemma 1 from [3], we can easily get

$$(28) \quad \left\| \left(\frac{1}{n+1} \sum_{k=0}^n \left| F'_k(re^{ix}) \right|^p \right)^{1/p} \right\|_{L^q} \leq \begin{cases} C_{32} W_m^q \left(\frac{1}{n+1} \right)_{L^q} & \text{when } 1 < q < \infty, \\ C_{33} \int_{1-r}^1 \frac{1}{t^2} W_n^p(t)_{L^q} dt & \text{when } q = 1, \infty. \end{cases}$$

Applying (28) in the proof of Lemma 4, we have

$$(29) \quad \left\| \left(\frac{1}{m+1} \sum_{k=0}^m \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_{\nu}^{\delta}(x; f_k) - \sigma_{\nu}^{\delta-1}(x; f_k) \right|^p \right)^{q/p} \right)^{1/q} \right\|_{L^q} \leq \begin{cases} C_{34} W_m^q \left(\frac{1}{n+1} \right)_{L^q} & \text{when } 1 < q < \infty, \\ C_{35} \left(\frac{1}{n+1} \sum_{\nu=0}^n W_m^q \left(\frac{1}{\nu+1} \right)_{L^q} \right) & \text{when } q = 1, \infty. \end{cases}$$

Because the inequality (29) is homogeneous with respect to the sums $\sum_{k=0}^m$, it can be easily observed that the sums can be replaced by integrals and thus we get

$$(30) \quad \left\| \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_{\nu}^{\delta}(y; f^{(2)}(y)) - \sigma_{\nu}^{\delta-1}(y; f^{(2)}(y)) \right|^p \right)^{1/p} \right\|_{L^{q\alpha}} \leq \begin{cases} C_{34} \omega_2 \left(\frac{1}{n+1} \right)_{L^{q\alpha}} & \text{when } 1 < q < \infty, \\ C_{35} \left(\frac{1}{n+1} \sum_{\nu=0}^n \omega_2 \left(\frac{1}{\nu+1} \right)_{L^{q\alpha}} \right) & \text{when } q = 1, \infty. \end{cases}$$

Finally, the estimate of G and (30) lead to (26). By symmetry (27) is also true and our proof is complete. \square

Lemma 6. Under the assumption $\gamma \geq 0$, $\delta > \frac{1}{2}$, $p > 0$, $(1 - \delta)p < 1$ and $1 \leq q < \infty$, we have

$$(31) \quad \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left\| \sigma_{\nu\nu}^{(\gamma, \delta)} - \sigma_{\nu\nu}^{(\gamma, \delta-1)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{30} L_n \psi_{2,n},$$

$$(32) \quad \left(\frac{1}{n+1} \sum_{\nu=n}^{2n} \left\| \sigma_{\nu\nu}^{(\delta, \gamma)} - \sigma_{\nu\nu}^{(\delta-1, \gamma)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{31} L_n \psi_{1,n}$$

for $n = 0, 1, 2, \dots$

Proof. For the above expressions we apply inequalities (25) and (16) when $q \leq \max(p, 2)$ or only (16) otherwise and Lemma 5 (see Lemma 2 from [3]). Thus our inequalities follow. \square

Lemma 7. If $\gamma \geq 0$ and $\delta > \frac{1}{2}$ and $p > 0$ are such that $(1 - \delta)p < 1$, then

$$(33) \quad \left\| \left((1-r) \sum_{\nu=N+1}^{\infty} r^\nu \left| \sigma_{\nu\nu}^{(\gamma, \delta)}(x, y) - \sigma_{\nu\nu}^{(\gamma, \delta-1)}(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\ \leq C_{36} \max(1, (1-r)^{1/p-1/q}) L_{N-2} \varphi_{2,N},$$

$$(34) \quad \left\| \left((1-r) \sum_{\nu=N+1}^{\infty} r^\nu \left| \sigma_{\nu\nu}^{(\delta, \gamma)}(x, y) - \sigma_{\nu\nu}^{(\delta-1, \gamma)}(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\ \leq C_{37} \max(1, (1-r)^{1/p-1/q}) L_{N-2} \varphi_{1,N},$$

when $r \in (0, 1)$.

Proof. Denote by G' the expression on the left-hand side of (33). Then Lemma 5 and inequality (18) imply

$$G' = \left\| \left((1-r) \sum_{l=\log_2(N+2)}^{\infty} \sum_{\nu=\alpha_l}^{\beta_l} r^\nu \left| \sigma_{\nu\nu}^{(\gamma, \delta)}(x, y) - \sigma_{\nu\nu}^{(\gamma, \delta-1)}(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\ \leq \left((1-r) \sum_{l=\log_2(N+2)}^{\infty} \left\| \left(\sum_{\nu=\alpha_l}^{\beta_l} r^{(p'/p)\nu} \left| \sigma_{\nu\nu}^{(\gamma, \delta)}(x, y) - \sigma_{\nu\nu}^{(\gamma, \delta-1)}(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}}^p \right)^{1/p} \\ \times \max(1, (1-r^{q/p})^{1/p-1/q})$$

$$\begin{aligned}
&\leq \max(1, (1 - r^{q/p})^{1/p-1/q}) \\
&\quad \times \left((1 - r) \sum_{l=\log_2(N+2)}^{\infty} 2^l r^{(p'/p)\alpha_l} \left\| \left(\frac{1}{2^l} \sum_{k=\alpha_l}^{\beta_l} \left| \sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y) \right|^p \right)^{1/p} \right\|_{L^{qq}}^p \right)^{1/p} \\
&\leq \max(1, (1 - r^{q/p})^{1/p-1/q}) \left((1 - r) \sum_{l=\log_2(N+2)}^{\infty} 2^l r^{(p'/p)\alpha_l} (C_{30} L_{\alpha_l} \varphi_{2,\alpha_l})^p \right)^{1/p}.
\end{aligned}$$

It is clear that

$$(35) \quad \varphi_{i,k}(q) \leq \varphi_{i,k'}(q) \quad \text{for } k \geq k', \quad i = 1, 2, \dots$$

In the case $\gamma > 0$ or $\gamma = 0$, $p = 2$, $1 < q < \infty$, using the elementary inequality

$$(36) \quad 1 - r^{1/p} \leq 1 - r < (p+1)(1 - r^{1/p}) \quad (p \geq 1)$$

and (35) we obtain

$$\begin{aligned}
G' &\leq C_{30} \left(\frac{p}{q} + 1 \right)^{1/q-1/p} \max(1, (1 - r)^{1/p-1/q}) \\
&\quad \times \left((1 - r) \sum_{l=\log_2(N+2)}^{\infty} 2^l r^{(p'/p)\alpha_l} \right)^{1/p} \varphi_{2,N+1} \\
&\leq 2^{1/p} \left(\frac{p}{p'} + 1 \right)^{1/p} \left(\frac{p}{q} + 1 \right)^{1/q-1/p} C_{30} \max(1, (1 - r)^{1/p-1/q}) \varphi_{2,N+1}.
\end{aligned}$$

In the other case (35) and (36) lead to

$$\begin{aligned}
G' &\leq C_{30} \left(\frac{q}{p} + 1 \right)^{1/q-1/p} \max(1, (1 - r)^{1/p-1/q}) \varphi_{2,N+1} \\
&\quad \times \left((1 - r) \sum_{l=\log_2(N+2)}^{\infty} 2^l r^{(p'/p)\alpha_l} \frac{\log^p(2^l + 1)}{(2^l + 1)^p} (2^l + 1)^p \right)^{1/p} \\
&\leq C_{30} \left(\frac{q}{p} + 1 \right)^{1/q-1/p} \max(1, (1 - r)^{1/p-1/q}) \varphi_{2,N+1} \\
&\quad \times \left(\frac{\log^p N}{N^p} (1 - r) \sum_{l=\log_2(N+2)}^{\infty} 2^l r^{(p'/p)\alpha_l} (2^l + 1)^p \right)^{1/p} \\
&\leq C_{30} \left(\frac{q}{p} + 1 \right)^{1/q-1/p} \max(1, (1 - r)^{1/p-1/q}) \log \frac{1}{1 - r} \varphi_{2,N+1} \\
&\quad \times \left(\frac{1 - r}{N^p} \sum_{k=[N/2]}^{\infty} k^p r^{(p'/p)k} 2^{2p+1} \right)^{1/p}
\end{aligned}$$

$$\leq C_{38} C_{30} 2^{3+1/p} \left(\frac{p}{p'} + 1 \right)^{1+1/p} \max(1, (1-r)^{1/p-1/q}) \log \frac{1}{1-r} \varphi_{2,N+1}.$$

This implies our inequality and by symmetry (34) holds, too. \square

Lemma 8. *Let $\gamma \geq 0$, $\delta > \frac{1}{2}$ and $p > 0$. Then, with $r \in (0, 1)$, we have*

$$(37) \quad \left((1-r) \sum_{\nu=N+1}^{\infty} r^{\nu} \left\| \sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{39} L_{N-2} \psi_{2,N+1},$$

$$(38) \quad \left((1-r) \sum_{\nu=N+1}^{\infty} r^{\nu} \left\| \sigma_{\nu\nu}^{(\delta,\gamma)} - \sigma_{\nu\nu}^{(\delta-1,\gamma)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{40} L_{N-2} \psi_{1,N+1}$$

for $1 \leq q < \infty$.

Proof. The proof runs along the same lines as the proof of Lemma 7. The difference, for the case $q > \max(p, 2)$, consists in the use of the inequality

$$(39) \quad \left(\sum_{k=0}^{\infty} r^{k \max(p,q)} \|g_k\|_{L^{qq}}^p \right)^{1/p} \leq \max(1, (1-r^q)^{1/p-1/q}) \left\| \left(\sum_{k=0}^{\infty} r^{kp} |g_k(x,y)|^p \right)^{1/p} \right\|_{L^{qq}}$$

instead of (18). As a consequence, the coefficient $\max(1, (1-r)^{1/p-1/q})$ changes to $(1-r)^{1/q-1/p}$.

In the case $q \leq \max(p, 2)$ we apply (25) and then (39) without any coefficient. \square

Lemma 9. *Suppose that $\gamma \geq 0$, $\delta > \frac{1}{2}$ and $p > 0$ are such that $(1-\delta)p < 1$. Then, for $1 \leq q < \infty$,*

$$(40) \quad \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^n (\nu+1)^{\delta p} \left\| \sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{41} L_n \psi_{2,n},$$

$$(41) \quad \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^n (\nu+1)^{\delta p} \left\| \sigma_{\nu\nu}^{(\delta,\gamma)} - \sigma_{\nu\nu}^{(\delta-1,\gamma)} \right\|_{L^{qq}}^p \right)^{1/p} \leq C_{42} L_n \psi_{1,n},$$

when $n = 0, 1, 2, \dots$

P r o o f. Denote by G'' the left-hand side of inequality (40). In view of Lemma 6 we obtain

$$\begin{aligned}
G'' &= \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} \sum_{\nu=\alpha_l}^{\beta_l} (\nu+1)^{\delta p} \left\| \sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)} \right\|_{L^{q,q}}^p \right)^{1/p} \\
&\leq \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} (2^{l+1}-1)^{\delta p} 2^l C_{30}^p L_{\alpha_l}^p \psi_{2,\alpha_l}^p \right)^{1/p} \\
&\leq 2^{1+\delta} C_{30} L_n \psi_{2,n} \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} 2^{l(p\delta+1)} \left(\frac{n+1}{2^l} \right)^p \right)^{1/p} \\
&= 2^{1+\delta} C_{30} L_n \psi_{2,n} \left(\frac{1}{(n+1)^{\delta p+1}} \frac{2^{(p\delta+1-p)\log_2[(n+2)/2]} - 1}{2^{p\delta+1-p} - 1} \right)^{1/p} \\
&\leq C_{43} L_n \psi_{2,n}
\end{aligned}$$

for $p\delta+1-p > 0$. Because the inequalities (41) and (40) are symmetric the desired results follow. \square

Lemma 10. *If $\gamma \geq 0$, $\delta > \frac{1}{2}$ and $p > 0$ are such that $(1-\delta)p < 1$, then*

$$\begin{aligned}
(42) \quad &\left\| \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^n (\nu+1)^{\delta p} \left| \sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y) \right|^p \right)^{1/p} \right\|_{L^{q,q}} \\
&\leq C_{44} \max(1, (n+1)^{1/q-1/p}) L_n \varphi_{2,n},
\end{aligned}$$

$$\begin{aligned}
(43) \quad &\left\| \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^n (\nu+1)^{\delta p} \left| \sigma_{\nu\nu}^{(\delta,\gamma)}(x,y) - \sigma_{\nu\nu}^{(\delta-1,\gamma)}(x,y) \right|^p \right)^{1/p} \right\|_{L^{q,q}} \\
&\leq C_{45} \max(1, (n+1)^{1/q-1/p}) L_n \varphi_{1,n}
\end{aligned}$$

for $n = 0, 1, 2, \dots$

P r o o f. The proof runs analogously to the proof of Lemma 9, but we have to apply Lemma 5 and inequality (17) instead of Lemma 6. \square

4. PROOFS OF THE THEOREMS

Proof of Theorem 1. We start with the obvious inequality

$$\begin{aligned} (P^p(r; f)_{L^{q,q}})^p &\leq 2^p \left(\frac{1}{A(r)} \sum_{k=0}^N \alpha_k(r) \left(\|\sigma_{kk}^{(\gamma, \delta-1)} - \sigma_{kk}^{(\gamma, \delta)}\|_{L^{q,q}} + \|\sigma_{kk}^{(\gamma, \delta)} - f\|_{L^{q,q}} \right) \right. \\ &\quad \left. + \frac{1}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r) \left(\|\sigma_{kk}^{(\gamma, \delta-1)} - \sigma_{kk}^{(\gamma, \delta)}\|_{L^{q,q}} + \|\sigma_{kk}^{(\gamma, \delta)} - f\|_{L^{q,q}} \right) \right) \\ &= 2^p (A^{II} + B^{II} + C^{II} + D^{II}). \end{aligned}$$

As is well known,

$$(44) \quad \left\| \sigma_{kl}^{(\chi, \eta)} - f \right\|_{L^{q,q}} \leq \begin{cases} C_{46} (\varphi_{1,k} + \varphi_{2,l}) & \text{when } \chi, \eta > 0, 1 \leq q \leq \infty \text{ or } \chi, \eta \geq 0, 1 < q < \infty, \\ C_{47} (\chi \log(l+2) + \eta \log(k+2)) (\varphi_{1,k} + \varphi_{2,l}) & \text{when } \eta = 0 \text{ or } \chi = 0, q = 1, \infty, \\ C_{48} \log(k+2) \log(l+2) (\varphi_{1,k} + \varphi_{2,l}) & \text{when } \chi = \eta = 0 \text{ and } q = 1, \infty \end{cases}$$

for $k, l = 0, 1, 2, \dots$ (see [7] pp. 178, 184).

Hence, in view of (25) and (1), we have

$$\begin{aligned} B^{II} + D^{II} &\leq C_{46}^p \frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \Phi_{kk}^p \\ &\leq 2^{1/\lambda} C_{46} \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) \Phi_{kk}^{\lambda p}}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \\ &\leq 2^{1/\lambda} C_{46}^p (1-r) \sum_{k=0}^{\infty} r^k \Phi_{kk}^p \end{aligned}$$

when $\gamma > 0$ or $\gamma = 0$ and $q \neq 1, \infty$.

If $\gamma = 0$ and $q = 1, \infty$, then

$$B^{II} + D^{II} \leq \delta^p 2^{1/\lambda} C_1 C_{47}^p (1-r) \sum_{k=0}^{\infty} r^k \log^p(k+2) \Phi_{k,k}^p.$$

In case (i) the Hölder inequality and Lemma 6 lead to

$$\begin{aligned}
A^{II} + C^{II} &= \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left\| \sigma_{kk}^{(\gamma, \delta-1)} - \sigma_{kk}^{(\gamma, \delta)} \right\|_{L^{qq}}^p \right) \\
&\leq \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \left(\frac{1}{2^l} \sum_{k=\alpha_l}^{\beta_l} \left\| \sigma_{kk}^{(\gamma, \delta-1)} - \sigma_{kk}^{(\gamma, \delta)} \right\|_{L^{qq}}^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leq \frac{C_{30}^p}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right)^{1/\lambda} L_{\alpha_l}^p \varphi_{2, \alpha_l}^p \\
&\leq \frac{2^p C_{30}^p}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \varphi_{2, k})^{p\lambda}}{(k+1)^{1-\lambda}} \right)^{1/\lambda}.
\end{aligned}$$

Next, from (1),

$$\begin{aligned}
A^{II} + C^{II} &\leq 2^p C_{30}^p C_1 (1-r) \sum_{k=0}^{\infty} r^k (L_k \varphi_{2, k})^p \\
&\leq 2^p C_{30}^p C_1 \left((1-r) \sum_{k=0}^{\infty} r^k (L_k \Phi_{k, k})^p \right).
\end{aligned}$$

In case (ii) we proceed similarly as before taking $\psi_{2, k}$ and $\Psi_{k, k}$ instead of $\varphi_{2, k}$ and $\Phi_{k, k}$, respectively.

Finally, by the inequality $\Phi_{k, k} \leq \Psi_{k, k}$ ($k = 0, 1, 2, \dots$), the desired assertion follows. \square

P r o o f of Theorem 2. Under the above notation, the condition (4) leads to

$$B^{II} \leq C_{49} L_{N-2}^p (1-r) \sum_{k=0}^N (\Phi_{k, k})^p$$

in all cases. However, in (i) we have

$$A^{II} \leq \begin{cases} C_{50} L_{N-2}^p (1-r) \sum_{k=0}^N (\Phi_{k, k})^p & \text{when } q \leq \max(\tilde{\lambda}p, 2), \\ C_{51} (1-r)^{\tilde{\lambda}p/q} \sum_{k=0}^N \left(\omega_1 \left(\frac{1}{k+1} \right)_{L^{qq}} + \omega_2 \left(\frac{1}{k+1} \right)_{L^{qq}} \right)^p & \text{when } q > \max(\tilde{\lambda}p, 2), \end{cases}$$

while in case (ii)

$$A^{II} \leq C_{52} L_{N-2}^p (1-r) \sum_{k=0}^N (\Phi_{k, k})^p,$$

and in case (iii)

$$A^{II} \leq C_{53} L_{N-2}^p (1-r) \sum_{k=0}^N \Psi_{k, k}^p(\tilde{\lambda}p, q).$$

From (44) and (5) it follows that in all cases

$$\begin{aligned}
D^{II} &\leq \frac{C_{46}^p + C_{47}^p}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r) L_k^p \Phi_{k,k}^p \\
&\leq \frac{C_{46}^p + C_{47}^p}{A(r)} \left(\sum_{k=N+1}^{\infty} \alpha_k^\lambda(r) r^{k(1-\lambda)} \right)^{1/\lambda} \left(\sum_{k=N+1}^{\infty} r^k (L_k \Phi_{k,k})^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leq C_{54} \left((1-r) \sum_{k=N+1}^{\infty} r^k (L_k \Phi_{k,k})^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leq C_{54} \Phi_{N,N}^p \left((1-r) \sum_{k=N+1}^{\infty} r^k L_k^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}}.
\end{aligned}$$

If $L_k = 1$, then the expression in brackets is smaller than 1, but if $L_k = \log(k+2)$ then, similarly as in the proof of Lemma 7, we obtain

$$\begin{aligned}
D^{II} &\leq C_{54} \Phi_{N,N}^p \left(\left(\frac{\log N}{N} \right)^{p\tilde{\lambda}} (1-r) \sum_{k=N+1}^{\infty} r^k (k+2)^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leq C_{55} \log^p \frac{1}{1-r} \Phi_{N,N}^p \left((1-r)^{p\tilde{\lambda}+1} \frac{1}{(1-r)^{p\tilde{\lambda}+1}} \right) \\
&= C_{55} \log^p \frac{1}{1-r} \Phi_{N,N}^p.
\end{aligned}$$

To estimate C^{II} we use the Hölder inequality, condition (5) and Lemma 8. Then

$$\begin{aligned}
C^{II} &\leq \frac{1}{(1-r)A(r)} \left((1-r) \sum_{k=N+1}^{\infty} \alpha_k^\lambda(r) r^{k(1-\lambda)} \right)^{1/\lambda} \\
&\quad \times \left((1-r) \sum_{k=N+1}^{\infty} r^k \left\| \sigma_{kk}^{(\gamma, \delta-1)} - \sigma_{kk}^{(\gamma, \delta)} \right\|_{L^{qq}}^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leq C_{39}^p C_5 L_{N-2}^p \psi_{2,N+1}^p(p\tilde{\lambda}, q).
\end{aligned}$$

When $(1-\delta)q < 1 - 1/\lambda$, we obtain

$$C^{II} \leq C_{39}^p C_5 L_{N-2}^p \varphi_{2,N+1}^p.$$

Because $\Phi_{k,k}$ is a non increasing function of k and $\Phi_{k,k} \leq \Psi_{k,k}$ for $k = 0, 1, 2, \dots$, then collecting the estimates, we obtain our result. \square

P r o o f of Theorem 3. We use the same notation as in the proof of Theorem 1. Then, by (10) and (44), we get

$$\begin{aligned} B^{II} + D^{II} &\leq (C_{46}^p + C_{47}^p) \frac{1}{A(r)} \left(\sum_{k=0}^N + \sum_{k=N+1}^{\infty} \right) \alpha_k(r) L_k^p \Phi_{k,k}^p \\ &\leq C_{10} (C_{46}^p + C_{47}^p) L_{N-2}^p (1-r) \sum_{k=0}^N \Phi_{k,k}^p + (C_{54} + C_{55}) L_{N-2}^p \Phi_{N,N}^p \\ &\leq \left(C_{10} (C_{46}^p + C_{47}^p) + 2(C_{54} + C_{55}) \right) L_{N-2}^p (1-r) \sum_{k=0}^N \Phi_{k,k}^p. \end{aligned}$$

To estimate A^{II} and C^{II} we use the Hölder inequality. By the condition (9) we obtain

$$\begin{aligned} A^{II} &\leq \frac{1}{A(r)} \left(\sum_{k=0}^N \alpha_k^\lambda(r) (k+1)^{-\delta\lambda p} \right)^{1/\lambda} \left(\sum_{k=0}^N (k+1)^{\delta\tilde{\lambda}p} \left\| \sigma_{k,k}^{(\gamma,\delta-1)} - \sigma_{k,k}^{(\gamma,\delta)} \right\|_{L^{q\tilde{\lambda}}}^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\ &\leq C_9 \left((1-r)^{p\delta\tilde{\lambda}+1} \sum_{k=0}^N (k+1)^{\delta\tilde{\lambda}p} \left\| \sigma_{k,k}^{(\gamma,\delta-1)} - \sigma_{k,k}^{(\gamma,\delta)} \right\|_{L^{q\tilde{\lambda}}}^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \end{aligned}$$

and by the condition (5) we have

$$\begin{aligned} C^{II} &\leq \frac{1}{A(r)} \left(\sum_{k=N+1}^{\infty} \alpha_k^\lambda(r) r^{k(1-\lambda)} \right)^{1/\lambda} \left(\sum_{k=N+1}^{\infty} r^k \left\| \sigma_{k,k}^{(\gamma,\delta-1)} - \sigma_{k,k}^{(\gamma,\delta)} \right\|_{L^{q\tilde{\lambda}}}^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\ &\leq C_5 \left((1-r) \sum_{k=N+1}^{\infty} r^k \left\| \sigma_{k,k}^{(\gamma,\delta-1)} - \sigma_{k,k}^{(\gamma,\delta)} \right\|_{L^{q\tilde{\lambda}}}^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}}. \end{aligned}$$

Further, applying Lemmas 8 and 9 and collecting the results, we get our assertion. \square

P r o o f of Theorem 4. Clearly,

$$\begin{aligned} Q^p(r; f)_{L^{q\tilde{\lambda}}} &\leq 4^{1/p} \left(\left\| \left(\frac{1}{A(r)} \sum_{k=0}^N \alpha_k(r) \left| \sigma_{k,k}^{(\gamma,\delta-1)}(x,y) - \sigma_{k,k}^{(\gamma,\delta)}(x,y) \right|^p \right)^{1/p} \right\|_{L^{q\tilde{\lambda}}} \right. \\ &\quad \left. + \left\| \left(\frac{1}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r) \left| \sigma_{k,k}^{(\gamma,\delta-1)}(x,y) - \sigma_{k,k}^{(\gamma,\delta)}(x,y) \right|^p \right)^{1/p} \right\|_{L^{q\tilde{\lambda}}} \right) \end{aligned}$$

$$\begin{aligned}
& + \left\| \left(\frac{1}{A(r)} \sum_{k=0}^N \alpha_k(r) \left| \sigma_{kk}^{(\gamma, \delta)}(x, y) - f(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\
& + \left\| \left(\frac{1}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r) \left| \sigma_{kk}^{(\gamma, \delta)}(x, y) - f(x, y) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\
& = 4^{1/p} (A^{III} + B^{III} + C^{III} + D^{III}).
\end{aligned}$$

In view of (44), (18) and (5), in the case $L_n = 1$ we have

$$D^{III} \leq \begin{cases} C_{46} \Phi_{N,N} & \text{when } q \geq p, \\ C_{46}^{1/p} (1-r)^{1/p-1/q} \Phi_{N,N} 2^{1-1/\lambda p} \left(1 + \frac{2(p/q-1)}{1-1/\lambda}\right)^{1/q-1/p} & \text{when } q < p. \end{cases}$$

In the case $L_n = \log(n+2)$, for $q \geq p$, we obtain

$$\begin{aligned}
D^{III} & \leq C_{47} \Phi_{N,N} \left(\frac{1}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r) \log^p(k+2) \right)^{1/p} \\
& \leq C_{47} C_5^{1/p} \Phi_{N,N} \left((1-r) \sum_{k=N+1}^{\infty} r^k \log^{\tilde{\lambda} p}(k+2) \right)^{1/p\tilde{\lambda}} \\
& \leq C_{47} C_5^{1/p} \log\left(\frac{1}{1-r}\right) \Phi_{N,N} \left(\frac{1-r}{N^{\tilde{\lambda} p}} \sum_{k=N+1}^{\infty} (k+2)^{\tilde{\lambda} p} r^k \right)^{1/p\tilde{\lambda}} \\
& \leq C_{56} C_{47} C_5^{1/p} \log\left(\frac{1}{1-r}\right) \Phi_{N,N},
\end{aligned}$$

and for $q < p$,

$$\begin{aligned}
D^{III} & \leq C_{47} \Phi_{N,N} (1-r_0)^{1/p-1/q} \left(\frac{1}{A(r)} \sum_{k=N+1}^{\infty} r_0^{k(\frac{q}{p}-1)} \alpha_k(r) \log^p(k+2) \right)^{1/p} \\
& \leq C_{47} C_5^{1/p} \Phi_{N,N} (1-r_0)^{1/p-1/q} \left((1-r) \sum_{k=N+1}^{\infty} r^k r_0^{k(\frac{q}{p}-1)\tilde{\lambda}} \log^{\tilde{\lambda} p}(k+2) \right)^{1/p\tilde{\lambda}} \\
& \leq C_5^{1/p} C_{47} (1-r_0)^{1/p-1/q} \log\left(\frac{1}{1-r}\right) \Phi_{N,N} \left(\frac{1-r}{N^{\tilde{\lambda} p}} \sum_{k=N+1}^{\infty} (k+2)^{\tilde{\lambda} p} r^k r_0^{k(\frac{q}{p}-1)\tilde{\lambda}} \right)^{1/p\tilde{\lambda}}.
\end{aligned}$$

Taking $r_0 = r^{1/\tilde{\lambda}}$, inequality (36) implies

$$D^{III} \leq C_{57} C_{47} C_5^{1/p} (1-r)^{1/p\tilde{\lambda}-1/q} \log\left(\frac{1}{1-r}\right) \Phi_{N,N}.$$

In view of condition (5) and Lemma 7 we have

$$\begin{aligned} B^{III} &\leq C_5^{1/p} \left\| \left((1-r) \sum_{k=N+1}^{\infty} r^k \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - \sigma_{kk}^{(\gamma, \delta)}(x, y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{q\tilde{\lambda}}} \\ &\leq C_5^{1/p} C_{36} \max(1, (1-r)^{1/p\tilde{\lambda}-1/q}) L_{N-2} \Phi_{N,N}. \end{aligned}$$

For the estimation of expressions A^{III} and C^{III} we first consider the case when the conditions (4) and (5) hold. Then, by (4), (17) and Lemma 5,

$$\begin{aligned} A^{III} &\leq \left\| \left(\frac{1}{A(r)} \sum_{i=0}^j \left(\sum_{k=\alpha_i}^{\beta'_i} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right) \left(\frac{1}{2^i} \sum_{k=\alpha_i}^{\beta'_i} \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - \sigma_{kk}^{(\gamma, \delta)}(x, y) \right|^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \right)^{1/p} \right\|_{L^{q\tilde{\lambda}}} \\ &\leq C_{30} \max(1, (1+j)^{1/q-1/p\tilde{\lambda}}) \left(\frac{1}{A(r)} \sum_{l=0}^j \left(\sum_{k=\alpha_l}^{\beta'_l} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right) L_{\alpha_l}^p \varphi_{2, \alpha_l}^p \right)^{1/p} \\ &\leq 2 C_4^{1/p} C_{30} \max(1, (1-r)^{1/p\tilde{\lambda}-1/q}) L_{N-2} \left((1-r) \sum_{k=0}^N \varphi_{2,k}^p \right)^{1/p}. \end{aligned}$$

Applying (4), (17) and (44) we obtain

$$\begin{aligned} C^{III} &\leq (C_{46} + C_{47}) \max(1, (1+N)^{1/q-1/p\tilde{\lambda}}) \left(\frac{1}{A(r)} \sum_{k=0}^N \alpha_k(r) (L_k \varphi_{2,k})^p \right)^{1/p} \\ &\leq (C_{46} + C_{47}) \max(1, (1-r)^{1/p\tilde{\lambda}-1/q}) L_N 2^{1/q} C_4^{1/p} \left((1-r) \sum_{k=0}^N \varphi_{2,k}^p \right)^{1/p}. \end{aligned}$$

In the other case, i. e. when the conditions (5), (9) and (10) hold, Lemma 10 yields

$$\begin{aligned} A^{III} &\leq \left(\frac{1}{A(r)} \right)^{1/p} \left(\sum_{k=0}^N \alpha_k^\lambda(r) (k+1)^{-\delta\lambda p} \right)^{1/\lambda p} \\ &\quad \times \left\| \left(\sum_{k=0}^N (k+1)^{\delta\tilde{\lambda} p} \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - \sigma_{kk}^{(\gamma, \delta)}(x, y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{q\tilde{\lambda}}} \\ &\leq C_9^{1/p} \left\| \left((1-r)^{p\delta\tilde{\lambda}+1} \sum_{k=0}^N (k+1)^{\delta\tilde{\lambda} p} \left| \sigma_{kk}^{(\gamma, \delta-1)}(x, y) - \sigma_{kk}^{(\gamma, \delta)}(x, y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{q\tilde{\lambda}}} \\ &\leq C_9^{1/p} C_{44} 2^{\delta+1/p\tilde{\lambda}} \max(1, (1-r)^{1/p\tilde{\lambda}-1/q}) L_N \varphi_{2,N} \\ &\leq C_9^{1/p} C_{44} 2^{\delta+1/p\tilde{\lambda}} \max(1, (1-r)^{1/p\tilde{\lambda}-1/q}) L_N \left((1-r) \sum_{k=0}^N \varphi_{2,k}^p \right)^{1/p}. \end{aligned}$$

In view of (44) we conclude

$$\begin{aligned} C^{III} &\leq (C_{46} + C_{47}) \max(1, (1 + N)^{1/q-1/p}) \left(\frac{1}{A(r)} \sum_{k=0}^N \alpha_k(r) (L_k \varphi_{2,k})^p \right)^{1/p} \\ &\leq (C_{46} + C_{47}) C_{25}^{1/p} \max(1, (1 - r)^{1/p\tilde{\lambda}-1/q}) L_N \left((1 - r) \sum_{k=0}^N \varphi_{2,k}^p \right)^{1/p} 2^{1/q}. \end{aligned}$$

Putting together our results we complete our proof. \square

Proof of Theorem 5. In the cases considered in Theorem 4 the proof proceeds similarly to the preceding ones and the factor

$$\max(1, (1 - r)^{1/p-1/q}) = \max(1, (1 - r)^{1/p-1/q'})$$

is equal to 1. Therefore, we concentrate our attention to the case when only the condition (1) holds. Then, by (44) and the Minkowski inequality, we obtain

$$\begin{aligned} &\left\| \left(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta)}(x, y) - f(x, y) \right|^p \right)^{1/p} \right\|_{L^{q'q'}} \\ &\leq (C_{46} + C_{47}) \left(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) (L_k \Phi_{k,k}(q'))^p \right)^{1/p} \\ &\leq (C_{46} + C_{47}) 2^{1/\lambda} \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \Phi_{k,k}(q'))^{p\lambda}}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \right)^{1/p} \\ &\leq 2^{1/\lambda} (C_{46} + C_{47}) C_1^{1/p} C_{58} L_{N-2} \left((1 - r) \sum_{k=0}^N \Phi_{kk}^p(q') \right)^{1/p}, \end{aligned}$$

and by Lemma 5,

$$\begin{aligned} &\left\| \left(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta-1)}(x, y) - \sigma_{kk}^{(\gamma,\delta)}(x, y) \right|^p \right)^{1/p} \right\|_{L^{q'q'}} \\ &\leq \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \left\| \left(\frac{1}{2^l} \sum_{k=\alpha_l}^{\beta_l} \left| \sigma_{kk}^{(\gamma,\delta-1)}(x, y) - \sigma_{k,k}^{(\gamma,\delta)}(x, y) \right|^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}p} \right\|_{L^{q'q'}}^p \\ &\leq 2 C_{30} \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \varphi_{2,k}(q'))^{p\lambda}}{(k+1)^{1-\lambda}} \right)^{1/\lambda} \right)^{1/p} \end{aligned}$$

$$\begin{aligned} &\leq 2 C_{30} C_1^{1/p} \left((1-r) \sum_{k=0}^{\infty} r^k (L_k \varphi_{2,k}(q'))^p \right)^{1/p} \\ &\leq 2 C_{58} C_{30} C_1^{1/p} L_{N-2} \left((1-r) \sum_{k=0}^N \Phi_{kk}^p(q') \right)^{1/p}. \end{aligned}$$

The proof of Theorem 5 is thus completed. \square

Proof of Theorem 6. We easily get the estimate

$$\begin{aligned} &(R^{p',s}(r; f)_{L^{q,q}})^s \\ &\leq 2^{1+s/p'} \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left(\sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y) \right|^{p'} \right)^{1/p'} \right\|_{L^{q,q}}^s \right. \\ &\quad \left. + \frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left(\sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta)}(x,y) - f(x,y) \right|^{p'} \right)^{1/p'} \right\|_{L^{q,q}}^s \right) \\ &= 2^{1+s/p'} (A^{IV} + B^{IV}). \end{aligned}$$

By (13) and Lemma 5 we obtain

$$\begin{aligned} A^{IV} &\leq \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r)}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'} (C_{30} L_{\alpha_l} \varphi_{2,\alpha_l})^s \right) \\ &\leq C_{30}^s \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) (L_k \varphi_{2,k})^{p'\lambda}}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'} \\ &\leq C_{30}^s C_{13} (1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k (L_k \varphi_{2,k})^{p'} \right)^{s/p'} \\ &\leq C_{30}^s C_{13} \left((1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k (L_k \Phi_{k,k})^{p'} \right)^{s/p'} \right). \end{aligned}$$

From (17) and (44) we have

$$\begin{aligned} B^{IV} &\leq \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left\| \sigma_{kk}^{(\gamma,\delta)}(x,y) - f(x,y) \right\|_{L^{q,q}}^{p'} \right)^{s/p'} \\ &\leq (C_{46}^s + C_{47}^s) \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) (L_k \Phi_{k,k})^{p'} \right)^{s/p'} \end{aligned}$$

$$\begin{aligned} &\leq 2^{s/\lambda p'} (C_{46}^s + C_{47}^s) \frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^\lambda(r) L_k^{\lambda p'} \Phi_{k,k}^{\lambda p'}}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'} \\ &\leq 2^{s/\lambda p'} (C_{46}^s + C_{47}^s) C_{13} \left((1-r) \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} r^k (L_k \Phi_{k,k})^{p'} \right)^{s/p'} \right). \end{aligned}$$

Hence, the desired result is now evident. \square

Proof of Theorem 7. The proof proceeds similarly to that of Theorem 6. We only apply the inequality (17) with exponents p and q' such that $q'/p \geq 1$. \square

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