CHARACTERIZING BINARY DISCRIMINATOR ALGEBRAS

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Abstract. The concept of the (dual) binary discriminator was introduced by R. Halaš, I. G. Rosenberg and the author in 1999. We study finite algebras having the (dual) discriminator as a term function. In particular, a simple characterization is obtained for such algebras with a majority term function.

Keywords: binary discriminator, majority function, compatible relation, finite algebra

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The concept of the discriminator was introduced by A. F. Pixley [7] and dualized by E. Fried and A. F. Pixley in [5]. Recall that a ternary function t on a set A is the *discriminator* if it satisfies

$$t(x,x,z)=z, \quad \text{and} \quad t(x,y,z)=x \quad \text{if} \quad x\neq y.$$

A ternary function d is the *dual discriminator* if it satisfies

$$d(x,x,z)=x, \quad \text{and} \quad d(x,y,z)=z \quad \text{if} \quad x\neq y.$$

The following concepts were introduced in [3]:

Definition. Let $A\neq\emptyset$ be a set. For a fixed element $0\in A$ the *binary discriminator* b and the *dual binary discriminator* h on A are binary functions defined by setting

$$b(x,y)=x \quad \text{if} \quad y=0 \quad \text{and} \quad b(x,y)=0 \ \text{ otherwise};$$
$$h(x,y)=0 \quad \text{if} \quad y=0 \quad \text{and} \quad h(x,y)=x \ \text{ otherwise};$$

For the foregoing concepts, it is easy to show that

$$
\begin{aligned}
b(x,y) &= t(0,y,x),\\
h(x,y) &= d(0,y,x),\\
h(x,y) &= b(x,b(x,y)),\\
d(x,y,z) &= t(z,t(x,y,z),x).
\end{aligned}
$$

Hence the dual discriminator is expressible by the discriminator and the dual binary discriminator is expressible by the binary discriminator (but not vice versa).

We are interested in (finite) algebras having the binary discriminator as a term function.

Definition. An algebra $\mathscr{A} = (A,F)$ with a constant 0 is a (*dually*) *binary discriminator algebra* if the (dual) binary discriminator is a term function on $\mathscr{A}$.

An element 0 of A is called a *constant* of an algebra $\mathscr{A} = (A,F)$ if it is either a nullary operation of F or a nullary (i.e. constant) term function of $\mathscr{A}$. We consider binary relations only, hence the attribute *binary* will always be omitted.

Let R be a relation on a set A. We say that an n-ary function $f\colon A^n \to A$ *preserves* R if

$$
(1) \qquad \langle a_1,b_1\rangle \in R,\dots,\langle a_n,b_n\rangle \in R \Rightarrow \langle f(a_1,\dots,a_n), f(b_1,\dots,b_n)\rangle \in R.
$$

A relation R is called *compatible (with an algebra $\mathscr{A}$)*, if (1) holds for each n-ary $f \in F$.

For our characterization, let us introduce the following concept:

Definition. Let 0 be a fixed element of a set A. A relation R on A is called 0-*projective* if $\langle 0,0\rangle \in R$ and for each $a,c,d \in A$ with $c \neq 0$ we have

$$
\langle a,d\rangle \in R \ \text{ and } \ \langle 0,c\rangle \in R \cup R^{-1} \ \text{ imply } \ \langle a,0\rangle \in R \ \text{ and } \ \langle 0,d\rangle \in R.
$$

Remark. A relation R on A is 0-projective if $\langle 0,0\rangle \in R$ and the graph of R is "conditionally" closed under projections onto the first *and* the second axis as shown in Fig. 1:

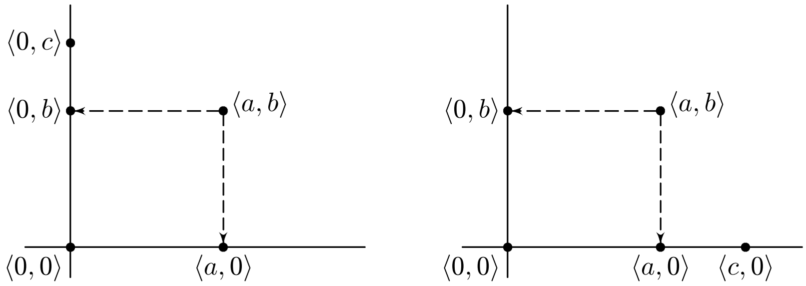

Fig. 1

Lemma 1. *If $\mathscr{A}$ is a binary discriminator algebra then every non-empty compatible relation on $\mathscr{A}$ is* 0*-projective.*

P r o o f. Let $b(x, y)$ be the binary discriminator which is a term function on $\mathscr{A}$. Let R be a non-empty relation on A which is compatible with $\mathscr{A}$. Suppose $\langle a, d\rangle \in R$ and $c \in \mathscr{A}, c \neq 0$. Then $b(\langle a, a\rangle, \langle d, d\rangle) = \langle 0, 0\rangle \in R$. Further, if $\langle c, 0\rangle \in R$ then

$$\langle 0, d\rangle = \langle b(a, c), b(d, 0)\rangle \in R,$$

which implies also

$$\langle a, 0\rangle = \langle b(a, 0), b(d, d)\rangle \in R.$$

If $\langle 0, c\rangle \in R$ then, analogously,

$$\langle a, 0\rangle = \langle b(a, 0), b(d, c)\rangle \in R,$$

which implies also

$$\langle 0, d\rangle = \langle b(a, a), b(d, 0)\rangle \in R.$$

Hence R is 0-projective. □

Lemma 2. *Let* 0 *be a fixed element of a set A. The binary discriminator on A preserves every* 0*-projective relation on A.*

P r o o f. Let $b(x, y)$ be the binary discriminator on A and let R be a 0-projective relation on A. For $\langle x_1, x_2\rangle \in R$ and $\langle y_1, y_2\rangle \in R$, we have just the following four possibilities:

(i) $b(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \langle x_1, x_2\rangle$ if $y_1 = 0 = y_2$,
(ii) $b(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \langle x_1, 0\rangle$ if $y_1 = 0$, $y_2 \neq 0$,
(iii) $b(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \langle 0, x_2\rangle$ if $y_1 \neq 0$, $y_2 = 0$,
(iv) $b(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \langle 0, 0\rangle$ if $y_1 \neq 0 \neq y_2$.

Of course, $\langle x_1, x_2\rangle \in R$ and $\langle 0, 0\rangle \in R$ by assumption. In the case (ii), $y_1 = 0$ gives $\langle 0, y_2\rangle \in R$ and, due to the 0-projectivity of R, also $\langle x_1, 0\rangle \in R$. Analogously in the case (iii), $y_2 = 0$ gives $\langle y_1, 0\rangle \in R$ and hence also $\langle 0, x_2\rangle \in R$. We conclude that $b(x, y)$ preserves R. □

Discriminator algebras having two constants 0 and 1 were characterized by B. A. Davey, V. J. Schumann and H. Werner in [4] by means of a set of certain term functions which are defined for any triplet of elements of A. A similar approach for algebras with one constant 0 was involved in [2]. We are going to apply this method to characterize algebras having every compatible relation 0-projective.

Lemma 3. *For an algebra $\mathscr{A} = (A, F)$ with 0, the following conditions are equivalent:*

(a) *every non-empty compatible relation on $\mathscr{A}$ is 0-projective;*

(b) *for every $a, c \in A$ there exist binary term functions $v_{ac}(x, y)$ and $w_{ac}(x, y)$ such that*

$$v_{ac}(a, a) = 0, \quad v_{ac}(c, 0) = c,$$

and

$$w_{ac}(a, a) = 0, \quad w_{ac}(c, 0) = 0.$$

Proof. (a)⇒(b): Let $a, b \in A$ and let R be a compatible relation on $\mathscr{A}$ generated by the pairs $\langle a, c\rangle$, $\langle a, 0\rangle$. Since R is 0-projective, we have $\langle 0, c\rangle$, $\langle 0, 0\rangle \in R$, i.e. there exists binary term functions $v_{ac}(x, y)$ and $w_{ac}(x, y)$ such that

$$\langle 0, c\rangle = v_{ac}(\langle a, c\rangle, \langle a, 0\rangle),$$
$$\langle 0, 0\rangle = w_{ac}(\langle a, c\rangle, \langle a, 0\rangle).$$

Writing it componentwise, we obtain (b).

(b)⇒(a): Let R be a compatible relation on $\mathscr{A}$ and $\langle a, c\rangle \in R$. If $\langle a, 0\rangle \in R$ then also

$$\langle 0, c\rangle = \langle v_{ac}(a, a), v_{ac}(c, 0)\rangle = v_{ac}(\langle a, c\rangle, \langle a, 0\rangle) \in R.$$

If $\langle 0, c\rangle \in R$ then also

$$\langle a, 0\rangle = \langle v_{ca}(a, a), v_{ca}(c, c)\rangle = v_{ca}(\langle a, c\rangle, \langle 0, c\rangle) \in R.$$

Finally,

$$\langle 0, 0\rangle = w_{ac}(\langle a, c\rangle, \langle a, 0\rangle) \in R.$$

Hence R is 0-projective. □

By a *majority function* we mean a ternary function $m(x, y, z)$ satisfying the identities

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

The following result of [1] will be used:

Baker-Pixley Theorem. *Let $\mathscr{A} = (A, F)$ be a finite algebra having a majority term function. Then every function $f\colon A^n \to A$, $n \geqslant 1$, which preserves all subalgebras of $\mathscr{A} \times \mathscr{A}$ is a term function.*

Let us remark that subalgebras of the direct product $\mathscr{A} \times \mathscr{A}$ are just the non-empty compatible relations on $\mathscr{A}$.

Now we can state our characterizations of finite binary discriminator algebras having a majority term function:

Theorem 1. *Let $\mathscr{A}$ be a finite algebra with 0 having a majority term function. The following conditions are equivalent:*

(1) *$\mathscr{A}$ is a binary discriminator algebra;*

(2) *every non-empty compatible relation on $\mathscr{A}$ is 0-projective;*

(3) *every function $f\colon A^n \to A$, $n \geqslant 1$, which preserves 0-projective compatible relations on $\mathscr{A}$ is a term function on $\mathscr{A}$;*

(4) *for each $a, c \in A$ there exist binary term functions $v_{ac}(x, y)$ and $w_{ac}(x, y)$ such that*

$$v_{ac}(a, a) = 0 \text{ and } v_{ac}(c, 0) = c,$$

and

$$w_{ac}(a, a) = 0 \text{ and } w_{ac}(c, 0) = 0.$$

P r o o f. (1) ⇒ (2) follows directly by Lemma 1. Let us prove (2) ⇒ (1): Let every non-empty compatible relation on an algebra $\mathscr{A} = (A, F)$ be 0-projective and let $b(x, y)$ be the binary discriminator on a set A. By Lemma 2, $b(x, y)$ preserves every 0-projective relation on A (and hence every compatible relation on $\mathscr{A}$). Since $\mathscr{A}$ has a majority term function, $b(x, y)$ is a term function on $\mathscr{A}$ by the Baker-Pixley Theorem.

(1) ⇒ (3): By Lemma 1, every non-empty compatible relation on $\mathscr{A}$ is 0-projective. Hence, every n-ary function which preserves 0-projective relations also preserves all subalgebras of $\mathscr{A} \times \mathscr{A}$. By the Baker-Pixley Theorem, we conclude (3).

(3) ⇒ (1): By Lemma 2, the binary discriminator $b(x, y)$ preserves 0-projective relations on $\mathscr{A}$. Thus, by (3), $b(x, y)$ is a term function on $\mathscr{A}$.

The equivalence (2) ⇔ (4) is shown by Lemma 3. □

To extend our considerations also to the dual binary discriminator, we need the following concept:

Definition. Let 0 be a fixed element of a set A. A binary relation S on A is called *weakly* 0*-projective* if for $a, b, c \in A$, $c \neq 0$, we have

$$\begin{array}{llllll} \text{if} & \langle a,b\rangle \in S & \text{and} & \langle 0,c\rangle \in S & \text{then} & \langle 0,b\rangle \in S, \\ \text{if} & \langle a,b\rangle \in S & \text{and} & \langle c,0\rangle \in S & \text{then} & \langle a,0\rangle \in S. \end{array}$$

Remark. Conditions of the foregoing definition can also be visualized by projections to the first *or* second axis as shown in Fig. 2 (we borrow the term "projective" from the quoted paper [5] where it is used in a more general case). Moreover, every 0-projective relation is also weakly 0-projective (but not vice versa).

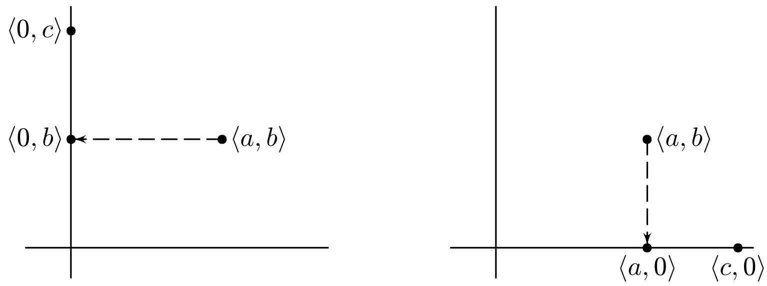

Fig. 2

Lemma 4. *If $\mathscr{A}$ is a dually binary discriminator algebra then every non-empty compatible relation on $\mathscr{A}$ is weakly* 0*-projective.*

Proof. Let S be a compatible relation on $\mathscr{A}$ and let the dual binary discriminator $h(x,y)$ be a term function on $\mathscr{A}$. Suppose $\langle a,b\rangle \in S$. If $\langle 0,c\rangle \in S$ for $c \neq 0$ then also $\langle 0,b\rangle = h(\langle a,b\rangle, \langle 0,c\rangle) \in S$. Analogously, if $\langle c,0\rangle \in S$ then $\langle a,0\rangle = h(\langle a,b\rangle, \langle c,0\rangle) \in S$. Thus S is weakly 0-projective. □

Lemma 5. *Let* 0 *be a fixed element of a set A. The dual binary discriminator on A preserves every weakly* 0*-projective relation on A.*

Proof. Let S be a weakly 0-projective relation on A and suppose $\langle x_1,x_2\rangle$, $\langle y_1,y_2\rangle \in S$. Then we have four possibilities:

(i) $h(\langle x_1,x_2\rangle, \langle y_1,y_2\rangle) = \langle x_1,x_2\rangle$ if $y_1 \neq 0 \neq y_2$,
(ii) $h(\langle x_1,x_2\rangle, \langle y_1,y_2\rangle) = \langle x_1,0\rangle$ if $y_1 \neq 0$, $y_2 = 0$,
(iii) $h(\langle x_1,x_2\rangle, \langle y_1,y_2\rangle) = \langle 0,x_2\rangle$ if $y_1 = 0$, $y_2 \neq 0$,
(iv) $h(\langle x_1,x_2\rangle, \langle y_1,y_2\rangle) = \langle 0,0\rangle$ if $y_1 = 0 = y_2$.

The first case is trivial since $\langle x_1, x_2\rangle \in S$, and, in the case (iv), $y_1 = 0 = y_2$ and $\langle y_1, y_2\rangle \in S$ imply $\langle 0, 0\rangle \in S$. For (ii), $y_2 = 0$ gives $\langle y_1, 0\rangle \in S$ and, by virtue of the weak 0-projectivity, also $\langle x_1, 0\rangle \in S$. For (iii) the situation is analogous, namely $y_1 = 0$ gives $\langle 0, y_2\rangle \in S$, thus also $\langle 0, x_2\rangle \in S$. In all cases the resulting pair belongs to S, i.e. $h(x, y)$ preserves S. □

Lemma 6. *For an algebra $\mathscr{A} = (A, F)$ with 0, the following conditions are equivalent:*

(a) *every non-empty compatible relation on $\mathscr{A}$ is weakly 0-projective;*

(b) *for every $a, b, c \in A$, $c \neq 0$, there exists a binary term function $w_{abc}(x, y)$ such that*

$$w_{abc}(a, c) = a \qquad \text{and} \qquad w_{abc}(b, 0) = 0.$$

Proof. (a)⇒(b): Let $a, b, c \in A$, $c \neq 0$ and let S be a compatible relation on $\mathscr{A}$ generated by the pairs $\langle a, b\rangle, \langle c, 0\rangle$. Since S is weakly 0-projective, hence also $\langle a, 0\rangle \in S$ and thus there exists a binary term function $w_{abc}(x, y)$ such that

$$\langle a, 0\rangle = w_{abc}(\langle a, b\rangle, \langle c, 0\rangle).$$

Writing it componentwise, we obtain (b).

(b)⇒(a): Let S be a compatible relation on $\mathscr{A}$ with $\langle a, b\rangle \in S$. If $c \neq 0$ and $\langle c, 0\rangle \in S$ then also

$$\langle a, 0\rangle = w_{abc}(\langle a, b\rangle, \langle c, 0\rangle) \in S,$$

and if $\langle 0, c\rangle \in S$ then

$$\langle 0, b\rangle = w_{abc}(\langle a, b\rangle, \langle 0, c\rangle) \in S,$$

thus S is weakly 0-projective. □

For algebras having a majority term function, we can state an assertion similar to Theorem 1 for binary discriminators and analogous to that for dual discriminators in [5]:

Theorem 2. *Let $\mathscr{A}$ be a finite algebra with 0 having a majority term function. The following conditions are equivalent:*

(1) *$\mathscr{A}$ is a dually binary discriminator algebra;*

(2) *every non-empty compatible relation on $\mathscr{A}$ is weakly 0-projective;*

(3) *every function $f\colon A^n \to A$, $n \geqslant 1$ which preserves weakly 0-projective compatible relations on $\mathscr{A}$ is a term function on $\mathscr{A}$;*

(4) *for each $a, b, c \in A$, $c \neq 0$ there exists a binary term function $v_{abc}(x, y)$ such that*

$$v_{abc}(a, c) = a \ \text{ and } v_{abc}(b, 0) = 0.$$

The proof is taylored by the same pattern as that of Theorem 1 with the only difference that Lemmas 4, 5 and 6 are used instead of 1, 2 and 3.

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References

[1] *Baker K. A., Pixley A. F.*: Polynomial interpolation and the Chinese remainder theorem for algebraic systems. Math. Z. *143* (1975), 165–174.
[2] *Chajda I., Rosenberg I. G.*: Discriminator algebras with one nullary operation. Contributions to General Algebra 10, Proc. of the Klagenfurt Conference 1997. Verlag Johannes Heyn, Klagenfurt, 1998, pp. 101–107.
[3] *Chajda I., Halaš R., Rosenberg I. G.*: Ideals and the binary discriminator in universal algebra. Algebra Universalis *42* (1999), 239–251.
[4] *Davey B. A., Schumann V. J., Werner H.*: From the subalgebra of the square to the discriminator. Algebra Universalis *28* (1991), 500–519.
[5] *Fried E., Pixley A. F.*: The dual discriminator function in universal algebra. Acta Sci. Math. *41* (1979), 83–100.
[6] *Pixley A. F.*: Functionally complete algebras generating distributive and permutable classes. Math. Z. *114* (1970), 361–372.
[7] *Pixley A. F.*: The ternary discriminator function in universal algebra. Math. Ann. *191* (1971), 167–180.

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