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# SOME BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

The linear hyperbolic system

$$\frac{\partial^2 u}{\partial x \partial y} = \mathcal{P}_0(x,y)u + \mathcal{P}_1(x,y)\frac{\partial u}{\partial x} + \mathcal{P}_2(x,y)\frac{\partial u}{\partial y} + q(x,y) \tag{1}$$

is considered, where $\mathcal{P}_0,\ \mathcal{P}_1,\ \mathcal{P}_2$ and $q$ are respectively the $n\times n$ matrices and the $n$-dimensional vector whose components are measurable and essentially bounded functions in the rectangle $\mathcal{D}_{ab} = [0,a]\times[0,b]$ or in the strip $\mathcal{D}_b = \mathbb{R}\times[0,b]$.

For system (1) problems with general functional boundary conditions are investigated in the rectangle $\mathcal{D}_{ab}$ and problems on bounded, almost-periodic and periodic solutions in the strip $\mathcal{D}_b$.

Optimal in a certain sense conditions are established, guaranteeing the unique solvability of the problems and the stability of their solutions with respect to small perturbations of the coefficients of system (1) and of the boundary conditions.

35L55

System of partial differential equations of hyperbolic type, boundary value problem in the rectangle, classical solution, absolutely continuous solution, generalized solution, periodic solution, almost-periodic solution, bounded solution.

**რეზიუმე.** განხილულია წრფივი ჰიპერბოლური სისტემა

$$\frac{\partial^2 u}{\partial x \partial y} = \mathcal{P}_0(x,y)u + \mathcal{P}_1(x,y)\frac{\partial u}{\partial x} + \mathcal{P}_2(x,y)\frac{\partial u}{\partial y} + q(x,y), \tag{1}$$

სადაც $\mathcal{P}_0,\ \mathcal{P}_1,\ \mathcal{P}_2$ და $q$, სათანადოდ, $n\times n$ მატრიცები და $n$-განზომილებიანი ვექტორია, რომელთა კომპონენტები $\mathcal{D}_{ab} = [0,a]\times[0,b]$ სწორკუთხედში ან $\mathcal{D}_b = \mathbb{R}\times[0,b]$ ზოლში განსაზღვრული ზომადი და არსებითად შემოსაზღვრული ფუნქციებია.

$\mathcal{D}_{ab}$ სწორკუთხედში (1) სისტემისათვის გამოკვლეულია ამოცანები ზოგადი ფუნქციონალური სასაზღვრო პირობებით, ხოლო $\mathcal{D}_b$ ზოლში —ამოცანები შემოსაზღვრული, თითქმის პერიოდული და პერიოდული ამონახსნების შესახებ.

დადგენილია გარკვეული აზრით ოპტიმალური პირობები, რომლებიც უზრუნველყოფენ აღნიშნულ ამოცანათა ცალსახად ამოხსნადობას და ამონახსნის მდგრადობას (1) სისტემის კოეფიციენტებისა და სასაზღვრო მნიშვნელობების მცირე შეშფოთებათა მიმართ.

Beginning from the 60ies, problems on periodic solutions in a strip or in the large as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see, e.g., [1-4, 6-10, 16, 17, 29, 31-36, 38-47,51]). These problems naturally lead us to boundary value problems in a rectangle with general functional boundary conditions, and also to problems on bounded and almost-periodic in a strip solutions. This work deals just with such a class of problems for the linear hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y), \tag{0.1}$$

where $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ and $q$ are, respectively, $n \times n$ matrices and an $n$-dimensional vector whose components are real measurable and essentially bounded functions given in the rectangle

$$\mathcal{D}_{ab} = [0,a] \times [0,b]$$

or in the strip

$$\mathcal{D}_b = \mathbb{R} \times [0,b]$$

($a$ and $b$ are positive numbers, $\mathbb{R}$ is the set of real numbers).

We start with the following two definitions.

A vector function $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ ($u : \mathcal{D}_b \to \mathbb{R}^n$) is called a solution of system (0.1) if it is absolutely continuous in $\mathcal{D}_{ab}$ [1] (in every rectangle contained in $\mathcal{D}_b$) and satisfies system (0.1) almost everywhere in $\mathcal{D}_{ab}$ (in $\mathcal{D}_b$).

A solution $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ ($u : \mathcal{D}_b \to \mathbb{R}^n$) of system (0.1) is called classical if it has the continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial x \partial y}$ in $\mathcal{D}_{ab}$ (in $\mathcal{D}_b$).

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[1] According to the well-known definition of an absolutely continuous function of many variables (see [5, §570], [11] and [49]), a vector function $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ is absolutely continuous if and only if it admits the representation

$$u(x,y) = v_1(x) + v_2(y) + \int_0^x \int_0^y v(s,t)\,ds\,dt \quad \text{for} \quad (x,y) \in \mathcal{D}_{ab},$$

where $v_1 : [0,a] \to \mathbb{R}^n$ and $v_2 : [0,b] \to \mathbb{R}^n$ are absolutely continuous and $v : \mathcal{D}_{ab} \to \mathbb{R}^n$ is summable.

Definition 0.1 was earlier used when investigating initial value problems for hyperbolic systems with discontinuous coefficients [11-13,49,50]. As for boundary value problems for system (0.1), they were studied, as a rule, in terms of the concept of a classical solution for $\mathcal{P}_i$ $(i = 0,1,2)$ and $q$ being continuous or even smooth. In fact, the concepts either of a classical solution or of an absolutely continuous one, do not appear to be sufficient for investigating even very simple boundary value problems. [2] Therefore we have to extend the concept of a solution on the basis of Picone's canonic representation of system (0.1) [37].

For an arbitrary $x \in [0,a]$ $(y \in [0,b])$ by $Z_1(x,\cdot)$ $(Z_2(\cdot,y))$ we denote the fundamental matrix of the system of ordinary differential equations

$$\frac{dz(x,y)}{dy} = \mathcal{P}_1(x,y)z(x,y) \quad \Big(\frac{dz(x,y)}{dx} = \mathcal{P}_2(x,y)z(x,y)\Big),$$

satisfying the initial condition

$$Z_1(x,0) = E \quad \big(Z_2(0,y) = E\big),$$

where $E$ is the unit $n \times n$ matrix.

According to Lemma 3.2 below, if the matrix function $\mathcal{P}_2(x,\cdot)$ is absolutely continuous, then a solution $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ of system (0.1) is also a solution of the system

$$\begin{aligned} &\frac{\partial}{\partial y}\Big[Z_1^{-1}(x,y)Z_2(x,y)\frac{\partial}{\partial x}\Big(Z_2^{-1}(x,y)u(x,y)\Big)\Big] = \\ &\qquad = Z_1^{-1}(x,y)\big(\mathcal{P}(x,y)u(x,y) + q(x,y)\big), \end{aligned} \tag{0.1'}$$

where

$$\mathcal{P}(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_2(x,y)}{\partial y}$$

and vice versa. System $(0.1')$ is called the Picone canonic form of system (0.1).

A vector function $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ $(u : \mathcal{D}_b \to \mathbb{R}^n)$ is called a generalized solution of system (0.1) if: i) $u$ admits the representation

$$u(x,y) = Z_2(x,y)[v_0(x,y) + v_1(y)],$$

where $v_0 : \mathcal{D}_{ab} \to \mathbb{R}^n$ $(v_0 : \mathcal{D}_b \to \mathbb{R}^n)$ is absolutely continuous (locally absolutely continuous) and $v_1 : [0,b] \to \mathbb{R}^n$ is summable; ii) equality $(0.1')$ holds almost everywhere in $\mathcal{D}_{ab}$ (in $\mathcal{D}_b$).

[2] For the periodic boundary value problem this fact was taken into account in [51].

In the rectangle $\mathcal{D}_{ab}$ for system (0.1) we study boundary value problems of four types

$$u(x,0)=\varphi_0(x), \quad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y)=\varphi_1(y); \tag{0.2}$$

$$\frac{\partial u(x,0)}{\partial x}-\mathcal{P}_2(x,0)u(x,0)=\psi_0(x), \quad h\Big(\frac{\partial(\cdot,y)}{\partial y}\Big)(y)=\psi_1(y); \tag{0.3}$$

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\Big)=\psi_0(x), \quad h\big(u(\cdot,y)\big)(y)=\psi_1(y); \tag{0.4}$$

$$\begin{gathered}\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\Big)=\psi_0(x),\\ h\Big(\frac{\partial}{\partial y}\big(u(\cdot,y)-Z_2(\cdot,y)u(0,y)\big)\Big)(y)=\psi_1(y),\end{gathered} \tag{0.5}$$

where $\varphi_0:[0,a]\to\mathbb{R}^n$ is absolutely continuous, $\psi_0:[0,a]\to\mathbb{R}^n$, $\varphi_1$ and $\psi_1:[0,b]\to\mathbb{R}^n$ are summable and $h$ is a linear continuous operator acting from the space of absolutely continuous in $[0,a]$ $n$-dimensional vector functions to the space of measurable and essentially bounded in $[0,b]$ $n$-dimensional vector functions. We are especially interested in the case when $h(v)(y)\equiv v(a)-v(0)$, i.e. when boundary conditions (0.2)-(0.5) are periodic,

$$u(x,0)=\varphi_0(x), \qquad \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}+\varphi_1(y); \tag{$0.2_1$}$$

$$\begin{gathered}\frac{\partial u(x,0)}{\partial x}-\mathcal{P}_2(x,0)u(x,0)=\psi_0(x),\\ \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}+\psi_1(y);\end{gathered} \tag{$0.3_1$}$$

$$\begin{gathered}\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\Big)=\psi_0(x),\\ u(a,y)=u(0,y)+\psi_1(y);\end{gathered} \tag{$0.4_1$}$$

$$\begin{gathered}\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\Big)=\psi_0(x),\\ \frac{\partial}{\partial y}\big(u(a,y)-u(0,y)\big)=\psi_1(y).\end{gathered} \tag{$0.5_1$}$$

For any $k\in\{2,3,4,5\}$ under a solution (a classical solution, a generalized solution) of problem (0.1),(0.$k$) we understand a solution (a classical solution, a generalized solution) of system (0.1) satisfying boundary conditions (0.k) almost everywhere in $\mathcal{D}_{ab}$.

The behaviour of the matrix function $M_0:[0,b]\to\mathbb{R}^{n\times n}$ induced by the operator $h$ and the fundamental matrix $Z_2$, [3] affects essentially the

[3]I.e., $M_0$ is the matrix function satisfying the equality $h(Z_2(\cdot,y)c)=M_0(y)c$ almost for all $y\in[0,b]$ and an arbitrary $c\in\mathbb{R}^n$.

solvability and correctness of boundary value problems of type (0.1),(0.$k$) and differential properties of solutions of these problems. For example: 1) if $M_0$ is singular at isolated points, then despite the smoothness of $\mathcal{P}_i$ ($i = 0, 1, 2$) and $q$ and the unique solvability of problem (0.1),(0.2) the latter's solution may be non-classical (see Remark 4.2); 2) if $M_0$ is singular in the set of positive measure, then problem (0.1),(0.2) may be non-Fredholmian (see Theorems 4.1 and 4.1′) or may have the unique classical solution and an infinite set of absolutely continuous solutions (see Remark 4.3); 3) if $M_0(y) \equiv \Theta$, where $\Theta$ is the zero $n \times n$ matrix, then problem (0.1),(0.4) may have a unique generalized solution which is not absolutely continuous (see Remark 4.13).

Two fundamentally different cases

(i) $\det(M_0(y)) \not\equiv 0$

and

(ii) $M_0(y) \equiv \Theta$.

will be treated by us separately.

The optimal in a certain sense conditions for the existence and uniqueness of classical, absolutely continuous and generalized solutions of problems (0.1),(0.2) and (0.1),(0.4) (problems (0.1),(0.3) and (0.1),(0.5)) and for the stability of these solutions with respect to small, in an integral sense, perturbations of coefficients of system (0.1) are obtained in case (i) (case (ii)). Moreover, the effective methods for constructing a solution of problem (0.1),(0.2) are developed.

The sufficient conditions for the unique solvability of problems (0.1),(0.2) and (0.1),(0.4) are obtained also in case (ii). In this case, however, the latter problems are ill-posed, since for their solvability it is necessary that certain integral equalities (see equalities (4.49) and (4.84)) be fulfilled; these equalities can be violated at arbitrarily small perturbations of the coefficients of system (0.1) or the vector functions $\varphi_k$ and $\psi_k$ ($k = 0, 1$).

The above results and their particular cases for problems (0.1),(0.$k_1$) ($k = 2, 3, 4, 5$) are stated in Chapter II (§§4-6). They are proved by the unified method that consists in reducing problems under consideration to the modified characteristic initial value problem with either of the two conditions below

$$\begin{gathered} u(x,0) = \varphi_0(x), \qquad \frac{\partial u(0,y)}{\partial y} = \psi(y) + \\ + \int_0^a \Big[ Q_0(s,y)u(s,y) + \gamma^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s} \Big] ds \end{gathered} \tag{0.6}$$

and

$$\begin{gathered} \lim_{y\to 0} \Big( \frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y) \Big) = \varphi(x), \\ u(0,y) = \psi(y) + \int_0^y \int_0^a q(y,s,t)u(s,t)ds\,dt. \end{gathered} \tag{0.7}$$

The material required to realize the method in question is stated in Chapter I. More exactly, in this chapter: (a) the sufficient conditions for the existence and uniqueness of a solution of a system of special type operator equations are established, on whose basis the unique solvability and correctness of auxiliary problems (0.1),(0.6) and (0.1),(0.7) are proved (see §§1 and 3); (b) new conditions for the unique solvability of the general linear boundary value problem for a system of ordinary differential equations with a parameter are obtained and the properties of matrix functions induced by this problem are studied (see §2).

The problem

$$u(x,0)=\varphi(x), \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b}\left(\left\|\frac{\partial u(x,y)}{\partial x}\right\|+\left\|\frac{\partial u(x,y)}{\partial y}\right\|\right)<+\infty \qquad (0.8)$$

in the strip $\mathcal{D}_b$ is considered for system (0.1) in Chapter III, where the sufficient conditions for the existence and uniqueness of its solution are also given (see §7). When $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ are periodic in the first argument with a period $a$ and $\varphi(x+a)\equiv\varphi(x)$, this solution is also a solution of the periodic problem

$$u(x,0)=\varphi(x), \quad u(x+a,y)=u(x,y). \qquad (0.9)$$

Other conditions for the unique solvability of problem (0.1),(0.9) follow from the results of Chapter II that deals with problems (0.1),$(0.k_1)$ $(k=2,3,4,5)$ (see §8). In the last §9 the problem of the almost-periodicity of a solution of problem (0.1),(0.8) is considered assuming that $\varphi$ is almost periodic, while $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ are almost-periodic in the first argument. Theorem 9.1 proved here is an analogue of Favard's well-known theorem [16]. Applying this theorem, one can obtain from the results of §7 the effective conditions ensuring the almost periodicity in the first argument of a solution of problem (0.1),(0.8).

$$\mathbb{R}=(-\infty,+\infty), \quad \mathbb{R}_+=[0,+\infty).$$

$\mathbb{R}^{m\times n}$ the space of $m\times n$ matrices $X=(x_{ij})$ with real components $x_{ij}$ $(i=1,\dots,m; j=1,\dots,n)$ and the norm

$$\|X\|=\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}|.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

$(x_{ij})_{i,j=1}^n$ is the square matrix with components $x_{ij}$ $(i,j=1,\dots,n)$ and $(x_i)_{i=1}^n$ is the $n$-dimensional column vector with components $x_i$ $(i==1,\dots,n)$.

By an absolute value of the matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$ we understand the matrix $|X|=(|x_{ij}|)\in\mathbb{R}^{m\times n}$ with components $|x_{ij}|$ $(i=1,\dots,m; j==1,\dots,n)$.

A matrix $X = (x_{ij}) \in \mathbb{R}^{m\times n}$ is called non-negative if $x_{ij} \geq 0$ ($i = 1,\dots,m; j = 1,\dots,n$).

The inequalities between the matrices $X = (x_{ij})$ and $Y = (y_{ij}) \in \mathbb{R}^{m\times n}$ are understood componentwise, i.e.,

$$X \leq Y \Leftrightarrow x_{ij} \leq y_{ij} \quad (i = 1,\dots,m; j = 1,\dots,n).$$

If $X_k = (x_{ijk}) \in \mathbb{R}^{m\times n}$ $(k = 1,\dots,k_0)$, then

$$\max_{1\leq k\leq k_0} X_k = \big(\max_{1\leq k\leq k_0} x_{ijk}\big).$$

$\det(X)$ is the determinant of the matrix $X \in \mathbb{R}^{n\times n}$.

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$.

$X^{-1}$ is the matrix reciprocal to $X \in \mathbb{R}^{n\times n}$.

$\operatorname{diag}(x_1,\dots,x_n)$ is the diagonal $n\times n$ matrix with diagonal elements $x_1,\dots,x_n$.

$E$ is the unit matrix.

$\Theta$ is the zero matrix.

A matrix function $Z : \mathcal{D} \to \mathbb{R}^{m\times n}$ is called measurable, summable, continuous, etc., if its components are such.

Let $\mathcal{D}$ be a $k$-dimensional segment, $k_0 \in \{1,\dots,k\}$, $\mathcal{D}^0$ be the projection of $\mathcal{D}$ onto $\mathbb{R}^{k_0}$ and $M_0 : \mathcal{D}^0 \to \mathbb{R}^{m\times m}$. A matrix function $Z : \mathcal{D} \to \mathbb{R}^{m\times m}$ is called $M_0$-continuous ($M_0$-summable) if it admits in $\mathcal{D}$ the representation

$$Z(x_1,\dots,x_k) = M_0(x_1,\dots,x_{k_0})Z_0(x_1,\dots,x_k),$$

where $Z_0 : \mathcal{D} \to \mathbb{R}^{m\times n}$ is continuous (summable).

Let $M_0 : [0,b] \to \mathbb{R}^{m\times m}$. We say that a matrix function $Z : \mathcal{D}_{ab} \to \mathbb{R}^{m\times n}$ satisfies the Carathéodory condition with $M_0$ weight if it admits the reprezentation

$$Z(x,y) = M_0(y)Z_0(x,y)$$

in $\mathcal{D}_{ab}$, where $Z_0(\cdot,y) : [0,a] \to \mathbb{R}^{m\times n}$ is measurable for all $y \in [0,b]$, $Z_0(x,\cdot) : [0,b] \to \mathbb{R}^{m\times n}$ is continuous almost for all $x \in [0,a]$ and $\max\limits_{0\leq y\leq b} \|Z_0(\cdot,y)\|$ is summable in $[0,a]$.

$\mathbb{C}(\mathcal{D};\mathbb{R}^{m\times n})$, $L_\infty(\mathcal{D};\mathbb{R}^{m\times n})$ and $L(\mathcal{D};\mathbb{R}^{m\times n})$ are the spaces of continuous, measurable and essentially bounded and summable functions $Z : \mathcal{D} \to \mathbb{R}^{m\times n}$ with the norms

$$\|Z\|_{\mathbb{C}} = \max_{(x_1,\dots,x_k)\in\mathcal{D}} \|Z(x_1,\dots,x_k)\|,$$

$$\|Z\|_{L_\infty} = \operatorname*{ess\,sup}_{(x_1,\dots,x_k)\in\mathcal{D}} \|Z(x_1,\dots,x_k)\|,$$

$$\|Z\|_L = \int\cdots\int_{\mathcal{D}} \|Z(x_1,\dots,x_k)\| dx_1\cdots dx_k.$$

If $Z=(z_{ij})\in L_\infty(\mathcal{D};\mathbb{R}^{m\times n})$, then

$$\operatorname*{ess\,sup}_{(x_1,\dots,x_k)\in\mathcal{D}} Z(x_1,\dots,x_k)=\Big(\operatorname*{ess\,sup}_{(x_1,\dots,x_k)\in\mathcal{D}} z_{ij}(x_1,\dots,x_k)\Big).$$

If $Z\in L(\mathcal{D}_{ab};\mathbb{R}^{m\times n})$, then

$$\|Z\|_L^{(0)}=\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}}\Big\|\int_0^y\int_0^x Z(s,t)dsdt\Big\|,$$

$$\|Z\|_L^{(1)}=\max_{(x,y)\in\mathcal{D}_{ab}}\Big[\int_0^y\Big\|\int_0^x Z(s,t)ds\Big\|dt+\int_0^x\Big\|\int_0^y Z(s,t)dt\Big\|ds\Big],$$

$$\|Z\|_L^{(2)}=\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}}\Big[\Big\|\int_0^x Z(s,y)ds\Big\|+\Big\|\int_0^y Z(x,t)dt\Big\|\Big].$$

$\mathbb{C}^k([0,a];\mathbb{R}^{m\times n})$ is the space of $k$ times continuously differentiable matrix functions $Z:[0,a]\to\mathbb{R}^{m\times n}$ with the norm

$$\|Z\|_{\mathbb{C}^k}=\max_{x\in[0,a]}\sum_{i=0}^k\|Z^{(i)}(x)\|.$$

$\widetilde{\mathbb{C}}^{k-1}([0,a];\mathbb{R}^{m\times n})$ is the space of matrix functions $Z:[0,a]\to\mathbb{R}^{m\times n}$ which are absolutely continuous together with their derivatives up to the $k-1$ order inclusive, with the norm

$$\|Z\|_{\widetilde{\mathbb{C}}^{k-1}}=\sum_{i=1}^{k-1}\|Z^{(i)}(0)\|+\int_0^a\|Z^{(k)}(x)\|dx.$$

$\widetilde{\mathbb{C}}_\infty(\mathbb{R};\mathbb{R}^{m\times n})$ is the space of bounded and Lipschitz continuous matrix functions $Z:\mathbb{R}\to\mathbb{R}^{m\times n}$.

$\widetilde{\mathbb{C}}_\infty^{k-1}([0,a];\mathbb{R}^{m\times n})$ is the space of matrix functions $Z:[0,a]\to\mathbb{R}^{m\times n}$ which are Lipschitz continuous together with their derivatives up to the $k-1$ order inclusive, with the norm

$$\|Z\|_{\widetilde{\mathbb{C}}_\infty^{k-1}}=\sum_{i=1}^{k-1}\|Z^{(i)}(0)\|+\operatorname*{ess\,sup}_{x\in[0,a]}\|Z^{(k)}(x)\|dx.$$

$\widetilde{\mathbb{C}}([0,a];\mathbb{R}^{m\times n})=\widetilde{\mathbb{C}}^0([0,a];\mathbb{R}^{m\times n})$, $\widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^{m\times n})=\widetilde{\mathbb{C}}_\infty^0([0,a];\mathbb{R}^{m\times n})$.

$\widetilde{\mathbb{C}}(\mathcal{D}_{ab};\mathbb{R}^{m\times n})$ is the space of absolutely continuous matrix functions $Z:\mathcal{D}_{ab}\to\mathbb{R}^{m\times n}$ with the norm

$$\|Z\|_{\widetilde{\mathbb{C}}}=\|Z(0,0)\|+\int_0^a\Big\|\frac{\partial Z(x,0)}{\partial x}\Big\|dx+\int_0^b\Big\|\frac{\partial Z(0,y)}{\partial y}\Big\|dy+\int_0^a\int_0^b\Big\|\frac{\partial^2 Z(x,y)}{\partial x\partial y}\Big\|dx\,dy.$$

$\widetilde{\mathbb{C}}_{loc}(\mathcal{D}_b;\mathbb{R}^{m\times n})$ is the space of locally absolutely continuous matrix functions $Z:\mathcal{D}_b\to\mathbb{R}^{m\times n}$.

If $Z \in \widetilde{\mathbb{C}}(\mathcal{D}_{ab}; \mathbb{R}^{m\times n})$, then

$$\|Z\|^{(1)}_{\widetilde{\mathbb{C}}} = \|Z(0,0)\| + \max_{(x,y)\in\mathcal{D}_{ab}} \Big( \int_0^x \Big\| \frac{\partial Z(s,y)}{\partial s} \Big\| ds + \int_0^y \Big\| \frac{\partial Z(x,t)}{\partial t} \Big\| dt \Big),$$

$$\|Z\|^{(2)}_{\widetilde{\mathbb{C}}} = \|Z(0,0)\| + \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \Big( \Big\| \frac{\partial Z(x,y)}{\partial x} \Big\| + \Big\| \frac{\partial Z(x,y)}{\partial y} \Big\| \Big).$$

$\widetilde{\mathbb{C}}^{(-1,k-1)}_{\infty}(\mathcal{D}_{ab}; \mathbb{R}^{m\times n})$ is the space of measurable matrix functions $Z : \mathcal{D}_{ab} \to \mathbb{R}^{m\times n}$ such that $Z(x,\cdot) \in \widetilde{\mathbb{C}}^{k-1}_{\infty}([0,b]; \mathbb{R}^{m\times n})$ almost for every $x \in$ $\in [0,a]$, and

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \sum_{i=0}^{k} \Big\| \frac{\partial^i Z(x,y)}{\partial y^i} \Big\| < +\infty.$$

$\widetilde{\mathbb{C}}^{(k-1,-1)}_{\infty}(\mathcal{D}_{ab}; \mathbb{R}^{m\times n})$ is the space of measurable matrix functions $Z : \mathcal{D}_{ab} \to \mathbb{R}^{m\times n}$ such that $Z(\cdot,y) \in \widetilde{\mathbb{C}}^{k-1}_{\infty}([0,a]; \mathbb{R}^{m\times n})$ almost for every $y \in$ $\in [0,b]$, and

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \sum_{i=0}^{k} \Big\| \frac{\partial^i Z(x,y)}{\partial x^i} \Big\| < +\infty.$$

CHAPTER I

## § 1.

Let $n$, $n_1$ and $n_2$ be natural numbers, $I$ be a finite or an infinite interval of the real axis,

$$\mathcal{D} = I \times [0, b]$$

and $\Lambda_1, \Lambda_2$ and $\Lambda$ be some nonempty closed subsets of the spaces $L_\infty(\mathcal{D};\mathbb{R}^{n_1})$, $L_\infty(\mathcal{D};\mathbb{R}^{n_2})$ and $L_\infty(\mathcal{D};\mathbb{R}^{n})$. The boundary value problems considered in this work are reduced to a system of operator equations of the type

$$z_i(x,y) = g_i(z_1, z_2)(x,y) \quad (i = 1,2) \tag{1.1}$$

or to the equation

$$z(x,y) = g(z)(x,y), \tag{1.2}$$

where $g_i : \Lambda_1 \times \Lambda_2 \to \Lambda_i$ $(i = 1,2)$ and $g : \Lambda \to \Lambda$ are continuous operators.

By a solution of system (1.1) (equation (1.2)) we shall understand a pair of vector functions $(z_i)_{i=1}^2 \in \Lambda_1 \times \Lambda_2$ (vector function $z \in \Lambda$), which satisfies system (1.1) (equation (1.2)) almost everywhere in $\mathcal{D}$.

For arbitrary $y \in [0, b]$ and $z \in L_\infty(\mathcal{D}, \mathbb{R}^k)$ assume

$$|z|_{I,y} = \operatorname*{ess\,sup}_{(x,t)\in I\times[0,y]} |z(x,t)|, \quad |z(\cdot,y)|_I = \operatorname*{ess\,sup}_{x\in I} |z(x,y)|,$$
$$\|z\|_{I,y} = \operatorname*{ess\,sup}_{(x,t)\in I\times[0,y]} \|z(x,t)\|, \quad \|z(\cdot,y)\|_I = \operatorname*{ess\,sup}_{x\in I} \|z(x,y)\|.$$

Let $\Lambda_0$ be a subspace of the space $L_\infty(I;\mathbb{R}^{n_2})$ satisfying the following conditions:

1) if $v(x) = c \in \mathbb{R}^{n_2}$ for $x \in I$ , then $v \in \Lambda_0$;

2) if $v_1$ and $v_2 \in \Lambda_0$ and $v(x) = \max\limits_{1\le i\le 2} v_i(x)$, then $v \in \Lambda_0$;

3) if $\zeta_0$ and $\zeta \in \Lambda_2$, then $|\zeta(\cdot,y) - \zeta_0(\cdot,y)| \in \Lambda_0$ almost for all $y \in [0,b]$.

An operator $l : \Lambda_0 \to \mathbb{R}^{n_2}$ is called non-negative if for any non-negative vector function $v \in \Lambda_0$ the vector $l(v)$ is non-negative.

If $A \in L_\infty(I;\mathbb{R}^{n_2\times n_2})$ and $A(\cdot)c \in \Lambda_0$ for $c \in \mathbb{R}^{n_2}$, then by $l\big(A\big)$ we shall denote a matrix such that

$$l\big(A(\cdot)c\big) = l\big(A\big)c \quad \text{for} \quad c \in \mathbb{R}^{n_2}.$$

*Let for any $\zeta_i$ and $\bar\zeta_i \in \Lambda_i$ $(i = 1,2)$ the inequalities*

$$\|g_1(\zeta_1,\zeta_2)(x,y) - g_1(\bar\zeta_1,\bar\zeta_2)(x,y)\| \le$$
$$\le \int_0^y g_0(t)(\|\zeta_1 - \bar\zeta_1\|_{I,t} + \|\zeta_2 - \bar\zeta_2\|_{I,t})\,dt \tag{1.3}$$

*and*

$$|g_2(\zeta_1,\zeta_2)(x,y)-g_2(\bar\zeta_1,\bar\zeta_2)(x,y)|\le$$
$$\le A_{01}|\zeta_1-\bar\zeta_1|_{I,y}+\int_0^y A_{02}(t)|\zeta_2-\bar\zeta_2|_{I,t}\,dt+$$
$$+A_1(x,y)|\zeta_2(\cdot,y)-\bar\zeta_2(\cdot,y)|_I+A_2(x,y)l(|\zeta_2(\cdot,y)-\bar\zeta_2(\cdot,y)|) \tag{1.4}$$

*hold almost evrywhere in* $\mathcal{D}_{ab}$ *where* $g_0:[0,b]\to\mathbb{R}_+$ *is a summable function,* $A_{01}$ *is a non-negative constant* $n_2\times n_1$ *matrix,* $A_{02}\in L([0,b];\mathbb{R}^{n_2\times n_2})$ *and* $A_i\in L_\infty(\mathcal{D};\mathbb{R}^{n_2\times n_2})$ $(i=1,2)$ *are non-negative matrix functions,*

$$A_i(\cdot,y)c\in\Lambda_0 \quad \text{for } y\in[0,b],\ \ c\in\mathbb{R}^{n_2} \quad (i=1,2)$$

*and* $l:\Lambda_0\to\mathbb{R}^{n_2}$ *is a non-negative linear operator. Besides, let*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\big(l(A_2(\cdot,y))\big)<1 \tag{1.5}$$

*and*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\big(A(y)\big)<1, \tag{1.6}$$

*where*

$$A(y)=\operatorname*{ess\,sup}_{x\in I}\big[A_1(x,y)+A_2(x,y)\big(E-l(A_2(\cdot,y))\big)^{-1}l(A_1(\cdot,y))\big]. \tag{1.7}$$

*Then system* (1.1) *has the unique solution* $(z_i)_{i=1}^2\in\Lambda_1\times\Lambda_2$. *Moreover, for an arbitrary* $(z_{i0})_{i=1}^2\in\Lambda_1\times\Lambda_2$ *we have*

$$z_{im}(x,y)\rightrightarrows z_i(x,y) \quad \text{for } m\to+\infty \ \ (i=1,2), \tag{1.8}$$

*where*

$$z_{im}(x,y)=g_i(z_{1\,m-1},z_{2\,m-1})(x,y) \quad (i=1,2;\ \ m=1,2,\dots). \tag{1.9}$$

To prove this lemma we need

*Let* $B_{01}$ *be a non-negative constant* $n_2\times n_1$ *matrix,* $B_{02}\in L([0,b];\mathbb{R}^{n_2\times n_2})$ *and* $B_i\in L_\infty(\mathcal{D};\mathbb{R}^{n_2\times n_2})$ $(i=1,2)$ *be non-negative matrix functions,*

$$B_i(\cdot,y)c\in\Lambda_0 \quad \text{for } y\in[0,b],\ \ c\in\mathbb{R}^{n_2} \ \ (i=1,2)$$

*and* $l:\Lambda_0\to\mathbb{R}^{n_2}$ *be a non-negative linear operator. Moreover, let*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\big(l(B_2(\cdot,y))\big)<1 \tag{1.10}$$

*and*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\big(B(y)\big)<1, \tag{1.11}$$

*where*

$$B(y) = \operatorname*{ess\,sup}_{x\in I} \big[B_1(x,y) + B_2(x,y)\big(E - l(B_2(\cdot,y))\big)^{-1} l(B_1(\cdot,y))\big]. \quad (1.12)$$

*Then there exists a positive number $\rho$ such that for vector functions $w_i \in L_\infty(\mathcal{D};\mathbb{R}^{n_i})$ $(i=1,2)$ satisfying the condition*

$$|w_2(\cdot,y)| \in \Lambda_0$$

*almost everywhere in $[0,b]$ and the inequalities*

$$\|w_1(x,y)\| \le \delta + \int_0^y g_0(t)(\|w_1\|_{I,t} + \|w_2\|_{I,t})\,dt \quad (1.13)$$

*and*

$$|w_2(x,y)| \le \eta + B_{01}|w_1|_{I,y} + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt + \\ + B_1(x,y)|w_2(\cdot,y)|_I + B_2(x,y)l(|w_2(\cdot,y)|), \quad (1.14)$$

*almost everywhere in $\mathcal{D}$, the estimates*

$$\|w_i\|_{I,b} \le \rho \exp\Big(\rho\int_0^b g_0(t)\,dt\Big)(\delta + \|\eta\|) \quad (i=1,2) \quad (1.15)$$

*are valid for any $\delta \in \mathbb{R}_+$, $\eta \in \mathbb{R}^{n_2}$ and $g_0 : [0,b] \to \mathbb{R}_+$.*

*Proof.* Because of the non-negativity of $l$ and the restrictions imposed on the space $\Lambda_0$, from (1.14) we have

$$l(|w_2(\cdot,y)|) \le l\Big(\eta + B_{01}|w_1|_{I,y} + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt\Big) + \\ + l(B_1(\cdot,y))|w_2(\cdot,y)|_I + l(B_2(\cdot,y))l(|w_2(\cdot,y)|)$$

and hence

$$\big[E - l(B_2(\cdot,y))\big]l(|w_2(\cdot,y)|) \le l\Big(\eta + B_{01}|w_1|_{I,y} + \\ + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt\Big) + l(B_1(\cdot,y))|w_2(\cdot,y)|_I,$$

which by virtue of condition (1.10) implies

$$l(|w_2(\cdot,y)|) \le \big(E - l(B_2(\cdot,y))\big)^{-1} l\Big(\eta + B_{01}|w_1|_{I,y} + \\ + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt\Big) + \big(E - l(B_2(\cdot,y))\big)^{-1} l(B_1(\cdot,y))|w_2(\cdot,y)|_I.$$

According to the latter estimate and equality (1.12) we find from (1.14) that

$$|w_2(\cdot,y)|_I \le B_0\Big(\eta + B_{01}|w_1|_{I,y} + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt\Big) + \\ + B(y)|w_2(\cdot,y)|_I,$$

where $B_0$ is the non-negative constant $n_2 \times n_2$ matrix depending on $l$ and $B_2$ only. From this and taking into account (1.11), we get

$$|w_2(\cdot,y)|_I \le \big(E - B(y)\big)^{-1} B_0\Big(\eta + B_{01}|w_1|_{I,y} + \int_0^y B_{02}(t)|w_2|_{I,t}\,dt\Big)$$

and

$$\|w_2\|_{I,y} \le \rho_1\Big(\|\eta\| + \|w_1\|_{I,y} + \int_0^y \|B_{02}(t)\|\,\|w_2\|_{I,t}\,dt\Big),$$

where

$$\rho_1 = (1 + \|B_{01}\|)\|B_0\| \operatorname*{ess\,sup}_{0\le y\le b} \|(E - B(y))^{-1}\|.$$

By Gronwall's lemma ([19],p.37) from the latter inequality it follows that

$$\|w_2\|_{I,y} \le \rho_2(\|\eta\| + \|w_1\|_{I,y}), \tag{1.16}$$

where

$$\rho_2 = \rho_1 \exp\Big(\rho_1 \int_0^b \|B_{02}(t)\|\,dt\Big).$$

In view of (1.13) and (1.16)

$$\|w_1\|_{I,y} \le \delta + \rho_2\|\eta\| \int_0^b g_0(t)\,dt + (1+\rho_2)\int_0^y g_0(t)\|w_1\|_{I,t}\,dt.$$

From this again by Gronwall's lemma we obtain

$$\|w_1\|_{I,y} \le \Big(\delta + \rho_2\|\eta\| \int_0^b g_0(t)\,dt\Big) \exp\Big((1+\rho_2)\int_0^y g_0(t)\,dt\Big) \le \\ \le (1+\rho_2)\exp\Big((2+\rho_2)\int_0^b g_0(t)\,dt\Big)(\delta + \|\eta\|).$$

Estimates (1.15) follow from the latter inequality and (1.16), where $\rho = 2(1+\rho_2)^2$ is the number depending on $B_{0i}$, $B_i$ $(i = 1, 2)$ and $l$ only. ∎

*Proof of Lemma 1.1.* In view of (1.5)-(1.7) there exists $\gamma \in (0,1)$ such that the matrix functions

$$B_i(x,y) = \gamma^{-1} A_i(x,y) \quad (i = 1,2) \tag{1.17}$$

satisfy conditions (1.10) and (1.11), where $B$ is the matrix given by (1.12).

First let us prove the existence of a solution of system (1.1).

Let us choose arbitrarily $(z_{i0})_{i=1}^2 \in \Lambda_1 \times \Lambda_2$, compose the sequence $(z_{im})_{i=1}^2$ $(m = 1,2,\dots)$ by formulas (1.9) and show that this sequence is fundamental in $L_\infty(\mathcal{D};\mathbb{R}^{n_1}) \times L_\infty(\mathcal{D};\mathbb{R}^{n_2})$. Put

$$v_{im}(x,y) = \gamma^{-m}\big[z_{im}(x,y) - z_{i\,m-1}(x,y)\big] \quad (i = 1,2;\ \ m = 1,2,\dots).$$

Then in view of (1.3),(1.4) and (1.17) we obtain

$$\|v_{1m}(x,y)\| \le \gamma^{-1}\int_0^y g_0(t)\big(\|v_{1\,m-1}\|_{I,t} + \|v_{2\,m-1}\|_{I,t}\big)\,dt \qquad (m = 2,3,\dots) \tag{1.18}$$

and

$$|v_{2m}(x,y)| \le B_{01}|v_{1\,m-1}|_{I,y} + \int_0^y B_{02}(t)|v_{2\,m-1}|_{I,t}\,dt + \\ +B_1(x,y)|v_{2\,m-1}(\cdot,y)|_I + B_2(x,y)l(|v_{2\,m-1}(\cdot,y)|) \quad (m = 2,3,\dots), \tag{1.19}$$

where

$$B_{01} = \gamma^{-1}A_{01}, \quad B_{02}(t) = \gamma^{-1}A_{02}(t). \tag{1.20}$$

Introducing the notation

$$w_{im}(x,y) = \max_{1\le k\le m}|v_{ik}(x,y)| \quad (i = 1,2;\ \ m = 1,2,\dots),$$
$$\delta = \|v_{11}\|_{I,b}, \quad \eta = |v_{21}|_{I,b}, \quad \overline{g}_0(t) = n_1\gamma^{-1}g_0(t),$$

from (1.18) and (1.19) we get

$$\|w_{1m}(x,y)\| \le \delta + \int_0^y \overline{g}_0(t)\big(\|w_{1m}\|_{I,t} + \|w_{2m}\|_{I,t}\big)\,dt \ \ (m = 1,2,\dots),$$
$$|w_{2m}(x,y)| \le \eta + B_{01}|w_{1m}|_{I,y} + \int_0^y B_{02}(t)|w_{2m}|_{I,t}\,dt + \\ + B_1(x,y)|w_{2m}(\cdot,y)|_I + B_2(x,y)l(|w_{2m}(\cdot,y)|) \ \ (m = 1,2,\dots),$$

whence according to Lemma 1.2 it follows that

$$\|w_{im}\|_{I,b} \le \rho_0 \quad (i = 1,2;\ \ m = 1,2,\dots),$$

where $\rho_0$ is a positive constant independent of $m$.

Therefore

$$\|z_{im} - z_{i\,m-1}\|_{I,b} \le \rho_0\gamma^m \quad (i = 1,2;\ \ m = 1,2,\dots).$$

Consequently, $\big((z_{im})_{i=1}^2\big)_{m=1}^{+\infty}$ is a fundamental sequence in $L_\infty(\mathcal{D};\mathbb{R}^{n_1})\times$ $\times L_\infty(\mathcal{D};\mathbb{R}^{n_2})$. Put

$$z_i(x,y) = \lim_{m\to+\infty} z_{im}(x,y) \quad (i = 1,2).$$

Since the sets $\Lambda_i$ $(i=1,2)$ are complete and the operators $g_i:\Lambda_1\times\Lambda_2\to\Lambda_i$ $(i=1,2)$ are continuous, it becomes obvious that $(z_i)_{i=1}^2\in\Lambda_1\times\Lambda_2$ is a solution of system (1.1).

To complete the proof of the lemma we have to show that system (1.1) has no solution different from $(z_i)_{i=1}^2$. Let $(\bar z_i)_{i=1}^2$ be an arbitrary solution of this system. Then in view of (1.3) and (1.4)

$$\|z_1(x,y)-\bar z_1(x,y)\|\le\int_0^y g_0(t)(\|z_1-\bar z_1\|_{I,t}+\|z_2-\bar z_2\|_{I,t})dt$$

and

$$|z_2(x,y)-\bar z_2(x,y)|\le A_{01}|z_1-\bar z_1|_{I,y}+\int_0^y A_{02}(t)|z_2-\bar z_2|_{I,t}dt+$$
$$+A_1(x,y)|z_2(\cdot,y)-\bar z_2(\cdot,y)|_I+A_2(x,y)l(|z_2(\cdot,y)-\bar z_2(\cdot,y)),$$

whence according to conditions (1.5) and (1.6) and Lemma 1.2 it follows that

$$\|z_i-\bar z_i\|_{I,b}=0\quad(i=1,2).\ \blacksquare$$

When $l$ is the zero operator Lemma 1.1 takes the form of

*Let for any $\zeta_i$ and $\bar\zeta_i\in\Lambda_i$ $(i=1,2)$ inequality* (1.3) *hold almost everywhere in $\mathcal D$ and*

$$|g_2(\zeta_1,\zeta_2)(x,y)-g_2(\bar\zeta_1,\bar\zeta_2)(x,y)|\le A_{01}|\zeta_1-\bar\zeta_1|_{I,y}+$$
$$+\int_0^y A_{02}(t)|\zeta_2-\bar\zeta_2|_{I,t}dt+A(y)|\zeta_2(\cdot,y)-\bar\zeta_2(\cdot,y)|,\tag{1.21}$$

*where $g_0:[0,b]\to\mathbb R_+$ is a summable function, $A_{01}$ is a non-negative constant $n_2\times n_1$ matrix, $A_{02}\in L([0,b];\mathbb R^{n_2\times n_2})$ and $A\in L_\infty(\mathcal D;\mathbb R^{n_2\times n_2})$ are non-negative matrix functions. Moreover, it is assumed that $A$ satisfies condition* (1.6)*. Then system* (1.1) *has the unique solution $(z_i)_{i=1}^2\in\Lambda_1\times\Lambda_2$ and for arbitrary $(z_{i0})_{i=1}^2\in\Lambda_1\times\Lambda_2$ conditions* (1.8) *hold, where*

$$z_{im}(x,y)=g_i(z_{1\,m-1},z_{2\,m-1})(x,y)\quad(i=1,2;\ m=1,2,\dots).$$

Equation (1.2) is equivalent to system (1.1), where

$$n_1=n,\quad n_2=1,\quad \Lambda_1=\Lambda,\quad \Lambda_2=L_\infty(\mathcal D;\mathbb R),$$
$$g_1(z_1,z_2)(x,y)\equiv g(z_1)(x,y),\quad g_2(z_1,z_2)(x,y)\equiv 0.$$

Therefore from Lemma 1.3 immediately follows

*Let for any $\zeta$ and $\bar\zeta\in\Lambda$ the inequality*

$$\|g(\zeta)(x,y)-g(\bar\zeta)(x,y)\|\le\int_0^y g_0(t)\|\zeta-\bar\zeta\|_{I,t}dt\tag{1.22}$$

*hold almost everywhere in $\mathcal{D}$, where $g_0 : [0,b] \to \mathbb{R}_+$ is a summable function. Then equation* (1.2) *has the unique solution $z \in \Lambda$ and for arbitrary $z_0 \in \Lambda$ we have*

$$z_m(x,y) \rightrightarrows z(x,y) \quad \textit{for } \ m \to +\infty, \tag{1.23}$$

*where*

$$z_m(x,y) = g(z_{m-1})(x,y) \quad (m = 1,2,\dots). \tag{1.24}$$

*Remark* 1.1. From the above arguments it becomes clear that if conditions of Lemma 1.1 take place, then

$$\|z_i - z_{im}\|_{I,b} \le \rho_0 \gamma^m \quad (m = 1,2,\dots),$$

where $\gamma \in (0,1)$ and $\rho_0 > 0$ are constants independent of $m$. If conditions of Lemma 1.4 hold, then

$$\|z - z_m\| \le \frac{\rho_0^m}{m!} \quad (m = 1,2,\dots).$$

## § 2.

In this section for a system of linear differential equations

$$\frac{dz(x,y)}{dx} = A(x,y)z(x,y) + c(x,y) \tag{2.1}$$

depending on a parameter $y \in [0,b]$, we investigate boundary value problems of the type

$$h(z(\cdot,y))(y) = \varphi(y), \tag{2.2}$$

where

$$\begin{gathered} A(\cdot,y) = (a_{ij}(\cdot,y)) \in L_\infty([0,a];\mathbb{R}^{n\times n}) \quad \text{for } \ y \in [0,b], \\ \gamma = \sup_{(x,y)\in\mathcal{D}_{ab}} \|A(x,y)\| < +\infty, \\ c(\cdot,y) \in L_\infty([0,a];\mathbb{R}^n), \quad \varphi(y) \in \mathbb{R}^n \quad \text{for } \ y \in [0,b] \end{gathered} \tag{2.3}$$

and $h$ is a linear continuous operator acting from the space $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to a subspace of the space $L_\infty([0,b];\mathbb{R}^n)$.

The operator $h$ uniquely defines matrix functions

$$H_0 \in L_\infty([0,b];\mathbb{R}^{n\times n}), \quad H \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \tag{2.4}$$

such that

$$h(u)(y) = H_0(y)u(0) + \int_0^a H(s,y)u'(s)\,ds \qquad \text{for } u \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad y \in [0,b].\text{[4]} \tag{2.5}$$

Along with (2.1),(2.2) let us consider the corresponding homogeneous boundary value problem

$$\frac{dz(x,y)}{dx} = A(x,y)z(x,y), \tag{2.1$_0$}$$

$$h(z(\cdot,y))(y) = 0. \tag{2.2$_0$}$$

By $Z_0(\cdot,y)$ will be meant a fundamental matrix of system (2.1$_0$), satisfying the initial condition

$$Z_0(0,y) = E.$$

By $M_0$ and $M$ we denote matrix functions given by the equalities

$$\begin{aligned} M_0(y) &= H_0(y) + \int_0^a H(s,y)\frac{\partial Z_0(s,y)}{\partial s}\,ds, \\ M(x,y) &= H(x,y)Z_0(x,y) + \int_x^a H(s,y)\frac{\partial Z_0(s,y)}{\partial s}\,ds. \end{aligned} \tag{2.6}$$

Moreover, we shall use the notation

$$I_{M_0} = \{y \in [0,b] \,:\, \det M_0(y) \neq 0\}.$$

In view of (2.5) and (2.6),

$$h(Z_0(\cdot,y)u(\cdot))(y) = M_0(y)u(0) + \int_0^a M(s,y)u'(s)\,ds \qquad \text{for } u \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad y \in [0,b]. \tag{2.7}$$

It is easily seen that the condition

$$y \in I_{M_0} \tag{2.8}$$

is necessary and sufficient for problem (2.1$_0$),(2.2$_0$) to have only the trivial solution.

[4]See Lemma 2.1.

**h**

In this subsection we shall give some results of Dunford-Pettis' theorem ([20],Ch.XI, §1, Theorem 6) on the integral representation of the operator $h$ which will be necessary in the sequel.

1 *$h$ is the linear continuous operator acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $L_\infty([0,b];\mathbb{R}^n)$ if and only if representation* (2.5) *is valid. $H_0$ and $H$ here are the matrix functions satisfying conditions* (2.4).

*Proof.* It is easy to see that if the matrix functions $H_0$ and $H$ satisfy conditions (2.4), then the operator $h$ given by equality (2.5) is a linear continuous operator acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $L_\infty([0,b];\mathbb{R}^n)$. Let us now show the converse: if $h:\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)\to L_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator, then representation (2.5) is valid. For an arbitrary vector function $v\in L([0,a];\mathbb{R}^n)$ assume that

$$l(v)(x)=\int_0^x v(s)\,ds$$

and

$$\bar{h}(v)(y)=h(l(v))(y).$$

It follows from the continuity of operators $l:L([0,a];\mathbb{R}^n)\to\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ and $h:\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)\to L_\infty([0,b];\mathbb{R}^n)$ that $\bar{h}$ is a linear continuous operator acting from $L([0,a];\mathbb{R}^n)$ to $L_\infty([0,b];\mathbb{R}^n)$. Therefore, according to the above-mentioned Dunford-Pettis theorem, there exists a unique matrix function $H\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ such that

$$\bar{h}(v)(y)=\int_0^a H(s,y)v(s)\,ds \quad\text{for}\quad v\in L([0,a];\mathbb{R}^n),\ \ y\in[0,b]. \tag{2.9}$$

On the other hand, since $c\to h(c)(y)$ is a linear continuous operator acting from $\mathbb{R}^n$ to $L_\infty([0,b];\mathbb{R}^n)$, there exists a unique matrix function $H_0\in \mathrm{L}_\infty([0,b];\mathbb{R}^{n\times n})$ such that

$$h(c)(y)=H_0(y)c \quad\text{for}\quad c\in\mathbb{R}^n,\ \ y\in[0,b]. \tag{2.10}$$

Let $u\in\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ be an arbitrary vector function. Then

$$u(x)=u(0)+l(u')(x)\quad\text{for}\quad x\in[0,a].$$

Therefore, in view of (2.9) and (2.10), we have

$$h(u)(y)=h(u(0))(y)+\bar{h}(u')(y)=$$
$$=H_0(y)u(0)+\int_0^a H(s,y)u'(s)\,ds\quad\text{for}\quad y\in[0,b].$$

Consequently, representation (2.5) is valid, where $H_0$ and $H$ are the matrix functions satisfying conditions (2.4). ■

$_2$[5] *$h$ is a linear continuous operator acting from* $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ *to* $\mathbb{C}([0,b];\mathbb{R}^n)$ *if and only if representation* (2.5) *is valid, where*

$$H_0 \in \mathbb{C}([0,b];\mathbb{R}^{n\times n}), \quad H \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \tag{2.11}$$

*and*

$$\int_0^x H(s,\cdot)\,ds \in \mathbb{C}([0,b];\mathbb{R}^{n\times n}) \quad \textit{for } x \in [0,a]. \tag{2.12}$$

*Proof.* Let $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}([0,b];\mathbb{R}^n)$ be a linear continuous operator. Then, according to Lemma $2.1_1$ representation (2.5) is valid, where $H_0$ and $H$ are matrix functions satisfying conditions (2.4). On the other hand,

$$H_0(\cdot)(c) = h(c) \in \mathbb{C}([0,b];\mathbb{R}^n) \quad \text{for } c \in \mathbb{R}^n$$

and

$$\Big(\int_0^x H(s,\cdot)\,ds\Big)c = h(\chi_{c,x}) \in \mathbb{C}([0,b];\mathbb{R}^n) \quad \text{for } c \in \mathbb{R}^n, \ \ x \in [0,a],$$

where

$$\chi_{c,x}(s) = \begin{cases} sc & \text{for } 0 \le s \le x \\ xc & \text{for } s > x \end{cases}.$$

Consequently conditions (2.11) and (2.12) hold. Thus the first part of the lemma is proved.

Assume now that $H_0$ and $H$ are the matrix functions satisfying conditions (2.11) and (2.12), and $h$ is an operator admitting representation (2.5). First of all let us show that $h$ acts from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $\mathbb{C}([0,b];\mathbb{R}^n)$. To do this, it is enough to show that for an arbitrary $v \in L([0,a];\mathbb{R}^n)$ the condition

$$w(\cdot) = \int_0^a H(s,\cdot)v(s)\,ds \in \mathbb{C}([0,b];\mathbb{R}^n) \tag{2.13}$$

holds.

Without loss of generality we may assume that

$$\gamma_0 = \sup_{(x,y)\in\mathcal{D}_{ab}} \|H(x,y)\| < +\infty. \tag{2.14}$$

Let $v_k : [0,a] \to \mathbb{R}^n$ $(k = 1,2,\dots)$ be a sequence of continuously differentiable vector functions satisfying conditions

$$\int_0^a \|v_k(s) - v(s)\|\,ds \to 0 \quad \text{for } k \to +\infty. \tag{2.15}$$

Put

$$w_k(y) = \int_0^a H(s,y)v_k(s)\,ds.$$

[5] This lemma can also be obtained from B.Z.Vulikh's theorem ([48], Theorem 2).

Then

$$w_k(y) = H_1(a,y)v_k(a) - \int_0^a H_1(s,y)v_k'(s)\,ds,$$

where

$$H_1(x,y) = \int_0^x H(s,y)\,ds,$$

whence, in view of (2.11) and (2.12), it follows that

$$w_k \in \mathbb{C}([0,b];\mathbb{R}^n) \quad (k = 1,2,\dots).$$

On the other hand, in view of (2.14) and (2.15),

$$\sup_{y\in[0,b]} \|w_k(y) - w(y)\| \le \gamma_0 \int_0^a \|v_k(s) - v(s)\|\,ds \to 0 \quad \text{for} \quad k \to +\infty.$$

Now, applying the well-known Weierstrass theorem, the validity of condition (2.13) becomes evident. Hence $h$ is a continuous operator acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $\mathbb{C}([0,b];\mathbb{R}^n)$. ■

3 *Let $k$ be an arbitrary positive integer. $h$ is a linear continuous operator acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $\widetilde{\mathbb{C}}^{k-1}_{\infty}([0,b];\mathbb{R}^n)$ if and only if condition* (2.5) *holds, where*

$$H_0 \in \widetilde{\mathbb{C}}^{k-1}_{\infty}([0,b];\mathbb{R}^{n\times n}) \tag{2.16}$$

*and*

$$H \in \widetilde{\mathbb{C}}^{(-1,k-1)}_{\infty}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}). \tag{2.17}$$

*Proof.* It is evident that if the matrix functions $H_0$ and $H$ satisfy conditions (2.16) and (2.17), respectively, then operator the $h$, given by the equality (2.5), transforms $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ into $\widetilde{\mathbb{C}}^{k-1}_{\infty}([0,b];\mathbb{R}^n)$ and is linear and continuous. Let us show the converse: if $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}^{k-1}_{\infty}([0,b];\mathbb{R}^n)$ is a linear continuous operator, then (2.5) is valid and $H_0$ and $H$ satisfy conditions (2.16) and (2.17), respectively.

For an arbitrary vector function $u \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ we put

$$h^{(i)}(u)(y) = \frac{d^i}{dy^i} h(u)(y) \quad (i = 0,\dots,k).$$

It is evident that $h^{(k)}$ and $h^{(i)}(\cdot)(0)$ $(i = 0,\dots,k-1)$ are linear continuous operators acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ respectively to $L_\infty([0,b];\mathbb{R}^n)$ and $\mathbb{R}^n$. According to Lemma $2.1_1$,

$$h^{(k)}(u)(y) = H_{0k}(y)u(0) + \int_0^a H_k(s,y)u'(s)\,ds$$
$$\text{for} \quad u \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad y \in [0,b],$$

and

$$h^{(i)}(u)(0) = H_{0i}u(0) + \int_0^a H_i(s)u'(s)\,ds \quad (i = 0, 1, \dots, k-1),$$

where

$$H_{0k} \in L_\infty([0,b];\mathbb{R}^{n\times n}), \quad H_k \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$
$$H_{0i} \in \mathbb{R}^{n\times n}, \quad H_i \in L_\infty([0,a];\mathbb{R}^{n\times n}) \quad (i = 0, \dots, k-1).$$

Therefore from the identity

$$h(u)(y) = \sum_{i=0}^{k-1} \frac{y^i}{i!} h^{(i)}(u)(0) + \frac{1}{(k-1)!}\int_0^y (y-t)^{k-1} h^{(k)}(u)(t)\,dt$$

there follows representation (2.5), where

$$H_0(y) = \sum_{i=0}^{k-1} \frac{y^i}{i!} H_{0i} + \frac{1}{(k-1)!}\int_0^y (y-t)^{k-1} H_{0k}(t)dt,$$

and

$$H(x,y) = \sum_{i=0}^{k-1} \frac{y^i}{i!} H_i(x) + \frac{1}{(k-1)!}\int_0^y (y-t)^{k-1} H_k(x,t)\,dt$$

are the matrix functions satisfying conditions (2.16) and (2.17). ■

4 *Let $k$ be an arbitrary positive integer. $h$ is a linear continuous operator acting from $\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ to $\mathbb{C}^k([a,b];\mathbb{R}^n)$ if and only if* (2.5) *is valid, where*

$$H_0 \in \mathbb{C}^k([0,b];\mathbb{R}^{n\times n}),$$

*the matrix function $H$ satisfies condition* (2.17), *and*

$$\int_0^x H(s,\cdot)\,ds \in \mathbb{C}^k([0,b];\mathbb{R}^{n\times n}) \quad \textit{for } x \in [0,a].$$

This lemma can be proved similarly to Lemma $2.1_3$, with the only difference that instead of Lemma $2.1_1$ we use Lemma $2.1_2$.

$Z_0$ $M_0$ $M$

1 *If*

$$A \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \tag{2.18}$$

*then*

$$Z_0 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \tag{2.19}$$

*and*

$$\frac{\partial Z_0}{\partial x} \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}). \tag{2.20}$$

*In the case when*

$$\int_0^x A(s,\cdot)\,ds \in \mathbb{C}([0,b]; \mathbb{R}^{n\times n}) \quad \textit{for} \quad x \in [0,a], \tag{2.21}$$

*then instead of* (2.19) *we have*

$$Z_0 \in \mathbb{C}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}). \tag{2.22}$$

*Proof.* In view of (2.3) and (2.18) it is clear that the matrix function $Z_0$ is measurable and satisfies inequalities

$$\Big\|\frac{\partial Z_0(x,y)}{\partial x}\Big\| \le \gamma \big\|Z_0(x,y)\big\|$$

and

$$\|Z_0(x,y)\| \le n + \gamma \int_0^x \|Z_0(s,y)\|\,ds$$

almost everywhere in $\mathcal{D}_{ab}$. From this, using Gronwall's lemma, we obtain

$$\|Z_0(x,y)\| \le n\exp(\gamma x), \quad \Big\|\frac{\partial Z_0(x,y)}{\partial x}\Big\| \le n\gamma\exp(\gamma x). \tag{2.23}$$

Hence $Z_0$ satisfies conditions (2.19) and (2.20).

Assume now that (2.21) holds. Then, in view of (2.3), the matrix function

$$A_1(x,y) = \int_0^x A(s,y)\,ds$$

is continuous in $\mathcal{D}_{ab}$. Therefore it is clear that the function

$$\omega(\delta) = \max_{\substack{0\le x\le a,\, 0\le y,\bar{y}\le b \\ |y-\bar{y}|\le\delta}} \|A_1(x,y) - A_1(x,\bar{y})\| \tag{2.24}$$

satisfies the condition

$$\lim_{\delta\to 0}\omega(\delta) = 0. \tag{2.25}$$

In view of (2.3), (2.23) and (2.24), from the equality

$$Z_0(x,y) - Z_0(x,\bar{y}) = \int_0^x [A(s,y) - A(s,\bar{y})]Z_0(s,y)\,ds +$$

$$+ \int_0^x A(s,\bar{y})[Z_0(s,y) - Z_0(s,\bar{y})]\,ds = [A_1(x,y) - A_1(x,\bar{y})]Z_0(x,y) -$$

$$- \int_0^x [A_1(s,y) - A_1(s,\bar{y})]A(s,y)Z_0(s,y)\,ds + \int_0^x A(s,\bar{y})[Z_0(s,y) - Z_0(s,\bar{y})]ds$$

it follows that

$$\|Z_0(x,y)-Z_0(x,\bar y)\| \le \gamma_1\omega(|y-\bar y|)+\gamma\int_0^x \|Z_0(s,y)-Z_0(s,\bar y)\|\,ds$$
$$\text{for} \quad 0\le x\le a, \;\; 0\le y,\bar y\le b,$$

where

$$\gamma_1 = n(1+a\gamma)\exp(\gamma a),$$

whence, according to Gronwall's lemma, we get

$$\|Z_0(x,y)-Z_0(x,\bar y)\| \le \gamma_0\omega(|y-\bar y|) \quad \text{for} \;\; 0\le x\le a, \;\; 0\le y,\bar y\le b, \tag{2.26}$$

where $\gamma_0 = \gamma_1\exp(\gamma a)$. The validity of (2.22) becomes evident if we take into account the above estimate and conditions (2.20) and (2.25). ■

2 *If*

$$A\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \tag{2.27}$$

*then*

$$Z_0\in\widetilde{\mathbb{C}}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \tag{2.28}$$

$$\frac{\partial Z_0}{\partial x}, \quad \frac{\partial Z_0}{\partial y} \quad \textit{and} \quad \frac{\partial^2 Z_0}{\partial x\partial y}\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}). \tag{2.29}$$

*When*

$$A \quad \textit{and} \quad \frac{\partial A}{\partial x}\in\mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \tag{2.30}$$

*then instead of* (2.29) *we have*

$$\frac{\partial Z_0}{\partial x}, \quad \frac{\partial Z_0}{\partial y} \quad \textit{and} \quad \frac{\partial^2 Z_0}{\partial x\partial y}\in\mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}). \tag{2.31}$$

*Proof.* In view of (2.27) there exists a positive number $l_0$ such that the function $\omega$ given by (2.24) satisfies the inequality

$$\omega(\delta)\le l_0\delta \quad \text{for} \quad \delta\ge 0.$$

Therefore from (2.26) we get

$$\|Z_0(x,y)-Z_0(x,\bar y)\| \le \gamma_0 l_0|y-\bar y| \quad \text{for} \;\; 0\le x\le a, \;\; 0\le y,\bar y\le b. \tag{2.32}$$

Taking into account the above arguments and conditions (2.3),(2.23) and (2.27), it follows from the identity

$$\frac{\partial Z_0(x,y)}{\partial x} = A(x,y)Z_0(x,y) \tag{2.33}$$

that

$$\frac{\partial Z_0}{\partial x} \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}), \tag{2.34}$$

$$\frac{\partial^2 Z_0(x,y)}{\partial x \partial y} = A(x,y)\frac{\partial Z_0(x,y)}{\partial y} + \frac{\partial A(x,y)}{\partial y} Z_0(x,y) \tag{2.35}$$

and

$$\frac{\partial Z_0(x,y)}{\partial y} = Z_0(x,y)\int_0^x Z_0^{-1}(s,y)\frac{\partial A(s,y)}{\partial y} Z_0(s,y)\,ds. \tag{2.36}$$

The validity of conditions (2.28) and (2.29) becomes evident if we take into consideration (2.32)-(2.35) together with (2.3),(2.20) and (2.27).

In the case when conditions (2.30) hold, conditions (2.31) follow from equalities (2.33),(2.35) and (2.36). [6] ■

By virtue of Lemmas $2.1_1$ - $2.1_4$, $2.2_1$ and $2.2_2$, from (2.6) we get the following propositions:

$_1$ *If* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to L_\infty([0,b]; \mathbb{R}^n)$ *is a linear continuous operator and the matrix function* $A$ *satisfies condition* (2.18), *then*

$$M_0 \in L_\infty([0,b]; \mathbb{R}^{n\times n}), \quad M \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}).$$

$_2$ *Let* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \mathbb{C}([0,b]; \mathbb{R}^n)$ *be a linear continuous operator and let the matrix function* $A$ *satisfy conditions* (2.18),(2.21). *Then*

$$M_0 \in \mathbb{C}([0,b]; \mathbb{R}^{n\times n}), \quad M \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$$

*and*

$$\int_0^x M(s,\cdot)\,ds \in \mathbb{C}([0,b]; \mathbb{R}^{n\times n}) \quad \textit{for} \quad x \in [0,a].$$

$_3$ *Let* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b]; \mathbb{R}^n)$ *be a linear continuous operator and let the matrix function* $A$ *satisfy condition* (2.27). *Then*

$$M_0 \in \widetilde{\mathbb{C}}_\infty([0,b]; \mathbb{R}^{n\times n})$$

*and*

$$M \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}). \tag{2.37}$$

---

[6] This fact is a consequence of the well-known Peano theorem on the differentiability of a solution of the Cauchy problem with respect to a parameter ([19], Theorem 3.1).

4 *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}^1([0,b];\mathbb{R}^n)$ *be a linear continuous operator and let the matrix function* $A$ *be continuous and have a continuous partial derivative in the second argument. Then*

$$M_0 \in \mathbb{C}^1([0,b];\mathbb{R}^{n\times n}),$$

*the matrix function* $M$ *satisfies condition* (2.37) *and*

$$\int_0^x M(s,\cdot)\,ds \in \mathbb{C}^1([0,b];\mathbb{R}^{n\times n}) \quad \textit{for} \quad x \in [0,a].$$

*Let the parameter* $y \in [0,b]$ *be such that*

$$\begin{gathered} M(x,y)Z_0^{-1}(x,y)c(x,y) = M_0(y)c_0(x,y) \quad \textit{for} \ 0 \le x \le a, \\ \varphi(y) = M_0(y)\varphi_0(y), \end{gathered} \tag{2.38}$$

*where*

$$c(\cdot,y) \in L([0,a];\mathbb{R}^n) \quad \textit{and} \quad \varphi_0(y) \in \mathbb{R}^n.$$

*Then the vector function*

$$z(x,y) = Z_0(x,y)\Big[\varphi_0(y) - \int_0^a c_0(s,y)\,ds + \int_0^x Z_0^{-1}(s,y)c(s,y)\,ds\Big] \tag{2.39}$$

*is a solution of problem* (2.1),(2.2), *which is unique if and only if condition* (2.8) *holds.*

*Proof.* According to the Cauchy formula, [7] the vector function

$$z(x,y) = Z_0(x,y)\Big[\alpha + \int_0^x Z_0^{-1}(s,y)c(s,y)\,ds\Big] \tag{2.40}$$

is a solution of system (2.1) for every $\alpha \in \mathbb{R}^n$ and, vice versa, for every solution of this system there exists $\alpha \in \mathbb{R}^n$ such that representation (2.40) is valid.

If we substitute (2.40) into (2.2), then with regard to (2.7) and (2.38) we shall find

$$M_0(y)\alpha + \int_0^a M(s,y)Z_0^{-1}(s,y)c(s,y)\,ds = \varphi(y)$$

and

$$M_0(y)\alpha = M_0(y)\Big[\varphi_0(y) - \int_0^a c_0(s,y)\,ds\Big]. \tag{2.41}$$

[7] See [19], Ch.4, §2, Corollary 2.1.

Consequently the vector function (2.40) is a solution of problem (2.1),(2.2) if and only if $\alpha$ is a solution of the system of linear algebraic equations (2.41). It is obvious that

$$\alpha = \varphi_0(y) - \int_0^a c_0(s,y)\,ds$$

is a solution of this system which is unique if and only if condition (2.8) holds. ∎

According to Lemmas $2.2_1$, $2.2_2$ and $2.3_1$ -$2.3_4$, from Lemma 2.4 we get the following assertions.

$_1$ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to L_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator, the matrix function* $A$ *satisfy condition* (2.18) *and*

$$mes\, I_{M_0} = b.$$

*Moreover, let*

$$c \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n), \quad \varphi \in L([0,b];\mathbb{R}^n), \quad \eta \in L([0,b];\mathbb{R}^n),$$

*where*

$$\eta(y) = \|M_0^{-1}(y)\| + \|M_0^{-1}(y)\varphi(y)\|.$$

*Then for almost every* $y \in [0,b]$ *problem* (2.1),(2.2)*has the unique solution* $z(\cdot,y)$, *and*

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \Big(\|z(x,y)\| + \Big\|\frac{\partial z(x,y)}{\partial x}\Big\|\Big)\Big/\eta(y) < +\infty.$$

$_2$ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}([0,b];\mathbb{R}^n)$ *be a linear continuous operator,* $A \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ *and* $I_{M_0} = [0,b]$. *Moreover, let*

$$c \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^n), \quad \varphi \in \mathbb{C}([0,b];\mathbb{R}^n).$$

*Then for every* $y \in [0,b]$ *problem* (2.1),(2.2) *has the unique solution* $z(\cdot,y)$, *and*

$$z \quad and \quad \frac{\partial z}{\partial x} \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^n).$$

$_3$ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator, the matrix function* $A$ *satisfy condition* (2.27), *and* $I_{M_0} =$ $= [0,b]$. *Let, in addition,*

$$c \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^n), \quad \varphi \in \widetilde{\mathbb{C}}([0,b];\mathbb{R}^n).$$

*Then for every* $y \in [0,b]$ *problem* (2.1),(2.2) *has the unique solution* $z(\cdot,y)$ *and*

$$z \in \widetilde{\mathbb{C}}(\mathcal{D}_{ab};\mathbb{R}^n).$$

4 *Let $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}^1([0,b];\mathbb{R}^n)$ be a linear continuous operator, $A : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ and $c : \mathcal{D}_{ab} \to \mathbb{R}^n$ be continuous and have the continuous partial derivatives $\frac{\partial A}{\partial y}$ and $\frac{\partial c}{\partial y}$, $\varphi : [0,b] \to \mathbb{R}^n$ be continuously differentiable and $I_{M_0} = [0,b]$. Then for every $y \in [0,b]$ problem (2.1),(2.2) has the unique solution $z(\cdot,y)$, where $z : \mathcal{D}_{ab} \to \mathbb{R}^n$ is continuous and has the continuous partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}$.*

$M_0^{-1}(y)$ For every $k \in \{1,\dots,n\}$ let $e_k$ denote an $n$-dimensional column vector whose $k$-th component is unity and the remainder are zero. We have

*If $I_{M_0} \neq \varnothing$, then*

$$\|M_0^{-1}(y)\| = \sum_{k=1}^{n} \|z_k(0,y)\| \quad \textit{for} \quad y \in I_{M_0}, \tag{2.42}$$

*where every $z_k(\cdot,y)$ is a solution of system $(2.1_0)$ satisfying the condition*

$$h(z_k(\cdot,y))(y) = e_k. \tag{2.43}$$

*Proof.* According to Lemma 2.4, problem $(2.1_0)$,(2.43) has the unique solution $z_k(\cdot,y)$ for every $k \in \{1,\dots,n\}$ and $y \in I_{M_0}$. Let $Z(x,y)$ be a matrix with columns $z_1(x,y), \dots, z_n(x,y)$. Then

$$Z(x,y) = Z_0(x,y)Z(0,y).$$

Hence, in view of (2.7) and (2.43), we have

$$E = M_0(y)Z(0,y) \quad \text{for} \quad y \in I_{M_0}.$$

Consequently, equality (2.42) is valid. ■

Let us introduce the notation

$$\begin{gathered} A_0(s,x,y) = \Theta, \quad A_1(s,x,y) = A(s,y), \\ A_{j+1}(s,x,y) = \int_s^x A(\xi,y)A_j(s,\xi,y)\,d\xi \quad (j = 1,2,\dots), \end{gathered} \tag{2.44}$$

$$\begin{gathered} M_{00}(y) = H_0(y), \qquad M_{0j}(y) = H_0(y) + \\ + \int_0^a H(s,y)A(s,y)\Big[E + \sum_{i=0}^{j-1} \int_0^s A_i(\xi,s,y)\,d\xi\Big]\,ds \;\; (j = 1,2,\dots). \end{gathered} \tag{2.45}$$

If the inequality

$$\det M_{0\,k-1}(y) \neq 0$$

holds for any $y \in [0,b]$ and natural $k$, then for every natural $m$ we put

$$B_{1m}(s,x,y) = \Big[E + \sum_{i=0}^{m-1} \int_0^x A_i(\xi,x,y)\,d\xi\Big] M_{00}^{-1}(y) H(s,y) A(s,y), \quad (2.46)$$

$$B_{km}(s,x,y) = \Big[E + \sum_{i=0}^{m-1} \int_0^x A_i(\xi,x,y)\,d\xi\Big] M_{0\,k-1}^{-1}(y) \times$$
$$\times \int_s^a H(\xi,y) A(\xi,y) A_{k-1}(s,\xi,y)\,d\xi \quad \text{for} \ \ k > 1 \quad (2.47)$$

and

$$B_{km}^0(y) = \max_{0 \le x \le a} \Big[ \int_0^x |A_m(s,x,y) - B_{km}(s,x,y)|\,ds +$$
$$+ \int_x^a |B_{km}(s,x,y)|\,ds\Big]. \quad (2.48)$$

*Let there exist natural $k$ and $m$ and a nonempty set $I_0 \subset [0,b]$ such that*

$$\det M_{0\,k-1}(y) \neq 0 \quad \textit{for} \quad y \in I_0 \quad (2.49)$$

*and*

$$r\big(B_{km}^0(y)\big) < 1 \quad \textit{for} \quad y \in I_0. \quad (2.50)$$

*Then $I_0 \subset I_{M_0}$ and*

$$\|M_0^{-1}(y)\| \le \rho_0 \|(E - B_{km}^0(y))^{-1}\| \, \|M_{0\,k-1}^{-1}(y)\| \quad \textit{for} \quad y \in I_0, \quad (2.51)$$

*where $\rho_0$ is a positive constant.*

*Proof.* Let $z(\cdot,y)$ be an arbitrary solution of system $(2.1_0)$. Then for any natural $j$ we have

$$z(x,y) = \Big[E + \sum_{i=0}^{j-1} \int_0^x A_i(s,x,y)\,ds\Big] z(0,y) + \int_0^x A_j(s,x,y) z(s,y)\,ds. \quad (2.52_j)$$

In view of (2.5),(2.44) and (2.45), from $(2.52_k)$ we find that

$$h\big(z(\cdot,y)\big)(y) = M_{00}(y) z(0,y) + \int_0^a H(s,y) A(s,y) z(s,y)\,ds \quad \text{for} \ \ k = 1$$

and

$$h\big(z(\cdot,y)\big)(y) = M_{0\,k-1}(y) z(0,y) +$$
$$+ \int_0^a \Big( \int_s^a H(\xi,y) A(\xi,y) A_{k-1}(s,\xi,y)\,d\xi\Big) z(s,y)\,ds \quad \text{for} \ \ k > 1.$$

Hence, by virtue of (2.49) it follows that

$$z(0,y) = M_{00}^{-1}(y)h\big(z(\cdot,y)\big)(y) - M_{00}^{-1}(y)\int_0^a H(s,y)A(s,y)z(s,y)\,ds$$
$$\text{for} \quad k=1, \ \ y \in I_0$$

and

$$z(0,y) = M_{0\,k-1}^{-1}(y)h\big(z(\cdot,y)\big)(y) - M_{0\,k-1}^{-1}(y)\int_0^a \Big(\int_s^a H(\xi,y)\times$$
$$\times A(\xi,y)A_{k-1}(s,\xi,y)\,d\xi\Big)z(s,y)ds \quad \text{for} \ \ k>1, \ \ y \in I_0.$$

If we substitute the value $z(0,y)$ into the formula $(2.52_m)$, then, in view of the notation (2.46) and (2.47), we shall get

$$z(x,y) = \Big[E + \sum_{i=0}^{m-1}\int_0^x A_i(s,x,y)\,ds\Big]M_{0\,k-1}^{-1}(y)h\big(z(\cdot,y)\big)(y) +$$
$$+\int_0^x [A_m(s,x,y) - B_{km}(s,x,y)]z(s,y)\,ds - \int_x^a B_{km}(s,x,y)z(s,y)\,ds \text{ for } y \in I_0,$$

whence, with regard to (2.48), we obtain

$$\bar z(y) \le B_{km}^0(y)\bar z(y) + A_m^0(y)|M_{0\,k-1}^{-1}(y)|\big|h\big(z(\cdot,y)\big)(y)\big| \quad \text{for} \ \ y \in I_0,$$

where

$$\bar z(y) = \max_{0\le x\le a}|z(x,y)|$$

and

$$A_m^0(y) = \max_{0\le x\le a}\Big|E + \sum_{i=0}^{m-1}\int_0^x A_i(\xi,x,y)\,d\xi\Big|.$$

By condition (2.50) it follows that

$$\bar z(y) \le \big(E - B_{km}^0(y)\big)^{-1}A_m^0(y)|M_{0\,k-1}^{-1}(y)|\big|h\big(z(\cdot,y)\big)(y)\big| \quad \text{for} \ \ y \in I_0,$$

and, consequently,

$$\|z(x,y)\| \le \|A_m^0(y)\|\,\big\|\big(E - B_{km}^0(y)\big)^{-1}\big\|\,\|M_{0\,k-1}^{-1}(y)\|\,\big\|h\big(z(\cdot,y)\big)(y)\big\|$$
$$\text{for} \quad 0 \le x \le a, \ \ y \in I_0.$$

However, in view of (2.3),

$$\rho_0 = n\sup_{0\le y\le b}\|A_m^0(y)\| < +\infty.$$

Thus we have proved that an arbitrary solution $z(\cdot,y)$ of system $(2.1_0)$ admits the estimate

$$\|z(x,y)\| \le \frac{\rho_0}{n}\big\|\big(E - B_{km}^0(y)\big)^{-1}\big\|\,\|M_{0\,k-1}^{-1}(y)\|\big\|h\big(z(\cdot,y)\big)(y)\big\| \tag{2.53}$$
$$\text{for} \quad 0 \le x \le a, \ \ y \in I_0,$$

where $\rho_0$ is independent of $I_0$ and $z(\cdot,y)$. It is easily seen that for every $y \in I_0$ problem $(2.1_0)$,$(2.2_0)$ has only the trivial solution, i.e. $I_0 \subset I_{M_0}$.

By virtue of Lemma 2.4, for every $y \in I_0$ and $k \in \{1,\dots,n\}$ problem $(2.1_0)$,(2.43) has the unique solution $z_k(\cdot,y)$. On the other hand, in view of (2.53)

$$\|z_k(x,y)\| \le \frac{\rho_0}{n}\big\|\big(E - B^0_{km}(y)\big)^{-1}\big\|\,\|M^{-1}_{0\,k-1}(y)\|$$
$$\text{for } \ 0 \le x \le a, \ \ y \in I_0 \ \ (k = 1,\dots,n).$$

With regard to these inequalities, from (2.42) we get (2.51). ■

*Remark* 2.1. Let $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}([0,b];\mathbb{R}^n)$ be a linear continuous operator, the matrix function $A$ satisfy condition (2.21) and let $I_0 \subset [0,b]$ be a closed set. Then, as follows from Lemmas $2.2_1$ and $2.3_2$, for conditions (2.49) and (2.50) to be fulfilled for some $k$ and $m$, it is not only sufficient but also necessary that

$$I_0 \subset I_{M_0}.$$

*Let*

$$h(u)(y) \equiv \big(u_i(a_i(y))\big)_{i=1}^n, \tag{2.54}$$

*where $a_i : [0,b] \to [0,a]$ $(i = 1,\dots,n)$ are measurable functions. Moreover, let there exist a nonempty set $I_0 \subset [0,b]$ such that the matrix function $A$ is bounded in $[0,a]\times I_0$ and*

$$r\big(A^0(y)\big) < \frac{\pi}{2a} \quad \textit{for} \quad y \in I_0, \tag{2.55}$$

*where*

$$A^0(y) = \operatorname*{ess\,sup}_{0\le x\le a} |A(x,y)|.$$

*Then $I_0 \subset I_{M_0}$ and*

$$\|M_0^{-1}(y)\| \le \rho_0\big\|\big(E - \frac{2a}{\pi}A^0(y)\big)^{-1}\big\| \quad \textit{for} \quad y \in I_0, \tag{2.56}$$

*where $\rho_0$ is a positive constant.*

*Proof.* Let $z(\cdot,y) = (z_i(\cdot,y))_{i=1}^n$ be an arbitrary solution of $(2.1_0)$. Then

$$z_i(x,y) = z_i(a_i(y),y) + \sum_{j=1}^n \int_{a_i(y)}^x a_{ij}(s,y)z_j(s,y)\,ds \quad (i = 1,\dots,n)$$

and

$$|z_i(x,y)| \le |z_i(a_i(y),y)| + \sum_{j=1}^n a^0_{ij}(y)\Big|\int_{a_i(y)}^x |z_j(s,y)|\,ds\Big| \quad (i = 1,\dots,n), \tag{2.57}$$

where

$$a_{ij}^0(y) = \operatorname*{ess\,sup}_{0\le x\le a} |a_{ij}(x,y)|.$$

By Minkovsky's inequality it follows that

$$\Big(\int_0^a z_i^2(x,y)\,dx\Big)^{1/2} \le a^{1/2}|z_i(a_i(y),y)| +$$

$$+\sum_{j=1}^n a_{ij}^0(y)\Big(\int_0^a \Big|\int_{a_i(y)}^x |z_j(s,y)|\,ds\Big|^2 dx\Big)^{1/2} \quad (i=1,\dots,n). \quad (2.58)$$

However, by Virtinger's inequality, [8]

$$\int_0^a \Big|\int_{a_i(y)}^x |z_j(s,y)|\,ds\Big|^2 dx \le \Big(\frac{2a}{\pi}\Big)^2 \int_0^a z_j^2(x,y)\,dx \quad (i,j=1,\dots,n).$$

Taking into account the above inequalities and condition (2.54), we get from (2.58)

$$\bar z(y) \le a^{1/2}|h(z(\cdot,y))(y)| + \frac{2a}{\pi}A^0(y)\bar z(y),$$

where

$$\bar z(y) = \Big(\Big(\int_0^a z_i^2(x,y)\,dx\Big)^{1/2}\Big)_{i=1}^n.$$

From this inequality, in view of condition (2.55), one arrives at the following estimate

$$\bar z(y) \le a^{1/2}\big(E - \frac{2a}{\pi}A^0(y)\big)^{-1}|h(z(\cdot,y))(y)| \quad \text{for } y\in I_0,$$

by which means from (2.57) we get

$$|z(x,y)| \le |h(z(\cdot,y))(y)| + a^{1/2}A^0(y)\bar z(y) \le$$

$$\le |h(z(\cdot,y))(y)| + aA^0(y)\big(E - \frac{2a}{\pi}A^0(y)\big)^{-1}|h(z(\cdot,y))(y)| =$$

$$= \Big(E + \frac{(\pi-2)a}{\pi}A^0(y)\Big)\big(E - \frac{2a}{\pi}A^0(y)\big)^{-1}|h(z(\cdot,y))(y)|.$$

The fact that $A^0$ is bounded in the set $I_0$ results in

$$\|z(x,y)\| \le \frac{1}{n}\rho_0\big\|\big(E - \frac{2a}{\pi}A^0(y)\big)^{-1}\big\|\,\|h(z(\cdot,y))(y)\| \quad \text{for } y\in I_0, \quad (2.59)$$

where

$$\rho_0 = n\sup_{y\in I_0}\Big\|E + \frac{(\pi-2)a}{\pi}A^0(y)\Big\|$$

is a constant independent of $z(\cdot,y)$.

By virtue of (2.59) it follows from Lemmas 2.4 and 2.6 that $I_0\subset I_{M_0}$ and estimate (2.56) is valid. ■

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[8] See [18], Ch.VI, Theorem 256.

If problem $(2.1_0)$,$(2.2_0)$ is periodic, i.e.

$$h\big(z(\cdot,y)\big)(y) = z(a,y) - z(0,y), \tag{2.60}$$

then $H_0(y) \equiv 0$, $H(x,y) \equiv E$ and, consequently,

$$M_0(y) \equiv Z_0(a,y) - E.$$

The lemmas below have something to do with the considered case.

*Let condition (2.60) hold and there exist a nonempty set $I_0 \subset [0,b]$, a diagonal matrix function*

$$A_0(x,y) = \operatorname{diag}[a_{01}(x,y),\dots,a_{0n}(x,y)] \tag{2.61}$$

*and a non-negative matrix function $B : I_0 \to \mathbb{R}^{n\times n}$ such that for every $y \in I_0$ $a_{0i}(\cdot,y) : [0,a] \to \mathbb{R}$ $(i = 1,\dots,n)$ are summable and of constant sign,*

$$\sup_{y\in I_0} \int_0^a |a_{0i}(s,y)|\, ds < +\infty \quad (i = 1,\dots,n), \tag{2.62}$$

$$\int_0^a a_{0i}(s,y)\, ds \neq 0 \quad \textit{for } y \in I_0 \ \ (i = 1,\dots,n), \tag{2.63}$$

$$r\big(B(y)\big) < 1 \quad \textit{for } y \in I_0 \tag{2.64}$$

*and*

$$|A(x,y) - A_0(x,y)| \le |A_0(x,y)|B(y) \quad \textit{for} \ \ 0 \le x \le a, \ \ y \in I_0. \tag{2.65}$$

*Then $I_0 \subset I_{M_0}$ and*

$$\|M_0^{-1}(y)\| \le \rho_0 \Big\| \big(E - B(y)\big)^{-1} \Big(\int_0^a A_0(s,y)\, ds\Big)^{-1} \Big\| \quad \textit{for } y \in I_0, \tag{2.66}$$

*where $\rho_0$ is a positive constant.*

*Proof.* First of all let us note that in view of conditions (2.62),

$$\rho_1 = \sup_{y\in I_0} \Big\| \exp\Big(\int_0^a |A_0(s,y)|\, ds\Big) \Big\| < +\infty,$$

$$\rho_0 = \sup_{0<|u|\le\rho_1} \frac{n|u|}{|\exp(u) - 1|} < +\infty.$$

Let $y \in I_0$ and $z(\cdot,y)$ be an arbitrary solution of $(2.1_0)$. Then

$$\frac{dz(x,y)}{dx} = A_0(x,y)z(x,y) + [A(x,y) - A_0(x,y)]z(x,y).$$

Therefore, according to conditions (2.61),(2.63) and Lemma 2.4, we have

$$z(x,y) = \exp\Big(\int_0^x A_0(\xi,y)\, d\xi\Big)\Big[\exp\Big(\int_0^a A_0(s,y)\, ds\Big) - E\Big]^{-1} h\big(z(\cdot,y)\big)(y) +$$

$$+\Big[E-\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big]^{-1}\int_0^x \exp\Big(\int_s^x A_0(\xi,y)\,d\xi\Big)[A(s,y)-$$
$$-A_0(s,y)]z(s,y)\,ds+\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big[E-\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big]^{-1}\times$$
$$\times\int_x^a \exp\Big(\int_s^x A_0(\xi,y)\,d\xi\Big)[A(s,y)-A_0(s,y)]z(s,y)\,ds. \qquad (2.67)$$

If we assume

$$\bar z(y)=\max_{0\le x\le a}|z(x,y)|,$$

then, in view of conditions (2.61), (2.63) and (2.65) and taking into account the fact that functions $a_i(\cdot,y)$ $(i=1,\dots,n)$ are of constant sign, we obtain from (2.67) that

$$|z(x,y)|\le \frac{\rho_0}{n}\Big|\int_0^a A_0(s,y)\,ds\Big|^{-1}\big|h\big(z(\cdot,y)\big)(y)\big|+$$
$$+\Big|E-\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big|^{-1}\Big[\Big|\int_0^x \exp\Big(\int_s^x A_0(\xi,y)\,d\xi\Big)A_0(s,y)\,ds\Big|+$$
$$+\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big|\int_x^a \exp\Big(\int_s^x A_0(\xi,y)\,d\xi\Big)A_0(s,y)\,ds\Big|\Big]B(y)\bar z(y)=$$
$$=\frac{\rho_0}{n}\Big|\int_0^a A_0(s,y)\,ds\Big|^{-1}\big|h\big(z(\cdot,y)\big)(y)\big|+$$
$$+\Big|E-\exp\Big(\int_0^a A_0(s,y)\,ds\Big)\Big|^{-1}\Big[\Big|E-\exp\Big(\int_0^x A_0(\xi,y)\,d\xi\Big)\Big|+$$
$$+\Big|\exp\Big(\int_0^x A_0(\xi,y)\,d\xi\Big)-\exp\Big(\int_0^a A_0(\xi,y)\,d\xi\Big)\Big|\Big]B(y)\bar z(y)=$$
$$=\frac{\rho_0}{n}\Big|\int_0^a A_0(s,y)\,ds\Big|^{-1}\big|h\big(z(\cdot,y)\big)(y)\big|+B(y)\bar z(y)$$

and

$$\bar z(y)\le B(y)\bar z(y)+\frac{\rho_0}{n}\Big|\int_0^a A_0(s,y)\,ds\Big|^{-1}\big|h\big(z(\cdot,y)\big)(y)\big|.$$

By virtue of inequality (2.64), we have

$$|\bar z(y)|\le \frac{1}{n}\rho_0\Big|\big(E-B(y)\big)^{-1}\Big(\int_0^a A_0(s,y)\,ds\Big)^{-1}\Big|\,\big|h\big(z(\cdot,y)\big)(y)\big|.$$

Consequently,

$$\|z(x,y)\|\le\frac{\rho_0}{n}\Big\|\big(E-B(y)\big)^{-1}\Big(\int_0^a A_0(s,y)\,ds\Big)^{-1}\Big\|\,\big\|h\big(z(\cdot,y)\big)(y)\big\|$$
$$\text{for}\quad 0\le x\le a,\ \ y\in I_0.$$

According to this estimate, it follows from Lemmas 2.4 and 2.6 that $I_0\subset I_{M_0}$ and estimate (2.66) is valid. ■

*Let condition (2.60) hold and there exist a nonempty set $I_0 \subset [0,b]$ and functions $\sigma_i : I_0 \to \{-1,1\}$ $(i=1,\dots,n)$ such that*

$$\begin{gathered}\bar a_{ii}(y) = \operatorname*{ess\,sup}_{0\le x\le a} \big(\sigma_i(y)a_{ii}(x,y)\big) < +\infty,\\ \bar a_{ij}(y)=\operatorname*{ess\,sup}_{0\le x\le a} |a_{ij}(x,y)|<+\infty \ \ (i\neq j,\ \ i,j=1,\dots,n) \ \ \textit{for}\ \ y\in I_0\end{gathered} \tag{2.68}$$

*and for every $y\in I_0$ the real parts of eigenvalues of the matrix*

$$\bar A(y) = \big(\bar a_{ij}(y)\big)_{i,j=1}^{n}$$

*are negative. Then $I_0 \subset I_{M_0}$ and*

$$\|M_0^{-1}(y)\| \le \rho\|\bar A^{-1}(y)\| \quad \textit{for}\ \ y\in I_0, \tag{2.69}$$

*where $\rho>0$ is a constant.*

*Proof.* From the asymptotic stability of the matrix $\bar A(y)$ (the real parts of eigenvalues of $\bar A(y)$ are negative for $y\in I_0$) we have

$$\bar a_{ii}(y)<0 \quad \text{for} \quad y\in I_0 \ \ (i=1,\dots,n) \tag{2.70}$$

and the matrix

$$B(y) = \Big((1-\delta_{ij})\frac{\bar a_{ij}(y)}{|\bar a_{ii}(y)|}\Big)_{i,j=1}^{n},$$

where $\delta_{ij}$, Kronecker's symbol, satisfies condition (2.64).

Assume

$$A_0(x,y) = \operatorname{diag}[a_{11}(x,y),\dots,a_{nn}(x,y)], \quad \bar A_0(y) = \operatorname{diag}[\bar a_{11}(y),\dots,\bar a_{nn}(y)].$$

Then by (2.68) and (2.70),

$$|\bar A_0(y)| \le |A_0(x,y)| \quad \text{for} \quad 0\le x\le a, \ \ y\in I_0$$

and

$$\begin{aligned}|A(x,y)-A_0(x,y)| &\le \big((1-\delta_{ij})\bar a_{ij}(y)\big)_{i,j=1}^{n} = |\bar A_0(y)|B(y) \le\\ &\le |A_0(x,y)|B(y) \quad \text{for} \ \ 0\le x\le a, \ \ y\in I_0.\end{aligned}$$

Consequently, all conditions of Lemma 2.9 are fulfilled. Therefore $I_0\subset I_{M_0}$ and estimate (2.66), where $\rho_0>0$ is a constant, takes place. On the other hand,

$$\big(E-B(y)\big)^{-1} = \bar A^{-1}(y)\bar A_0(y)$$

and

$$\Big\|\bar A_0(y)\Big(\int_0^a A_0(s,y)\,ds\Big)^{-1}\Big\| = \sum_{i=1}^{n} |\bar a_{ii}(y)|\,\Big|\int_0^a a_{ii}(s,y)\,ds\Big|^{-1} \le \frac{n}{a} \ \ \text{for}\ y\in I.$$

According to these conditions, from (2.66) there follows estimate (2.69), where $\rho=\frac{n}{a}\rho_0$. ■

§ 3.

In this section for the linear hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y) \tag{3.1}$$

we shall investigate two modifications of the characteristic initial value problem

$$\begin{gathered} u(x,0) = \varphi_0(x), \qquad \frac{\partial u(0,y)}{\partial y} = \psi(y) + \\ + \int_0^a \Big( Q_0(s,y)u(s,y) + \gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s} \Big) ds \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} \lim_{y\to 0} \Big( \frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y) \Big) = \varphi(x), \\ u(0,y) = \psi(y) + \int_0^y \int_0^a Q(y,s,t)u(s,t)dsdt, \end{gathered} \tag{3.3}$$

where

$$\begin{gathered} \mathcal{P}_i \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^n), \\ \varphi_0 \in \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n), \quad \gamma_1(x) = 1 + \|\varphi_0'(x)\|, \end{gathered} \tag{3.4}$$

and the vector and matrix functions $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ and $Q_i : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ $(i = 0,1)$, $Q : [0,b] \times \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ are summable. Moreover, there exists a summable function $\eta : [0,b] \to [0,+\infty)$ such that

$$\|Q(y,s,t)\| \le \eta(y) \quad \text{for} \quad y \in [0,b], \ \ (s,t) \in \mathcal{D}_{ab}. \tag{3.5}$$

For an arbitrary $x \in [0,a]$ $(y \in [0,b])$ by $Z_1(x,\cdot)$ $\big(Z_2(\cdot,y)\big)$ we mean the fundamental matrix of the system of ordinary differential equations

$$\frac{dz}{dy} = \mathcal{P}_1(x,y)z \quad \Big(\frac{dz}{dx} = \mathcal{P}_2(x,y)z\Big),$$

which satisfies the initial condition

$$Z_1(x,0) = E \quad (Z_2(0,y) = E).$$

Before we proceed to formulating the conditions of the unique solvability of problems (3.1),(3.2) and (3.1),(3.3), we have to establish some properties of solutions of system (3.1).

*Every solution $u$ of system* (3.1) *satisfies the conditions*

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \Big[\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \Big/ \Big(1+\Big\|\frac{\partial u(x,0)}{\partial x}\Big\|\Big)\Big] < +\infty \tag{3.6}$$

*and*

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \Big[\Big\|\frac{\partial u(x,y)}{\partial y}\Big\| \Big/ \Big(1+\Big\|\frac{\partial u(0,y)}{\partial y}\Big\|\Big)\Big] < +\infty. \tag{3.7}$$

*Proof.* According to conditions (3.4) and due to the absolute continuity of $u$, there exists a positive number $\gamma$ such that

$$\|\mathcal{P}_1(x,y)\| \le \gamma$$

and

$$\int_0^y \Big\|\mathcal{P}_0(x,t)u(x,t) + \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t} + q(x,t)\Big\| dt \le \gamma$$

almost for every $(x,y)\in\mathcal{D}_{ab}$. Therefore from the equality

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial u(x,0)}{\partial x} + \int_0^y \big[\mathcal{P}_0(x,t)u(x,t) + \\ + \mathcal{P}_1(x,t)\frac{\partial u(x,t)}{\partial x} + \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t} + q(x,t)\big]\, dt$$

we get

$$\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \le (1+\gamma)\Big(1+\Big\|\frac{\partial u(x,0)}{\partial x}\Big\|\Big) + \gamma\int_0^y \Big\|\frac{\partial u(x,t)}{\partial x}\Big\| dt.$$

It follows from the above inequality and Gronwall's lemma that

$$\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \le (1+\gamma)\Big(1+\Big\|\frac{\partial u(x,0)}{\partial x}\Big\|\Big)\exp(\gamma b).$$

Hence, inequality (3.6) holds. The validity of inequality (3.7) can be proved similarly. ■

*If $\mathcal{P}_2(x,\cdot):[0,b]\to\mathbb{R}^{n\times n}$ is absolutely continuous almost for every $x\in[0,a]$, then every solution of* (3.1) *is a solution of the system*

$$\frac{\partial}{\partial y}\Big[Z_1^{-1}(x,y)Z_2(x,y)\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u\big)\Big] = \\ = Z_1^{-1}(x,y)\big(\mathcal{P}(x,y)u(x,y)+q(x,y)\big), \tag{3.8}$$

*where*

$$\mathcal{P}(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_2(x,y)}{\partial y}, \tag{3.9}$$

*and, vice versa, every solution of* (3.8) *is a solution of* (3.1).

*Proof.* Let $u : \mathcal{D}_{ab} \to \mathbb{R}^n$ be an arbitrary absolutely continuous vector function. Then, by virtue of the equalities

$$\frac{\partial Z_1^{-1}(x,y)}{\partial y} = -Z_1^{-1}(x,y)\mathcal{P}_1(x,y)$$

and

$$\frac{\partial Z_2^{-1}(x,y)}{\partial x} = -Z_2^{-1}(x,y)\mathcal{P}_2(x,y)$$

we have

$$\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big) = -Z_2^{-1}(x,y)\mathcal{P}_2(x,y)u(x,y) + Z_2^{-1}(x,y)\frac{\partial u(x,y)}{\partial x}$$

and

$$\frac{\partial}{\partial y}\big[Z_1^{-1}(x,y)Z_2(x,y)\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big)\big] =$$
$$= \frac{\partial}{\partial y}\big[Z_1^{-1}(x,y)\big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\big)\big] =$$
$$= Z_1^{-1}(x,y)\big[\frac{\partial^2 u(x,y)}{\partial x \partial y} + \big(\mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_2(x,y)}{\partial y}\big)u(x,y) -$$
$$-\mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y}\big].$$

The latter equality shows that $u$ is a solution of system (3.1) if and only if it is a solution of system (3.8). ■

*If*

$$\mathcal{P}_2 \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}), \tag{3.10}$$

*then an arbitrary generalized solution $u$ of system* (3.1) *admits the representation*

$$u(x,y) = Z_2(x,y)\Big[v_0(y) + \int_0^x Z_2^{-1}(s,y)Z_1(s,y)v_1(s)ds\Big] +$$
$$+Z_2(x,y)\int_0^x\int_0^y Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t) + q(s,t)\big)ds\,dt, \tag{3.11}$$

*where*

$$v_0(y) = u(0,y), \quad v_1(x) = \lim_{y\to 0}\big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\big) \tag{3.12}$$

*and, vice versa, whatever summable vector functions $v_0 : [0,b] \to \mathbb{R}^n$ and $v_1 : [0,a] \to \mathbb{R}^n$ might be, the summable vector function $u : \mathcal{D}_{ab} \to \mathbb{R}^n$,* *admitting representation* (3.11), *is a generalized solution of system* (3.1) *satisfying conditions* (3.12).

*Proof.* Let $u$ be an arbitrary generalized solution of (3.1). Then, in view of conditions (3.4),(3.9) and (3.10), the vector function

$$Z_1^{-1}(\cdot,\cdot)(\mathcal{P}(\cdot,\cdot)u(\cdot,\cdot)+q(\cdot,\cdot)):\mathcal{D}_{ab}\to\mathbb{R}^n$$

as well as the vector functions $v_0:[0,b]\to\mathbb{R}^n$ and $v_1:[0,a]\to\mathbb{R}^n$ defined by (3.12), is summable. As for the vector function

$$Z_1^{-1}(x,\cdot)Z_2(x,\cdot)\frac{\partial}{\partial x}\big(Z_2^{-1}(x,\cdot)u(x,\cdot)\big):[0,b]\to\mathbb{R}^n,$$

it is absolutely continuous almost for all $x\in[0,a]$ according to Lemma $2.2_2$. Moreover,

$$Z_2(x,y)\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big)=\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y). \quad (3.13)$$

Therefore, the integration of equality (3.8) from 0 to $y$ yields

$$Z_1^{-1}(x,y)Z_2(x,y)\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big)=$$
$$=v_1(x)+\int_0^y Z_1^{-1}(x,t)\big(\mathcal{P}(x,t)u(x,t)+q(x,t)\big)dt$$

and, consequently,

$$\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big)=Z_2^{-1}(x,y)Z_1(x,y)v_1(x)+$$
$$+\int_0^y Z_2^{-1}(x,y)Z_1(x,y)Z_1^{-1}(x,t)\big(\mathcal{P}(x,t)u(x,t)+q(x,t)\big)dt.$$

Then, because of the fact that the vector function

$$Z_2^{-1}(\cdot,y)u(\cdot,y):[0,a]\to\mathbb{R}^n$$

is absolutely continuous almost for every $y\in[0,b]$, we get equality (3.11).

Assume now that $v_0:[0,b]\to\mathbb{R}^n$, $v_1:[0,a]\to\mathbb{R}^n$ and $u:\mathcal{D}_{ab}\to\mathbb{R}^n$ are arbitrary summable vector functions. Moreover, $u$ admits representation (3.11), i.e.

$$u(x,y)=Z_2(x,y)[v_0(y)+v(x,y)],$$

where

$$v(x,y)=\int_0^x Z_2^{-1}(s,y)Z_1(s,y)v_1(s)ds+$$
$$+\int_0^x\int_0^y Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+q(s,t)\big)ds\,dt.$$

Then by Lemmas $2.2_1$ and $2.2_2$ and conditions (3.4),(3.9) and (3.10), the vector function $v:\mathcal{D}_{ab}\to\mathbb{R}^n$ is absolutely continuous. It is also evident that the vector function $u$ satisfies system (3.8) and conditions (3.12) almost everywhere in $\mathcal{D}_{ab}$. Consequently, $u$ is a generalized solution of system (3.1). ■

*Problem* (3.1),(3.2) *has one and only one solution* $u$, *and*

$$u_k(x,y) \rightrightarrows u(x,y), \quad \gamma_1^{-1}(x)\frac{\partial u_k(x,y)}{\partial x} \rightrightarrows \gamma_1^{-1}(x)\,\frac{\partial u(x,y)}{\partial x},$$
$$\gamma_2^{-1}(y)\frac{\partial u_k(x,y)}{\partial y} \rightrightarrows \gamma_2^{-1}(y)\frac{\partial u(x,y)}{\partial y} \;\; \text{for} \;\; k \to +\infty, \tag{3.14}$$

*where*

$$\gamma_2(y) = 1 + \|\psi(y)\| + \int_0^a [\|Q_0(s,y)\| + \|Q_1(s,y)\|]ds, \tag{3.15}$$

$u_0(x,y) \equiv 0$ *and*

$$u_k(x,y) = \varphi_0(x) + \int_0^y \psi(t)dt +$$
$$+\int_0^y\int_0^a \Big[Q_0(s,t)u_{k-1}(s,t) + \gamma_1^{-1}(s)Q_1(s,t)\frac{\partial u_{k-1}(s,t)}{\partial s}\Big]ds\,dt +$$
$$+\int_0^y\int_0^x \Big[\mathcal{P}_0(s,t)u_{k-1}(s,t) + \mathcal{P}_1(s,t)\frac{\partial u_{k-1}(s,t)}{\partial s} +$$
$$+\mathcal{P}_2(s,t)\frac{\partial u_{k-1}(s,t)}{\partial t} + q(s,t)\Big]ds\,dt. \tag{3.16}$$

*Proof.* Let $n_1 = 2n$, $n_2 = n$ and $\gamma_0 > 0$ be large such that

$$r(A) < 1, \quad \text{where } A \equiv \gamma_0^{-1} \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} |\mathcal{P}_2(x,y)|. \tag{3.17}$$

For arbitrary vector functions $z_1 = (z_1^i)_{i=0}^1 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})$, where $z_1^i \in$ $\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ $(i=0,1)$ and $z_2 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$, assume that

$$g_1^0(z_1,z_2)(x,y) = \varphi_0(x) + \int_0^y \psi(t)dt +$$
$$+\int_0^y\int_0^a [Q_0(s,t)z_1^0(s,t) + Q_1(s,t)z_1^1(s,t)]ds\,dt +$$
$$+\int_0^y\int_0^x [\mathcal{P}_0(s,t)z_1^0(s,t) + \gamma_1(s)\mathcal{P}_1(s,t)z_1^1(s,t) +$$
$$+\gamma_2(t)\exp(\gamma_0 s)\mathcal{P}_2(s,t)z_2(s,t) + q(s,t)]ds\,dt; \tag{3.18}$$

$$g_1^1(z_1,z_2)(x,y) = \gamma_1^{-1}(x)\frac{\partial}{\partial x}g_1^0(z_1,z_2)(x,y),$$
$$g_1(z_1,z_2)(x,y) = (g_1^i(z_1,z_2)(x,y))_{i=0}^1, \tag{3.19}$$

and

$$g_2(z_1,z_2)(x,y)=\gamma_2^{-1}(y)\exp(-\gamma_0 x)\frac{\partial}{\partial y}g_1^0(z_1,z_2)(x,y). \tag{3.20}$$

Let us show that problem (3.1),(3.2) is equivalent to the system of functional equations (1.1).

Indeed, let problem (3.1),(3.2) have a solution $u$. Introduce the notation

$$\begin{gathered} z_1^0(x,y)=u(x,y),\quad z_1^1(x,y)=\gamma_1^{-1}(x)\frac{\partial u(x,y)}{\partial x},\\ z_1(x,y)=(z_1^i(x,y))_{i=1}^1,\quad z_2(x,y)=\gamma_2^{-1}(y)\exp(-\gamma_0 x)\frac{\partial u(x,y)}{\partial y}. \end{gathered} \tag{3.21}$$

According to Lemma 3.1,

$$(z_i)_{i=1}^2\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})\times L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2}).$$

On the other hand, it becomes evident from equalities (3.18)-(3.21) that $(z_i)_{i=1}^2$ is a solution of system (1.1).

The converse is obvious: if $z_1^i\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ $(i=1,2)$, $z_1=(z_1^i)_{i=0}^1$, $z_2\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$ and $(z_i)_{i=1}^2$is a solution of system (1.1), then $u(\cdot,\cdot)=z_1^0(\cdot,\cdot)$ is a solution of problem (3.1),(3.2), and equalities (3.21) are valid.

To complete the proof of this lemma, it remains to show that system (1.1) in the space $L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})\times L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$ has the unique solution $(z_i)_{i=1}^2$ and conditions (1.8) hold, where $z_{i0}(x,y)\equiv 0$ $(i=1,2)$ and $(z_{ik})_{k=1}^{+\infty}$ $(i=1,2)$ are the sequences given by (1.9).

According to conditions (3.4) and the restrictions imposed on the vector and matrix functions $\varphi_0$, $\psi$ and $Q_i$ $(i=1,2)$, it follows from (3.18)-(3.20) that operators $g_i: L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})\times L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})\to L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_i})$ $(i=1,2)$ satisfy, for arbitrary $\zeta_i$ and $\bar\zeta_i\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_i})$ $(i=1,2)$, inequalities (1.3) and (1.21) almost everywhere in $\mathcal{D}_{ab}$, where $I=[0,a]$,

$$\begin{gathered} g_0(t)=\gamma_2(t)\Big[1+\Big(1+\int_0^a\gamma_1(s)ds\Big)\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}}\big(\|\mathcal{P}_0(x,y)\|+\\ +\|\mathcal{P}_1(x,y)\|+\exp(\gamma_0 x)\|\mathcal{P}_2(x,y)\|\big)\Big],\\ A_{01}=\operatorname*{ess\,sup}_{0\le y\le b}\Big[\gamma_2^{-1}(y)\int_0^a(|Q_0(s,y)|+|Q_1(s,y)|)ds\Big]+\\ +\operatorname*{ess\,sup}_{0\le y\le b}\int_0^a(|\mathcal{P}_0(s,y)|+\gamma_1(s)|\mathcal{P}_1(s,y)|)ds,\quad A_{02}(y)\equiv\Theta \end{gathered}$$

and

$$A(y)\equiv A.$$

The validity of Lemma 3.4 becomes evident if we take condition (3.17) into account and apply Lemma 1.3. ■

′ *If* $\mathcal{P}_i : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ $(i = 0,1,2)$, $q : \mathcal{D}_{ab} \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ *are continuous,* $\varphi_0 : [0,a] \to \mathbb{R}^n$ *is continuously differentiable and* $Q_i : \mathcal{D}_{ab} \to \mathbb{R}^n$ $(i = 0,1)$ *satisfy the Carathéodory conditions,* [9] *then problem* (3.1),(3.2) *is uniquely solvable, its solution* $u$ *is classical and*

$$\begin{gathered} u_k(x,y) \rightrightarrows u(x,y), \quad \frac{\partial u_k(x,y)}{\partial x} \rightrightarrows \frac{\partial u(x,y)}{\partial x}, \\ \frac{\partial u_k(x,y)}{\partial y} \rightrightarrows \frac{\partial u(x,y)}{\partial y} \quad \text{for } k \to \infty, \end{gathered} \tag{3.22}$$

*where* $u_0(x,y) = 0$ *and for an arbitrary natural* $k$ *the vector function* $u_k$ *is given by equality* (3.16).

*Proof.* By Lemma 3.4, problem (3.1),(3.2) has the unique solution $u$ and conditions (3.14) hold. On the other hand, according to the restrictions imposed on $\mathcal{P}_i$ $(i = 0,1,2)$, $\psi$, $\varphi_0$ and $Q_i$ $(i = 0,1)$, it becomes clear from (3.16) that

$$\begin{gathered} u_k, \ \frac{\partial u_k}{\partial x} \ \text{ and } \ \frac{\partial u_k}{\partial y} \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^n)\ (k = 1,2,\dots), \\ \gamma_1 \in \mathbb{C}([0,a];(1,+\infty)), \quad \gamma_2 \in \mathbb{C}([0,b];(1,+\infty)). \end{gathered}$$

Therefore, by virtue of the well-known Weierstrass theorem, it follows from (3.14) that $u$ is a classical solution and conditions (3.22) take place. ■

*If condition* (3.10) *holds, then problem* (3.1),(3.3) *has one and only one generalized solution* $u$ *and*

$$\begin{gathered} \gamma^{-1}(y)u_k(x,y) \rightrightarrows \gamma^{-1}(y)u(x,y), \quad \frac{\partial u_k(x,y)}{\partial x} - \mathcal{P}_2(x,y)u_k(x,y) \rightrightarrows \\ \rightrightarrows \frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y) \quad \text{for } k \to +\infty, \end{gathered} \tag{3.23}$$

*where* $\gamma(y) = 1 + \|\psi(y)\| + \eta(y)$, $u_0(x,y) = 0$ *and*

$$\begin{gathered} u_k(x,y) = Z_2(x,y)\Big[\psi(y) + \int_0^y \int_0^a Q(y,s,t)u_{k-1}(s,t)ds\,dt\Big] + \\ + Z_2(x,y)\int_0^x Z_2^{-1}(s,y)Z_1(s,y)\varphi(s)ds + Z_2(x,y)\int_0^x \int_0^y Z_2^{-1}(s,y) \times \\ \times Z_1(s,y)Z_1^{-1}(s,t)[\mathcal{P}(s,t)u_{k-1}(s,t) + q(s,t)]dsdt. \end{gathered} \tag{3.24}$$

*Proof.* According to Lemma 3.3, an arbitrary generalized solution of problem (3.1), (3.3) is a solution of the system of integral equations

$$u(x,y) = Z_2(x,y)\Big[\psi(y) + \int_0^y \int_0^a Q(y,s,t)u(s,t)ds\,dt\Big] +$$

---

[9] I.e., $Q_i(\cdot,y) : [0,a] \to \mathbb{R}^{n\times n}$ are measurable for all $y \in [0,b]$, $Q_i(x,\cdot) : [0,b] \to \mathbb{R}^{n\times n}$ are continuous almost for all $x \in [0,a]$ and there exists a summable function $\gamma : [0,a] \to$ $\to [0,+\infty)$ such that $\|Q_i(x,y)\| \le \gamma(x)$ almost for all $(x,y) \in \mathcal{D}_{ab}$ $(i = 0,1)$.

$$+Z_2(x,y)\int_0^x Z_2^{-1}(s,y)Z_1(s,y)\varphi(s)ds + Z_2(x,y)\int_0^x\int_0^y Z_2^{-1}(s,y)\times$$
$$\times Z_1(s,y)Z_1^{-1}(s,t)[\mathcal{P}(s,t)u(s,t)+q(s,t)]ds\,dt \tag{3.25}$$

and, vice versa, an arbitrary summable solution of this system is a generalized solution of problem (3.1),(3.3 ). On the other hand, in view of conditions (3.4)-(3.6), it becomes clear that a summable function $u:\mathcal{D}_{ab}\to\mathbb{R}^n$ is a solution of system (3.25) if and only if the vector function

$$z(x,y)=\gamma^{-1}(y)u(x,y)$$

belongs to $L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ and is a solution of functional equation (1.2), where

$$g(z)(x,y)=\gamma^{-1}(y)Z_2(x,y)\Big[\psi(y)+\int_0^y\int_0^a Q(y,s,t)\gamma(t)z(s,t)ds\,dt\Big]+$$
$$+\gamma^{-1}(y)Z_2(x,y)\int_0^x Z_2^{-1}(s,y)Z_1(s,y)\varphi(s)ds+\gamma^{-1}(y)Z_2(x,y)\times$$
$$\times\int_0^x\int_0^y Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)[\mathcal{P}(s,t)\gamma(t)z(s,t)+q(s,t)]ds\,dt.$$

It follows from conditions (3.4)-(3.6) that the operator $g$ transforms the space $L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ into itself and for every $\zeta$ and $\bar\zeta\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ satisfies inequality (1.22), where $I=[0,a]$, $g_0(t)=c_0\eta(t)$ and $c_0$ is a positive constant. Therefore, by virtue of Lemma 1.4,system (1.2) has the unique solution and condition (1.23) holds, where $z_0(x,y)\equiv 0$ and $(z_k)_{k=1}^\infty$ is a sequence given by (1.24). It is clear from the above arguments that problem (3.1),(3.3) has the unique solution

$$u(x,y)=\gamma(y)z(x,y)$$

and conditions (3.23) take place. ■

*Remark* 3.1. From the proof of Lemma 3.5 it is clear that the following assertions are valid:

a) if $\psi\in L_\infty([0,b];\mathbb{R}^n)$ and $Q\in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n})$, then the generalized solution $u$ of problem (3.1),(3.3) belongs to $L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$, and condition (3.23) takes the form

$$\begin{gathered} u_k(x,y)\rightrightarrows u(x,y),\quad \frac{\partial u_k(x,y)}{\partial x}-\mathcal{P}_2(x,y)u_k(x,y)\rightrightarrows\\ \rightrightarrows\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\ \text{ for }\ k\to+\infty;\end{gathered}\tag{3.26}$$

b) if $\psi\in\mathbb{C}([0,b];\mathbb{R}^n)$, $Q\in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ and $Q(\cdot,s,t):[0,b]\to$ $\to\mathbb{R}^{n\times n}$ is continuous almost for every $(s,t)\in\mathcal{D}_{ab}$, then the generalized solution $u$ of problem (3.1),(3.3) is continuous, and condition (3.23) takes the form of (3.26).

*′ Let the matrix function $\mathcal{P}_2$ satisfy condition* (3.10),

$$\psi \in \widetilde{\mathbb{C}}([0,b];\mathbb{R}^n), \quad Q \in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

$Q(\cdot,s,t) : [0,b] \to \mathbb{R}^{n\times n}$ *be absolutely continuous almost for every* $(s,t) \in \in \mathcal{D}_{ab}$ *and the inequality*

$$\Big\|\frac{\partial Q(y,s,t)}{\partial y}\Big\| \leq \eta(y)\eta_0(s,t)$$

*hold in* $[0,b]\times\mathcal{D}_{ab}$, *where* $\eta : [0,b] \to \mathbb{R}_+$ *and* $\eta_0 : \mathcal{D}_{ab} \to \mathbb{R}_+$ *are summable functions. Then problem* (3.1),(3.3) *has the unique generalized solution* $u$ *which is absolutely continuous, and*

$$\begin{gathered} u_k(x,y) \rightrightarrows u(x,y), \quad \frac{\partial u_k(x,y)}{\partial x} \rightrightarrows \frac{\partial u(x,y)}{\partial x}, \\ \gamma^{-1}(y)\frac{\partial u_k(x,y)}{\partial y} \rightrightarrows \gamma^{-1}(y)\frac{\partial u(x,y)}{\partial y} \quad \text{for } k \to +\infty, \end{gathered} \tag{3.27}$$

*where* $\gamma(y) = 1 + \|\psi'(y)\| + \eta(y)$, $u_0(x,y) = 0$ *and the vector function* $u_k$ *is given by equality* (3.24) *for an arbitrary natural* $k$.

From Lemmas $2.2_2$ and $3.5'$ there immediately follows

*″ Let* $\mathcal{P}_i : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ $(i = 0,1,2)$, $q : \mathcal{D}_{ab} \to \mathbb{R}^n$ *and* $\varphi : [0,a] \to \mathbb{R}^n$ *be continuous and* $\mathcal{P}_2$ *have a continuous partial derivative in the second argument. Moreover, let*

$$\psi \in \mathbb{C}^1([0,b];\mathbb{R}^n), \quad Q \in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

$Q(\cdot,s,t) : [0,b] \to \mathbb{R}^{n\times n}$ *be absolutely continuous almost for all* $(s,t) \in \mathcal{D}_{ab}$, $Q(\cdot,s,\cdot) : [0,b] \to \mathbb{R}^{n\times n}$ *be continuous almost for all* $s \in [0,a]$, $\int\limits_0^x Q(\cdot,s,t)ds :$ $[0,b] \to \mathbb{R}^{n\times n}$ *be continuously differentiable for* $x \in [0,a]$ *and almost for all* $t \in [0,b]$, *and the inequality*

$$\Big\|\frac{\partial Q(y,s,t)}{\partial y}\Big\| \leq \eta_0(s,t)$$

*hold in* $[0,b]\times\mathcal{D}_{ab}$, *where* $\eta_0 : \mathcal{D}_{ab} \to [0,\infty)$ *is a summable function. Then problem* (3.1),(3.3) *is uniquely solvable, its solution* $u$ *is classical and condition* (3.22) *takes place, where* $u_0(x,y) = 0$ *and the vector function* $u_k$ *is given by equality* (3.24) *for an arbitrary natural* $k$.

*For any* $\alpha > 0$ *there exists* $\beta > 0$ *such that if*

$$\begin{gathered} \|\mathcal{P}_0\|_L \leq \alpha, \quad \|\mathcal{P}_i\|_{L_\infty} \leq \alpha \quad (i = 1,2), \\ \|Q_0\|_L \leq \alpha, \quad \|\gamma_1^{-1}Q_1\|_{L_\infty} \leq \alpha, \end{gathered} \tag{3.28}$$

*then the solution $u$ of problem* (3.1),(3.2) *admits the estimate*

$$\|u\|_{\widetilde{\mathbb{C}}} \le \beta(\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|_L). \tag{3.29}$$

*Proof.* In view of (3.1) and (3.2)

$$\frac{\partial u(x,y)}{\partial x} = \varphi_0'(x) + \int_0^y \big(\mathcal{P}_0(x,t)u(x,t) +$$
$$+\mathcal{P}_1(x,t)\frac{\partial u(x,t)}{\partial x} + \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t} + q(x,t)\big)dt, \tag{3.30}$$
$$\frac{\partial u(x,y)}{\partial y} = Z_2(x,y)\Big[\psi(y) + \int_0^a \big(Q_0(s,y)u(s,y) +$$
$$+\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big)ds\Big] + Z_2(x,y)\int_0^x Z_2^{-1}(s,y)\times$$
$$\times\big[\mathcal{P}_0(s,y)u(s,y) + \mathcal{P}_1(s,y)\frac{\partial u(s,y)}{\partial s} + q(s,y)\big]ds \tag{3.31}$$

and

$$u(x,y) = \varphi_0(x) + \int_0^y \frac{\partial u(x,t)}{\partial t}dt. \tag{3.32}$$

Moreover, as follows from (3.28),

$$\|Z_2(x,y)\| \le \alpha_1, \quad \|Z_2^{-1}(x,y)\| \le \alpha_1, \tag{3.33}$$

where $\alpha_1 = n\exp(a\alpha)$.

If we assume

$$\rho_0(y) = \max_{0\le x\le a}\|u(x,y)\|, \quad \rho(y) = \int_0^a \|\frac{\partial u(s,y)}{\partial s}\|ds,$$

then taking into account (3.28) and (3.33), from (3.31) and (3.32) we obtain

$$\|\frac{\partial u(x,y)}{\partial y}\| \le \alpha_1\|\psi(y)\| + \Big[\int_0^a (\alpha_1\|Q_0(s,y)\| +$$
$$+\alpha_1^2\|\mathcal{P}_0(s,y)\|)ds\Big]\rho_0(y) + \alpha(\alpha_1+\alpha_1^2)\rho(y) + \alpha_1^2\int_0^a \|q(s,y)\|ds \tag{3.34}$$

and

$$\rho_0(y) \le \|\varphi_0\|_{\widetilde{\mathbb{C}}} + \alpha_1\|\psi\|_L + \int_0^y \Big[\int_0^a \Big(\alpha_1\|Q_0(s,t)\| +$$
$$+\alpha_1^2\|\mathcal{P}_0(s,t)\|\Big)ds\Big]\rho_0(t)dt + \alpha(\alpha_1+\alpha_1^2)\int_0^y \rho(t)dt + \alpha_1^2\|q\|_L,$$

whence according to Gronwall's lemma and inequalities (3.28)

$$\rho_0(y) \le \Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \alpha_1\|\psi\|_L + \alpha_1^2\|q\|_L + \alpha(\alpha_1+\alpha_1^2)\int_0^y \rho(t)dt\Big]\times$$

$$\times\exp\left(\alpha_1\|Q_0\|_L+\alpha_1^2\|\mathcal{P}_0\|_L\right)\leq\alpha_2\Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L+\int_0^y\rho(t)dt\Big],\ (3.35)$$

where

$$\alpha_2=(1+\alpha)(\alpha_1+\alpha_1^2)\exp(\alpha(\alpha_1+\alpha_1^2)).$$

In view of (3.28),(3.34) and (3.35) from (3.30) we get

$$\rho(y)\leq\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\int_0^y\left(\int_0^a\|\mathcal{P}_0(s,t)\|ds\right)\rho_0(t)dt+$$
$$+\alpha\int_0^y\rho(t)dt+\alpha\int_0^y\int_0^a\|\frac{\partial u(s,t)}{\partial t}\|dsdt+\|q\|_L\leq$$
$$\leq\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\alpha\alpha_2\left[\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L+\int_0^y\rho(t)dt\right]+\alpha\int_0^y\rho(t)dt+$$
$$+a\alpha\alpha_1\|\psi\|_L+a\alpha^2(\alpha_1+\alpha_1^2)\Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L+\int_0^y\rho(t)dt\Big]+$$
$$+a\alpha^2(\alpha_1+\alpha_1^2)\int_0^y\rho(t)dt+a\alpha\alpha_1^2\|q\|_L+\|q\|_L\leq$$
$$\leq\alpha_3\Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L+\int_0^y\rho(t)dt\Big],\qquad(3.36)$$

where

$$\alpha_3=\alpha\alpha_2+\alpha\alpha_1\alpha_2(\alpha_1+\alpha_1^2)+\alpha_1(1+\alpha)(\alpha_1+\alpha_1^2).$$

Applying again Gronwall's lemma, from (3.36) we obtain

$$\rho(y)\leq\alpha_4(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L),\qquad(3.37)$$

where $\alpha_4=\alpha_3\exp(b\alpha_3)$.

By (3.34),(3.35) and (3.37) we have

$$\int_0^a\|\frac{\partial u(s,y)}{\partial s}\|ds\leq\alpha_4(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L),$$
$$\|u(x,y)\|\leq\alpha_5(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L),$$
$$\int_0^b\|\frac{\partial u(x,t)}{\partial t}\|dt\leq\alpha_6(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L),$$

where $\alpha_5=\alpha_2(1+b\alpha_4)$,

$$\alpha_6=\alpha_1^2+\alpha_4(\alpha_1+\alpha_1^2)(\alpha_4+\alpha_5).$$

With regard to the latter estimates and conditions (3.28) we get

$$\|u\|_{\widetilde{\mathbb{C}}}=\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\int_0^b\|\frac{\partial u(0,t)}{\partial t}\|dt+\int_0^a\int_0^b\|\frac{\partial^2u(s,t)}{\partial s\partial t}\|dsdt\leq$$
$$\leq\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\alpha_6(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L)+\int_0^a\int_0^b\Big[\|\mathcal{P}_0(s,t)\|\|u(s,t)\|+$$

$$+\|\mathcal{P}_1(s,t)\|\|\frac{\partial u(s,t)}{\partial s}\|+\|\mathcal{P}_2(s,t)\|\|\frac{\partial u(s,t)}{\partial t}\|\Big]dsdt+\|q\|_L\le$$
$$\le(1+\alpha_6+\alpha\alpha_5+b\alpha\alpha_4+a\alpha\alpha_6)(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L).$$

Consequently, estimate (3.29) is valid, where $\beta=1+\alpha_6+\alpha\alpha_5+b\alpha\alpha_4+a\alpha\alpha_6$ depends on $a$, $b$ and $\alpha$ only. ■

*For any $\alpha>0$ there exists $\beta>0$ such that if*

$$\|\mathcal{P}_0\|_L^{(2)}\le\alpha,\quad \|\mathcal{P}_i\|_{L_\infty}\le\alpha\quad (i=1,2),$$
$$\|Q_0\|_L^{(2)}\le\alpha,\quad \|\gamma_1^{-1}Q_1\|_{L_\infty}\le\alpha, \tag{3.38}$$

*then the solution $u$ of problem* (3.1),(3.2) *admits the estimate*

$$\|u\|_{\widetilde{\mathbb{C}}}^{(1)}\le\beta(\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+\|q\|_L^{(1)}), \tag{3.39}$$

*but for $\varphi_0\in\widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^n)$ and $\psi\in L_\infty([0,b];\mathbb{R}^n)$, it admits the estimate*

$$\|u\|_{\widetilde{\mathbb{C}}}^{(2)}\le\beta(\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty}+\|\psi\|_{L_\infty}+\|q\|_L^{(2)}). \tag{3.40}$$

*Proof.* Put

$$\rho_1(x,y)=\|\varphi_0(0)\|+\max_{(s,t)\in\mathcal{D}_{xy}}\bigg(\int_0^s\|\frac{\partial u(\xi,t)}{\partial\xi}\|d\xi+\int_0^t\|\frac{\partial u(s,\tau)}{\partial\tau}\|d\tau\bigg).$$

It is clear that $\rho_1:\mathcal{D}_{ab}\to\mathbb{R}_+$ is continuous and

$$\rho_1(a,b)=\|u\|_{\widetilde{\mathbb{C}}}^{(1)}.$$

From (3.1) and (3.2) we obtain

$$\|\varphi_0(0)\|+\int_0^x\|\frac{\partial u(s,y)}{\partial s}\|ds+\int_0^y\|\frac{\partial u(x,t)}{\partial t}\|dt\le\|\varphi_0\|_{\widetilde{\mathbb{C}}}+\|\psi\|_L+$$
$$+\int_0^y\bigg\|\int_0^a\Big[Q_0(s,t)u(s,t)+\gamma_1^{-1}(s)Q_1(s,t)\frac{\partial u(s,t)}{\partial s}\Big]ds\bigg\|dt+$$
$$+\int_0^x\bigg\|\int_0^y\mathcal{P}_0(s,t)u(s,t)dt\bigg\|ds+\int_0^y\bigg\|\int_0^x\mathcal{P}_0(s,t)u(s,t)ds\bigg\|dt+$$
$$+\int_0^x\bigg\|\int_0^y\mathcal{P}_1(s,t)\frac{\partial u(s,t)}{\partial s}dt\bigg\|ds+\int_0^y\bigg\|\int_0^x\mathcal{P}_1(s,t)\frac{\partial u(s,t)}{\partial s}ds\bigg\|dt+$$
$$+\int_0^x\bigg\|\int_0^y\mathcal{P}_2(s,t)\frac{\partial u(s,t)}{\partial t}dt\bigg\|ds+$$
$$+\int_0^y\bigg\|\int_0^x\mathcal{P}_2(s,t)\frac{\partial u(s,t)}{\partial t}ds\bigg\|dt+\|q\|_L^{(1)}. \tag{3.41}$$

But

$$\int_0^a Q_0(s,t)u(s,t)ds=\bigg(\int_0^a Q_0(s,t)ds\bigg)\Big[\varphi_0(a)+\int_0^t\frac{\partial u(a,\tau)}{\partial\tau}d\tau\Big]-$$

$$-\int_0^a \left(\int_0^s Q_0(\xi,t)d\xi\right)\frac{\partial u(s,t)}{\partial s}ds, \tag{3.42}$$

$$\int_0^x \mathcal{P}_0(s,t)u(s,t)ds = \left(\int_0^x \mathcal{P}_0(s,t)ds\right)\Big[\varphi_0(x)+\int_0^t \frac{\partial u(x,\tau)}{\partial \tau}d\tau\Big] -$$
$$-\int_0^x \left(\int_0^s \mathcal{P}_0(\xi,t)d\xi\right)\frac{\partial u(s,t)}{\partial s}ds, \tag{3.43}$$

$$\int_0^y \mathcal{P}_0(s,t)u(s,t)dt = \left(\int_0^y \mathcal{P}_0(s,t)dt\right)\Big[\varphi_0(s)+\int_0^y \frac{\partial u(s,t)}{\partial t}dt\Big] -$$
$$-\int_0^y \left(\int_0^t \mathcal{P}_0(s,\tau)d\tau\right)\frac{\partial u(s,t)}{\partial t}dt. \tag{3.44}$$

If along with these inequalities we take into account conditions (3.38), then from (3.41) we shall get

$$\|\varphi_0(0)\| + \int_0^x \|\frac{\partial u(s,y)}{\partial s}\|ds + \int_0^y \|\frac{\partial u(x,t)}{\partial t}\|dt \le \|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L +$$
$$+\|q\|_L^{(1)} + \alpha\int_0^y \big[\|\varphi_0(a)\| + \int_0^t \|\frac{\partial u(a,\tau)}{\partial \tau}\|d\tau + 2\int_0^a \|\frac{\partial u(s,t)}{\partial s}\|ds\big]dt +$$
$$+\alpha\int_0^x \big[\|\varphi_0(s)\| + 2\int_0^y \|\frac{\partial u(s,t)}{\partial s}\|dt\big]ds +$$
$$+\alpha\int_0^y \big[\|\varphi_0(x)\| + \int_0^t \|\frac{\partial u(x,\tau)}{\partial \tau}\|d\tau + \int_0^x \|\frac{\partial u(s,t)}{\partial s}\|ds\big]dt +$$
$$+2\alpha\int_0^y\int_0^x \|\frac{\partial u(s,t)}{\partial s}\|dsdt + 2\alpha\int_0^x\int_0^y \|\frac{\partial u(s,t)}{\partial t}\|dtds \le$$
$$\le (1+2b\alpha+a\alpha)\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|_L^{(1)} + 2\alpha\int_0^y \rho_1(a,t)dt +$$
$$+3\alpha\int_0^y \rho_1(x,t)dt + 4\alpha\int_0^x \rho_1(s,y)ds$$

and, consequently,

$$\rho_1(x,y) \le (1+2b\alpha+a\alpha)(\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|_L^{(1)}) +$$
$$+5\alpha\int_0^y \rho_1(a,t)dt + 4\alpha\int_0^x \rho_1(s,y)ds.$$

Applying twice Gronwall's lemma, we find from the latter inequality that

$$\rho_1(x,y) \le (1+2b\alpha+a\alpha)\exp(4a\alpha)(\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|_L^{(1)}) +$$
$$+5\alpha\exp(4a\alpha)\int_0^y \rho_1(a,t)dt,$$
$$\rho_1(a,y) \le \beta_0(\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|_L^{(1)})$$

and, consequently,

$$\|u\|^{(1)}_{\widetilde{\mathbb{C}}} \leq \beta_0(\|\varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi\|_L + \|q\|^{(1)}_L), \tag{3.45}$$

where

$$\beta_0 = (1 + 2b\alpha + a\alpha)\exp[4a\alpha + 5b\alpha\exp(4a\alpha)].$$

Now consider the case when

$$\varphi_0 \in \widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^n), \quad \psi \in L_\infty([0,b];\mathbb{R}^n).$$

In this case according to Lemma 3.1 and conditions (3.2),(3.38) and (3.42), the vector functions $\frac{\partial u(x,y)}{\partial x}$ and $\frac{\partial u(x,y)}{\partial y}$ are essentially bounded. Assume

$$\rho_2(y) = \max_{0\leq x\leq a} \|\frac{\partial u(x,y)}{\partial y}\|.$$

Taking into account conditions (3.38) and (3.42)-(3.44) we obtain from (3.1) and (3.2)

$$\begin{aligned}
\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| &\leq \|\varphi_0'(x)\| + \Big\|\int_0^y \mathcal{P}_0(x,t)u(x,t)dt\Big\| + \\
&+\Big\|\int_0^y \mathcal{P}_1(x,t)\frac{\partial u(x,t)}{\partial x}dt\Big\| + \Big\|\int_0^y \mathcal{P}_2(x,t)\frac{\partial u(x,t)}{\partial t}dt\Big\| + \|q\|^{(2)}_L \leq \\
&\leq \|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|q\|^{(2)}_L + \alpha\Big[\|\varphi_0(x)\| + 2\int_0^y \Big\|\frac{\partial u(x,t)}{\partial t}\Big\|dt\Big] + \\
&+\alpha\int_0^y \Big\|\frac{\partial u(x,t)}{\partial x}\Big\|dt + \alpha\int_0^y \Big\|\frac{\partial u(x,t)}{\partial t}\Big\|dt \leq (1+\alpha+a\alpha)\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \\
&+\|q\|^{(2)}_L + 3\alpha\int_0^y \rho_2(t)dt + \alpha\int_0^y \Big\|\frac{\partial u(x,t)}{\partial x}\Big\|dt \leq \\
&\leq (1+3\alpha+a\alpha)\Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|q\|^{(2)}_L + \int_0^y \rho_2(t)dt\Big] + \alpha\int_0^y \Big\|\frac{\partial u(x,t)}{\partial x}\Big\|dt
\end{aligned}$$

and

$$\begin{aligned}
\Big\|\frac{\partial u(x,y)}{\partial y}\Big\| &\leq \|\psi\|_{L_\infty} + \Big\|\int_0^a \big(Q_0(s,y)u(s,y) + \\
&+\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big)ds\Big\| + \Big\|\int_0^x \mathcal{P}_0(s,y)u(s,y)ds\Big\| + \\
&+\Big\|\int_0^x \mathcal{P}_1(s,y)\frac{\partial u(s,y)}{\partial s}ds\Big\| + \Big\|\int_0^x \mathcal{P}_2(s,y)\frac{\partial u(s,y)}{\partial y}ds\Big\| + \\
&+\|q\|^{(2)}_L \leq \|\psi\|_{L_\infty} + \|q\|^{(2)}_L + \alpha\Big[\|\varphi_0(a)\| + \int_0^y \|\frac{\partial u(a,t)}{\partial t}\|dt + \\
&+2\int_0^a \|\frac{\partial u(s,y)}{\partial s}\|ds\Big] + \alpha\Big[\|\varphi_0(x)\| + \int_0^y \|\frac{\partial u(x,t)}{\partial t}\|dt + \\
&+\int_0^x \|\frac{\partial u(s,y)}{\partial s}\|ds\Big] + \alpha\int_0^x \|\frac{\partial u(s,y)}{\partial s}\|ds + \alpha\int_0^x \|\frac{\partial u(s,y)}{\partial y}\|ds \leq
\end{aligned}$$

$$\leq (2\alpha + 2a\alpha)\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_{L^{(2)}} + 4\alpha \int_0^a \|\frac{\partial u(s,y)}{\partial s}\|ds +$$
$$+2\alpha \int_0^y \rho_2(t)dt + \alpha \int_0^x \|\frac{\partial u(s,y)}{\partial y}\|ds \leq (1 + 4\alpha + 2a\alpha)\Big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} +$$
$$+\|\psi\|_{L_\infty} + \|q\|_L^{(2)} + \int_0^y \rho_2(t)dt + \int_0^a \|\frac{\partial u(s,y)}{\partial s}\|ds\Big] + \alpha \int_0^x \|\frac{\partial u(s,y)}{\partial y}\|ds,$$

whence by Gronwall's lemma

$$\|\frac{\partial u(x,y)}{\partial x}\| \leq \beta_1 \big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_L^{(2)} + \int_0^y \rho_2(t)dt\big] \tag{3.46}$$

and

$$\|\frac{\partial u(x,y)}{\partial y}\| \leq \beta_2 \big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_L^{(2)} + $$
$$+ \int_0^y \rho_2(t)dt + \int_0^a \|\frac{\partial u(s,y)}{\partial s}\|ds\big], \tag{3.47}$$

where

$$\beta_1 = (1 + 3\alpha + a\alpha)\exp(b\alpha), \quad \beta_2 = (1 + 4\alpha + 2a\alpha)\exp(a\alpha).$$

By virtue of (3.46) and (3.47) we get

$$\rho_2(y) \leq (1 + \beta_1 a)\beta_2 \big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_L^{(2)} + \int_0^y \rho_2(t)dt\big].$$

If we apply again Gronwall's lemma, then from the latter inequality we obtain

$$\rho_2(y) \leq \beta_3 \big[\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_L^{(2)}\big], \tag{3.48}$$

where $\beta_3 = (1 + \beta_1 a)\beta_2 \exp(\beta_2 b + \beta_1 \beta_2 ab)$.

The estimate

$$\|u\|_{\widetilde{\mathbb{C}}}^{(2)} \leq \beta_4 \big(\|\varphi_0\|_{\widetilde{\mathbb{C}}_\infty} + \|\psi\|_{L_\infty} + \|q\|_L^{(2)}\big) \tag{3.49}$$

follows from (3.46) and (3.48), where

$$\beta_4 = \beta_1(1 + \beta_3 b) + \beta_3 + 1.$$

According to (3.45) and (3.49) it is clear that estimates (3.39) and (3.40) are valid, where $\beta = \beta_0 + \beta_4$ depends on $a$, $b$ and $\alpha$ only. ■

*For any $\alpha > 0$, $\mathcal{P}_1 \in \widetilde{\mathbb{C}}_\infty^{(0,-1)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$ and $\mathcal{P}_2 \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$ there exists $\beta > 0$ such that if $\psi \in L_\infty([0,b]; \mathbb{R}^n)$, $Q \in L_\infty([0,b]\times \times\mathcal{D}_{ab}; \mathbb{R}^n)$,*

$$\|\mathcal{P}_0\|_L \leq \alpha, \quad \|Q\|_{L_\infty} \leq \alpha, \tag{3.50}$$

*then the generalized solution* $u$ *of problem* (3.1),(3.3) *admits the estimate*

$$\|u\|_{L_\infty} \le \beta(\|\varphi\|_L + \|\psi\|_{L_\infty} + \|q\|_L^{(0)}). \tag{3.51}$$

*Proof.* By Lemma 3.3 the generalized solution $u$ of problem (3.1),(3.3) admits the representation

$$\begin{aligned} u(x,y) = &\int_0^x Z(x,y,s,y,)Z_1(s,y)\varphi(s)ds + \\ &+Z_2(x,y)\big[\psi(y) + \int_0^y \int_0^a Q(y,s,t)u(s,t)dsdt\big] + \\ &+\int_0^y \int_0^x Z(x,y,s,t)\big[\mathcal{P}(s,t)u(s,t) + q(s,t)\big]dsdt, \end{aligned} \tag{3.52}$$

where

$$Z(x,y,s,t) = Z_2(x,y)Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t),$$

and $\mathcal{P}$ is a matrix function given by (3.9).

However,

$$\begin{aligned} \int_0^y \int_0^x Z(x,y,s,t)q(s,t)dsdt = &\int_0^y \int_0^x q(s,t)dsdt - \\ -&\int_0^y \frac{\partial Z(x,y,x,t)}{\partial t}\left(\int_0^t \int_0^x q(s,\tau)dsd\tau\right)dt - \\ -&\int_0^x \frac{\partial Z(x,y,s,y)}{\partial s}\left(\int_0^y \int_0^s q(\xi,t)d\xi dt\right)ds + \\ +&\int_0^y \int_0^x \left[\frac{\partial^2 Z(x,y,s,t)}{\partial s\partial t}\int_0^t \int_0^s q(\xi,\tau)d\xi d\tau\right]dsdt. \end{aligned}$$

On the other hand, by virtue of Lemma $2.2_2$ and the restrictions imposed on the matrix functions $\mathcal{P}_1$ and $\mathcal{P}_2$, there exists a positive number $\alpha_0$ such that the inequalities

$$\|\mathcal{P}_1(x,y)\mathcal{P}_2(x,y)\| + \|\frac{\partial \mathcal{P}_2(x,y)}{\partial y}\| \le \alpha_0, \ \ \|Z_i(x,y)\| \le \alpha_0 \ \ (i=1,2) \tag{3.53}$$

and

$$\begin{aligned} \|Z(x,y,s,t)\| + \left\|\frac{\partial Z(x,y,s,t)}{\partial s}\right\| + \left\|\frac{\partial Z(x,y,s,t,)}{\partial t}\right\| + \\ +\left\|\frac{\partial^2 Z(x,y,s,t)}{\partial s\partial t}\right\| \le \alpha_0 \end{aligned} \tag{3.54}$$

hold almost everywhere in $\mathcal{D}_{ab}$ and $\mathcal{D}_{ab}\times\mathcal{D}_{ab}$. Therefore

$$\left\|\int_0^x \int_0^y Z(x,y,s,t)q(s,t)dsdt\right\| \le \big[1 + (a+b+ab)\alpha_0\big]\|q\|_L^{(0)} \tag{3.55}$$

and

$$\|\mathcal{P}(x,y)\| \leq \|\mathcal{P}_0(x,y)\| + \alpha_0. \tag{3.56}$$

Put

$$\rho(y) = \max_{0\leq x\leq a} \|u(x,y)\|.$$

Then with regard to inequalities (3.50) and (3.53)-(3.56) we obtain from (3.52)

$$\rho(y) \leq \alpha_0^2\|\varphi\|_L + \alpha_0\Big[\|\psi\|_{L_\infty} + a\alpha\int_0^y \rho(t)dt\Big] +$$
$$+\alpha_0\int_0^y \Big[\int_0^a \|\mathcal{P}_0(s,t)\|ds + \alpha_0 a\Big]\rho(t)dt + [1+(a+b+ab)\alpha_0]\|q\|_L^{(0)} \leq$$
$$\leq (1+\alpha_0+\alpha_0^2)(1+a+b+ab)(\|\varphi\|_L + \|\psi\|_{L_\infty} + \|q\|_{L^{(0)}}) +$$
$$+\alpha_0\int_0^y \Big[a\alpha + \alpha_0 a + \int_0^a \|\mathcal{P}_0(s,t)\|ds\Big]\rho(t)dt,$$

which by Gronwall's lemma and (3.50) implies estimate (3.51), where $\beta = (1+\alpha_0+\alpha_0^2)(1+a+b+ab)\exp\big(\alpha_0\alpha + ab\alpha_0(\alpha+\alpha_0)\big)$ is a constant depending on $\mathcal{P}_1$, $\mathcal{P}_2$, $a$, $b$ and $\alpha$ only. ∎

Alongside with problems (3.1),(3.2) and (3.1),(3.3), for any natural $k$ let us consider the problems

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \mathcal{P}_{0k}(x,y)u(x,y) + \mathcal{P}_{1k}(x,y)\frac{\partial u(x,y)}{\partial x} +$$
$$+\mathcal{P}_{2k}(x,y)\frac{\partial u(x,y)}{\partial y} + q_k(x,y), \tag{3.57}$$

$$u(x,0) = \varphi_{0k}(x), \quad \frac{\partial u(0,y)}{\partial y} = \psi_k(y) +$$
$$+\int_0^a \Big[Q_{0k}(s,y)u(s,y) + \gamma_{1k}^{-1}(s)Q_{1k}(s,y)\frac{\partial u(s,y)}{\partial s}\Big]ds \tag{3.58}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \mathcal{P}_{0k}(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} +$$
$$+\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q_k(x,y), \tag{3.59}$$

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \varphi_k(x),$$
$$u(0,y) = \psi_k(y) + \int_0^y\int_0^a Q_k(y,s,t)u(s,t)dsdt, \tag{3.60}$$

where

$$\mathcal{P}_{ik} \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \ \ (i=0,1,2), \quad q_k \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n),$$
$$\varphi_{0k} \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \ \ \gamma_{1k}(x) = 1 + \|\varphi_{0k}'(x)\|,$$
$$\varphi_k \in L([0,a];\mathbb{R}^n), \quad \psi_k \in L([0,b];\mathbb{R}^n),$$
$$Q_{ik} \in L(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \quad (i=0,1)$$

and

$$Q_k \in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n}).$$

As above, by $\gamma_1$ and $\gamma_2$ we imply the functions

$$\gamma_1(x) = 1 + \|\varphi_0'(x)\|, \quad \gamma_2(y) = 1 + \|\psi(y)\| + \int_0^a \big[\|Q_0(s,y)\| + \|Q_1(s,y)\|\big]ds.$$

*Let*

$$\sup_{k\geq 1} \|\mathcal{P}_{ik}\|_{L_\infty} < +\infty \ \ (i=1,2), \ \ \sup_{k\geq 1} \|\gamma_{1k}^{-1}Q_{1k}\|_{L_\infty} < +\infty, \tag{3.61}$$
$$\lim_{k\to+\infty} \|\mathcal{P}_{ik} - \mathcal{P}_i\|_L = 0 \quad (i=0,1,2), \quad \lim_{k\to+\infty} \|q_k - q\|_L = 0 \tag{3.62}$$

*and*

$$\begin{aligned} &\lim_{k\to+\infty} \|\varphi_{0k} - \varphi_0\|_{\widetilde{\mathbb{C}}} = 0, \quad \lim_{k\to+\infty} \|\psi_k - \psi\|_L = 0, \\ &\lim_{k\to+\infty} \|Q_{0k} - Q_0\|_L = 0 \quad \lim_{k\to+\infty} \|\gamma_1\gamma_{1k}^{-1}Q_{1k} - Q_1\|_L = 0. \end{aligned} \tag{3.63}$$

*Then*

$$\lim_{k\to+\infty} \|u_k - u\|_{\widetilde{\mathbb{C}}} = 0, \tag{3.64}$$

*where $u$ and $u_k$ are, respectively, solutions of problems* (3.1),(3.2) *and* (3.57),(3.58).

*Proof.* For an arbitrary natural $k$ the vector function

$$v(x,y) = u_k(x,y) - u(x,y)$$

is a solution of the problem

$$\begin{aligned} \frac{\partial^2 v(x,y)}{\partial x\partial y} &= \mathcal{P}_{0k}(x,y)v(x,y) + \mathcal{P}_{1k}(x,y)\frac{\partial v(x,y)}{\partial x} + \\ &\quad + \mathcal{P}_{2k}(x,y)\frac{\partial v(x,y)}{\partial y} + \widetilde{q}_k(x,y), \end{aligned} \tag{3.65}$$

$$\begin{aligned} v(x,0) = \widetilde{\varphi}_{0k}(x), \quad \frac{\partial v(0,y)}{\partial y} &= \widetilde{\psi}_k(y) + \\ + \int_0^a \big[Q_{0k}(s,y)v(s,y) &+ \gamma_{1k}^{-1}(s)Q_{1k}(s,y)\frac{\partial v(s,y)}{\partial s}\big]ds, \end{aligned} \tag{3.66}$$

where

$$\widetilde{q}_k(x,y) = \big[\mathcal{P}_{0k}(x,y) - \mathcal{P}_0(x,y)\big]u(x,y) + \big[\mathcal{P}_{1k}(x,y) - \mathcal{P}_1(x,y)\big] \times \\ \times \frac{\partial u(x,y)}{\partial x} + \big[\mathcal{P}_{2k}(x,y) - \mathcal{P}_2(x,y)\big]\frac{\partial u(x,y)}{\partial y} + q_k(x,y) - q(x,y) \quad (3.67)$$

and

$$\widetilde{\varphi}_{0k}(x) = \varphi_{0k}(x) - \varphi_0(x), \quad \widetilde{\psi}_k(y) = \psi_k(y) - \psi(y) + \\ + \int_0^a \big[\big(Q_{0k}(s,y) - Q_0(s,y)\big)u(s,y) + \\ + \big(\gamma_{1k}^{-1}(s)Q_{1k}(s,y) - \gamma_1^{-1}(s)Q_1(s,y)\big)\frac{\partial u(s,y)}{\partial s}\big]ds. \quad (3.68)$$

In view of conditions (3.61)-(3.63) there exists $\alpha > 0$ such that

$$\|\mathcal{P}_{0k}\|_L \le \alpha, \quad \|\mathcal{P}_{ik}\|_{L_\infty} \le \alpha \quad (i = 1,2), \\ \|Q_{0k}\|_L \le \alpha, \quad \|\gamma_{1k}Q_{1k}\|_{L_\infty} \le \alpha \quad (k = 1,2,\dots). \quad (3.69)$$

By virtue of Lemma 3.1 we may consider without loss of generality that

$$\|u(x,y)\| \le \alpha, \quad \|\frac{\partial u(x,y)}{\partial x}\| \le \alpha\gamma_1(x), \quad \|\frac{\partial u(x,y)}{\partial y}\| \le \alpha\gamma_2(y). \quad (3.70)$$

By Lemma 3.6 conditions (3.69) guarantee the existence of a positive constant $\beta$ such that

$$\|u_k - u\|_{\widetilde{\mathbb{C}}} \le \beta(\|\widetilde{\varphi}_{0k}\|_{\widetilde{\mathbb{C}}} + \|\widetilde{\psi}_k\|_L + \|\widetilde{q}_k\|_L) \quad (k = 1,2,\dots). \quad (3.71)$$

However, in view of (3.67),(3.68) and (3.70),

$$\|\widetilde{q}_k\|_L \le \alpha\big[\|\mathcal{P}_{0k} - \mathcal{P}_0\|_L + \sum_{i=1}^{2}\|\gamma_i(\mathcal{P}_{ik} - \mathcal{P}_i)\|_L + \|q_k - q\|_L\big] \\ (k = 1,2,\dots) \quad (3.72)$$

and

$$\|\widetilde{\psi}_k\|_L \le \|\psi_k - \psi\|_L + a\big[\|Q_{0k} - Q_0\|_L + \|\gamma_1\gamma_{1k}^{-1}Q_{1k} - Q_1\|_L\big] \\ (k = 1,2,\dots). \quad (3.73)$$

According to conditions (3.61)-(3.63), from estimates (3.71)-(3.73) we get equality (3.64). ■

Lemmas 3.10 and 3.11 below alsi deal with the correctness of problem (3.1),(3.2).

*Let conditions* (3.61) *hold,*

$$\sup_{k\geq 1}\|\mathcal{P}_{0k}\|_L^{(2)}<+\infty,\quad \sup_{k\geq 1}\|Q_{0k}\|_L^{(2)}<+\infty, \tag{3.74}$$

$$\lim_{k\to+\infty}\|\mathcal{P}_{ik}-\mathcal{P}_i\|_L^{(1)}=0\quad (i=0,1,2),\quad \lim_{k\to+\infty}\|q_k-q\|_L^{(1)}=0, \tag{3.75}$$

$$\begin{gathered}\lim_{k\to+\infty}\|\varphi_{0k}-\varphi_0\|_{\widetilde{\mathbb{C}}}=0,\quad \lim_{k\to+\infty}\|\psi_k-\psi\|_L=0,\\ \lim_{k\to+\infty}\|Q_{0k}-Q_0\|_L^{(1)}=0,\quad \lim_{k\to+\infty}\|\gamma_1\gamma_{1k}^{-1}Q_{1k}-Q_1\|_L^{(1)}=0.\end{gathered} \tag{3.76}$$

*Then*

$$\lim_{k\to+\infty}\|u_k-u\|_{\widetilde{\mathbb{C}}}^{(1)}=0, \tag{3.77}$$

*where $u$ and $u_k$ are, respectively, the solutions of problems* (3.1),(3.2) *and* (3.57),(3.58).

*Let conditions* (3.61) *take place, $\varphi_0$ and $\varphi_{0k}\in\widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^n)$, $\psi$ and $\psi_k\in L_\infty([0,b];\mathbb{R}^n)$,*

$$\begin{gathered}\lim_{k\to+\infty}\|\mathcal{P}_{ik}-\mathcal{P}_i\|_L^{(2)}=0\quad (i=0,1,2),\\ \lim_{k\to+\infty}\|q_k-q\|_L^{(2)}=0,\end{gathered} \tag{3.78}$$

$$\begin{gathered}\lim_{k\to+\infty}\|\varphi_{0k}-\varphi_0\|_{\widetilde{\mathbb{C}}_\infty}=0,\quad \lim_{k\to+\infty}\|\psi_k-\psi\|_{L_\infty}=0,\\ \lim_{k\to+\infty}\|Q_{0k}-Q_0\|_L^{(2)}=0,\quad \lim_{k\to+\infty}\|Q_{1k}-Q_1\|_L^{(2)}=0.\end{gathered} \tag{3.79}$$

*Then*

$$\lim_{k\to+\infty}\|u_k-u\|_{\widetilde{\mathbb{C}}}^{(2)}=0. \tag{3.80}$$

To prove the above stated lemmas we need three auxiliary assertions.

*Let $p_k:\mathcal{D}_{ab}\to\mathbb{R}$ $(k=1,2,\dots)$ be a sequence of summable functions satisfying the condition*

$$\lim_{k\to+\infty}\|p_k\|_L^{(i)}=0 \tag{3.81}$$

*for some $i\in\{1,2\}$ and let the function $z:\mathcal{D}_{ab}\to\mathbb{R}$ be such that $z(\cdot,y)$ and $z(x,\cdot)$ are absolutely continuous almost for any $y\in[0,b]$ and $x\in[0,a]$, respectively, and the inequalities*

$$\Big|\frac{\partial z(x,y)}{\partial x}\Big|\leq z_1(x),\quad \Big|\frac{\partial z(x,y)}{\partial y}\Big|\leq z_2(y) \tag{3.82}$$

*take place almost everywhere in $\mathcal{D}_{ab}$, where $z_1:[0,a]\to\mathbb{R}_+$ and $z_2:[0,b]\to\mathbb{R}_+$ are summable functions. Then*

$$\lim_{k\to+\infty}\|p_kz\|_L^{(i)}=0. \tag{3.83}$$

*Proof.* In view of (3.82), from the equalities

$$\int_0^x p_k(s,y)z(s,y)ds = z(x,y)\int_0^x p_k(s,y)ds - \\ - \int_0^x \left(\int_0^s p_k(\xi,y)d\xi\right)\frac{\partial z(s,y)}{\partial s}ds$$

and

$$\int_0^y p_k(x,t)z(x,t)dt = z(x,y)\int_0^y p_k(x,t)dt - \\ - \int_0^y \left(\int_0^t p_k(x,\tau)d\tau\right)\frac{\partial z(x,t)}{\partial t}dt \tag{3.84}$$

we have

$$\|p_k z\|_L^{(i)} \leq \big[\max_{(x,y)\in\mathcal{D}_{ab}} \|z(x,y)\| + \int_0^a z_1(s)ds + \int_0^b z_2(t)dt\big]\,\|p_k\|_L^{(i)},$$

whence according to (3.81) we have equality (3.83). ■

*Let* $p_k : \mathcal{D}_{ab} \to \mathbb{R}$ $(k = 1, 2, \dots)$ *be a sequence of measurable and essentially bounded functions satisfying conditions* (3.81) *for some* $i \in \{0, 1\}$ *and*

$$\alpha = \sup_{k\geq 1} \|p_k\|_{L_\infty} < +\infty. \tag{3.85}$$

*Then equality* (3.83) *is valid for any summable function* $z : \mathcal{D}_{ab} \to \mathbb{R}$.

*Proof.* Since $z$ is summable, there exists a sequence of functions $z_m : \mathcal{D}_{ab} \to \to \mathbb{R}$ $(m = 1, 2, \dots)$ such that for any $m \geq 1$, $j$ and $l \in \{1, \dots, m\}$ the function $z_m$ is constant in the rectangle $(\frac{j-1}{m}a, \frac{j}{m}a) \times (\frac{l-1}{m}b, \frac{l}{m}b)$ and

$$\lim_{m\to+\infty} \|z - z_m\|_L = 0. \tag{3.86}$$

By (3.81) and (3.85)

$$\|p_k z\|_L^{(i)} \leq \|p_k(z - z_m)\|_L^{(i)} + \|p_k z_m\|_L^{(i)} \leq \alpha\|z - z_m\|_L^{(i)} + \|p_k z_m\|_L^{(i)}$$

and

$$\lim_{k\to+\infty} \|p_k z_m\|_L^{(i)} = 0 \quad (m = 1, 2, \dots).$$

Therefore

$$\limsup_{k\to+\infty} \|p_k z\|_L^{(i)} \leq \alpha\|z - z_m\|_L^{(i)} \quad (m = 1, 2, \dots),$$

from which with regard to (3.86) we get equality (3.83). ■

*Let $p_k : \mathcal{D}_{ab} \to \mathbb{R}$ $(k = 1, 2, \dots)$ be a sequence of measurable essentially bounded functions satisfying condition* (3.85) *and*

$$\lim_{k\to+\infty} \|p_k\|_L^{(2)} = 0. \tag{3.87}$$

*Then for any function $z$ belonging to $\widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R})$ or $\widetilde{\mathbb{C}}_{\infty}^{(0,-1)}(\mathcal{D}_{ab};\mathbb{R})$ we have*

$$\lim_{k\to+\infty} \|p_k z\|_L^{(2)} = 0. \tag{3.88}$$

*Proof.* To be more precise, we assume that $z \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R})$ since the case when $z \in \widetilde{\mathbb{C}}_{\infty}^{(0,-1)}(\mathcal{D}_{ab};\mathbb{R})$ is considered similarly.

Choose $\beta > 0$ such that the inequalities

$$|z(x,y)| \le \beta, \quad \Big|\frac{\partial z(x,y)}{\partial y}\Big| \le \beta \tag{3.89}$$

hold almost everywhere in $\mathcal{D}_{ab}$. Then from (3.84) we find that

$$\left| \int_0^y p_k(x,t) z(x,t)dt \right| \le (1+b)\beta \|p_k\|_L^{(2)}. \tag{3.90}$$

Let $\varepsilon$ be an arbitrarily small positive number. Choose a natural $m$ such that

$$\frac{a\alpha\beta}{m} < \frac{\varepsilon}{2}.$$

For every $j \in \{1,\dots,m\}$ we put

$$z_{0j}(x) = z(x, \frac{jb}{m})$$

and choose a step-function $z_j : [0,a] \to \mathbb{R}$ such that

$$\alpha \int_0^a |z_{0j}(s) - z_j(s)|ds < \frac{\varepsilon}{2}.$$

Then, taking into account (3.85) and (3.89),

$$\begin{aligned} \left| \int_0^x p_k(s,y) z(s,y)ds \right| &\le \int_0^x |p_k(s,y)||z(s,y) - z(s,\frac{jb}{m})|ds + \\ &+ \int_0^x |p_k(s,y)||z_{0j}(s) - z_j(s)|ds + |\int_0^x p_k(s,y) z_j(s)ds| \le \\ &\le \frac{a\alpha\beta}{m} + \alpha \int_0^a |z_{0j}(s) - z_j(s)|ds + \|p_k z_j\|_L^{(2)} \le \\ &\le \varepsilon + \|p_k z_j\|_L^{(2)} \quad \text{for} \quad \frac{j-1}{m} b \le y \le \frac{jb}{m}. \end{aligned} \tag{3.91}$$

It follows from (3.90) and (3.91) that

$$\|p_k z\|_L^{(2)} \le \varepsilon + (1+b)\beta \|p_k\|_L^{(2)} + \max_{1\le j\le m} \|p_k z_j\|_L^{(2)}.$$

On the other hand, in view of (3.87) we have

$$\lim_{k\to+\infty} \|p_k z_j\|_L^{(2)} = 0 \quad (j = 1, \dots, m).$$

Therefore

$$\limsup_{k\to+\infty} \|p_k z\|_L^{(2)} \le \varepsilon.$$

From this and in view of the arbitrariness of $\varepsilon$, we get equality (3.88). ■

*Proof of Lemma* 3.10. For every natural $k$ the vector function $v(x,y) = u_k(x,y) - u(x,y)$ is a solution of problem (3.65),(3.66), where $\widetilde{q}_k$, $\widetilde{\varphi}_{0k}$ and $\widetilde{\psi}_k$ are the vector functions given by equalities (3.67) and (3.68).

According to conditions (3.61),(3.74) and Lemma 3.7 there exists a positive number $\beta$ such that

$$\|u_k - u\|_{\widetilde{\mathbb{C}}}^{(1)} \le \beta \varepsilon_k \quad (k = 1, 2, \dots), \tag{3.92}$$

where $\varepsilon_k = \|\widetilde{q}_k\|_L^{(1)} + \|\varphi_{0k}\|_{\widetilde{\mathbb{C}}} + \|\widetilde{\psi}_k\|_L$. On the other hand, by virtue of Lemma 3.1 inequalities (3.70) take place for some $\alpha > 0$.

In view of (3.67) and (3.68)

$$\begin{aligned}\varepsilon_k \le \|(\mathcal{P}_{0k} - \mathcal{P}_0)u\|_L^{(1)} + \|(\mathcal{P}_{1k} - \mathcal{P}_1)w_1\|_L^{(1)} + \|(\mathcal{P}_{2k} - \mathcal{P}_2)w_2\|_L^{(1)} + \\ + \|q_k - q\|_L^{(1)} + \|\varphi_{0k} - \varphi_0\|_{\widetilde{\mathbb{C}}} + \|\psi_k - \psi\|_L + \\ + \|(Q_{0k} - Q_0)u\|_L^{(1)} + \|(\gamma_1 \gamma_{1k}^{-1} Q_{1k} - Q_0)\gamma_1^{-1} w_1\|_L^{(1)},\end{aligned}$$

where

$$w_1(x,y) = \frac{\partial u(x,y)}{\partial x}, \quad w_2(x,y) = \frac{\partial u(x,y)}{\partial y},$$

from which by virtue of conditions (3.61) and (3.74)-(3.76) and Lemmas 3.12 and 3.13 we have

$$\lim_{k\to+\infty} \varepsilon_k = 0.$$

Then taking into account estimate (3.92), we obtain equality (3.77). ■

Proof of Lemma 3.11 proceeds analogously to that of Lemma 3.10. The only difference is that instead of Lemma 3.13 we use Lemma 3.14.

*Let* $\mathcal{P}_1 \in \widetilde{\mathbb{C}}_\infty^{(0,-1)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$

$$\sup_{k\ge1} \|\mathcal{P}_{0k}\|_{L_\infty} < +\infty, \sup_{k\ge1} \|Q_k\|_{L_\infty} < +\infty, \tag{3.93}$$

$$\lim_{k\to+\infty} \|\mathcal{P}_{0k} - \mathcal{P}_0\|_L^{(0)} = 0, \quad \lim_{k\to+\infty} \|q_k - q\|_L^{(0)} = 0, \tag{3.94}$$

$$\lim_{k\to+\infty} \|\varphi_k - \varphi\|_L = 0, \quad \lim_{k\to+\infty} \|\psi_k - \psi\|_{L_\infty} = 0 \tag{3.95}$$

*and*

$$\lim_{k\to+\infty}\ \operatorname*{ess\,sup}_{\substack{0\le y_1<y_2\le b,\\ (x,y)\in\mathcal{D}_{ab}}}\left\|\int_{y_1}^{y_2}\int_0^x \Big[Q_k(y,s,t)-Q(y,s,t)\Big]dsdt\right\|=0 \quad (3.96)$$

*Then*

$$\lim_{k\to+\infty}\|u_k-u\|_{L_\infty}=0, \qquad (3.97)$$

*where $u$ and $u_k$ are, respectively, the generalized solutions of problems* (3.1), (3.3) *and* (3.59),(3.60).

*Proof.* For every natural $k$ the vector function $v(x,y)=u_k(x,y)-u(x,y)$ is a generalized solution of the problem

$$\frac{\partial^2 v(x,y)}{\partial x\partial y}=\mathcal{P}_{0k}(x,y)v(x,y)+\mathcal{P}_1(x,y)\frac{\partial v(x,y)}{\partial x}+$$
$$+\mathcal{P}_2(x,y)\frac{\partial v(x,y)}{\partial y}+\widetilde{q}_k(x,y),$$
$$\lim_{y\to 0}\Big(\frac{\partial v(x,y)}{\partial x}-\mathcal{P}_2(x,y)v(x,y)\Big)=\widetilde{\varphi}_k(x),$$
$$v(0,y)=\widetilde{\psi}_k(y)+\int_0^y\int_0^a Q_k(y,s,t)v(s,t)dsdt,$$

where

$$\widetilde{q}_k(x,y)=[\mathcal{P}_{0k}(x,y)-\mathcal{P}_0(x,y)]u(x,y)+q_k(x,y)-q(x,y),$$
$$\widetilde{\varphi}_k(x)=\varphi_k(x)-\varphi(x),\quad \widetilde{\psi}_k(y)=\psi_k(y)-\psi(y)+$$
$$+\int_0^y\int_0^a[Q_k(y,s,t)-Q(y,s,t)]u(s,t)dsdt.$$

By conditions (3.93) and Lemma 3.8 there exists $\beta>0$ such that

$$\|u_k-u\|_{L_\infty}\le\beta(\|\widetilde{\varphi}_k\|_L+\|\psi_k\|_{L_\infty}+\|\widetilde{q}_k\|_L^{(0)})\quad (k=1,2,\dots). \qquad (3.98)$$

By virtue of conditions (3.93),(3.94) and Lemma 3.13

$$\lim_{k\to+\infty}\|\widetilde{q}_k\|_L^{(0)}=0. \qquad (3.99)$$

On the other hand, in view of (3.93) and (3.96) we have

$$\lim_{k\to+\infty}\sup_{0\le y\le b}\left\|\int_0^y\int_0^a\big[Q_k(y,s,t)-Q(y,s,t)\big]u(s,t)dsdt\right\|=0 \quad (3.100)$$

Taking into account conditions (3.95),(3.99) and (3.100), we obatin equality (3.97) from (3.98). ■

## § 4.

In this chapter for the linear hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y) \tag{4.1}$$

we study boundary value problems of four types

$$u(x,0) = \varphi_0(x), \quad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \varphi_1(y), \tag{4.2}$$

$$\frac{\partial u(x,0)}{\partial x} - \mathcal{P}_2(x,0)u(x,0) = \psi_0(x), \quad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \psi_1(y), \tag{4.3}$$

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x), \quad h(u(\cdot,y))(y) = \psi_1(y) \tag{4.4}$$

and

$$\begin{gathered} \lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x), \\ h\Big(\frac{\partial}{\partial y}\big(u(\cdot,y)\big)(y) - Z_2(\cdot,y)u(0,y)\big)\Big)(y) = \psi_1(y), \end{gathered} \tag{4.5}$$

whose special cases are periodic boundary value problems

$$u(x,0) = \varphi_0(x), \quad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y} + \varphi_1(y), \tag{4.2$_1$}$$

$$\frac{\partial u(x,0)}{\partial x} - \mathcal{P}_2(x,0)u(x,0) = \psi_0(x), \quad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y} + \psi_1(y), \tag{4.3$_1$}$$

$$\begin{gathered} \lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x), \\ u(a,y) = u(0,y) + \psi_1(y) \end{gathered} \tag{4.4$_1$}$$

and

$$\begin{gathered} \lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x), \\ \frac{\partial}{\partial y}\big(u(a,y) - u(0,y)\big) = \psi_1(y). \end{gathered} \tag{4.5$_1$}$$

Here and everywhere below, unless stated otherwise, we assume

$$
\begin{gathered}
\mathcal{P}_i \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n), \\
\varphi_0 \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \ \ ; \ \varphi_1 \in L([0,b];\mathbb{R}^n), \\
\psi_0 \in L([0,a];\mathbb{R}^n), \quad \psi_1 \in L([0,b];\mathbb{R}^n)
\end{gathered}
\tag{4.6}
$$

and $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to L_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator. Additional restrictions on the coefficients of system (4.1) and on the boundary conditions will be given in the theorems formulated below.

By $Z_1 : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ and $Z_2 : \mathcal{D}_{ab} \to \mathbb{R}^{n\times n}$ will be meant solutions of the matrix differential equations

$$
\frac{\partial Z_1(x,y)}{\partial y} = \mathcal{P}_1(x,y)Z_1(x,y)
$$

and

$$
\frac{\partial Z_2(x,y)}{\partial x} = \mathcal{P}_2(x,y)Z_2(x,y),
$$

satisfying the initial conditions

$$
Z_1(x,0) = E \quad \text{for} \quad 0 \le x \le a
$$

and

$$
Z_2(0,y) = E \quad \text{for} \quad 0 \le y \le b.
$$

According to Lemmas $2.1_1$ and $2.3_1$,

$$
\begin{gathered}
h(v)(y) = H_0(y)v(0) + \int_0^a H(s,y)v'(s)ds \\
\text{for} \ \ v \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \ \ y \in [0,b]
\end{gathered}
\tag{4.7}
$$

and

$$
\begin{gathered}
h\big(Z_2(\cdot,y)v(\cdot)\big)(y) = M_0(y)v(0) + \int_0^a M(s,y)v'(s)ds \\
\text{for} \quad v \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \ \ y \in [0,b],
\end{gathered}
\tag{4.8}
$$

where

$$
H_0 \in L_\infty([0,b];\mathbb{R}^{n\times n}), \quad H \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),
\tag{4.9}
$$

$$
\begin{gathered}
M_0(y) = H_0(y) + \int_0^a H(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds, \\
M(x,y) = H(x,y)Z_2(x,y) + \int_x^a H(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds
\end{gathered}
\tag{4.10}
$$

and

$$
M_0 \in L_\infty([0,b];\mathbb{R}^{n\times n}), \quad M \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}).
\tag{4.11}
$$

Assume

$$I_{M_0} = \{y \in [0,b] : \det M_0(y) \neq 0\}.$$

$M_0(y) \not\equiv \Theta$ Alongside with (4.1),(4.2) we have to consider the homogeneous problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y}, \quad (4.1_0)$$

$$u(x,0) = 0, \qquad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = 0. \qquad (4.2_0)$$

*Let the vector and the matrix functions*

$$\varphi_1, \ MZ_2^{-1}q, \ MZ_2^{-1}\mathcal{P}_0, \ (1 + \|\varphi_0'\|)MZ_2^{-1}\mathcal{P}_1 \qquad (4.12)$$

*be $M_0$-summable. Then problem* (4.1),(4.2) *is solvable and its solution is unique if and only if*

$$\operatorname{mes} I_{M_0} = b. \qquad (4.13)$$

*Moreover, if condition* (4.13) *is violated, then the space of solutions of homogeneous problem* $(4.1_0)$,$(4.2_0)$ *is infinite dimensional.*

*Proof.* According to the $M_0$-summability of vector and matrix functions (4.12), there exist

$$\varphi_{10} \in L([0,b];\mathbb{R}^n), \ \ q_0 \in L(\mathcal{D}_{ab};\mathbb{R}^n), \ \ Q_i \in L(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \ (i = 0,1)$$

such that

$$\begin{gathered}\varphi_1(y) = M_0(y)\varphi_{10}(y), \ \ M(x,y)Z_2^{-1}(x,y)q(x,y) = -M_0(y)q_0(x,y), \\ M(x,y)Z_2^{-1}(x,y)\mathcal{P}_0(x,y) = -M_0(y)Q_0(x,y), \\ M(x,y)Z_2^{-1}(x,y)\mathcal{P}_1(x,y) = -\gamma_1^{-1}(x)M_0(y)Q_1(x,y),\end{gathered} \qquad (4.14)$$

where $\gamma_1(x) = 1 + \|\varphi_0'(x)\|$.

Let $u$ be an arbitrary solution of system (4.1). Then

$$\begin{aligned}\frac{\partial u(x,y)}{\partial y} = Z_2(x,y)\Big(\frac{\partial u(0,y)}{\partial y} + \int_0^x Z_2^{-1}(s,y)\big[\mathcal{P}_0(s,y)u(s,y) + \\ +\mathcal{P}_1(s,y)\frac{\partial u(s,y)}{\partial s} + q(s,y)\big]ds\Big),\end{aligned} \qquad (4.15)$$

whence, in view of representation (4.8), it is clear that the boundary condition

$$h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \varphi_1(y) \qquad (4.16)$$

holds if and only if

$$\varphi_1(y) = M_0(y)\frac{\partial u(0,y)}{\partial y} + \int_0^a M(s,y)Z_2^{-1}(s,y)\big(\mathcal{P}_0(s,y)u(s,y) +$$

$$+\mathcal{P}_1(s,y)\frac{\partial u(x,y)}{\partial s}+q(s,y)\big)ds \quad \text{for} \quad 0\le y\le b.$$

By (4.14) from the above equality we have

$$M_0(y)\frac{\partial u(0,y)}{\partial y}=M_0(y)\Big(\psi(y)+$$
$$+\int_0^a\big[Q_0(s,y)u(s,y)+\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds\Big),$$

where

$$\psi(y)=\varphi_{10}(y)+\int_0^a q_0(s,y)ds.$$

Consequently, for conditions (4.2) to be fulfilled, it is sufficient, and when (4.13) holds, it is necessary that

$$\begin{aligned}u(x,0)=\varphi_0(x),\qquad \frac{\partial u(0,y)}{\partial y}=\psi(y)+\\ +\int_0^a\big[Q_0(s,y)u(s,y)+\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds.\end{aligned}\tag{4.17}$$

However, according to Lemma 3.4, problem (4.1),(4.17) has the unique solution. Thus we have proved that problem (4.1),(4.2) is solvable and its solution is unique in the case if condition (4.13) is fulfilled.

For completion of the proof it remains to show that if

$$\operatorname{mes} I_{M_0}<b,\tag{4.18}$$

then homogeneous problem $(4.1_0)$,$(4.2_0)$ has an infinite dimensional space of solutions.

In view of (4.18) there exists a measurable function $c_0:[0,b]\to\mathbb{R}^n$ such that

$$\|c_0(y)\|=1 \quad \text{for} \quad y\notin I_{M_0},\quad c_0(y)=0 \quad \text{for} \quad y\in I_{M_0}\tag{4.19}$$

and

$$M_0(y)c_0(y)=0 \quad \text{for} \quad y\in[0,b].$$

As shown above, the solution $u$ of system $(4.1_0)$ satisfies the condition

$$h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y)=0$$

if and only if

$$M_0(y)\frac{\partial u(0,y)}{\partial y}=M_0(y)\int_0^a\big[Q_0(s,y)u(s,y)+\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds.$$

Therefore it is clear that for all natural $k$ the solution of system $(4.1_0)$ satisfying boundary conditions

$$u(x,0)=0,\quad \frac{\partial u(0,y)}{\partial y}=y^{k-1}c_0(y)+\int_0^a\big[Q_0(s,y)u(s,y)+ \\ +\gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds \tag{4.20}$$

is the solution of problem $(4.1_0)$, $(4.2_0)$.

According to Lemma 3.4, problem $(4.1_0)$,(4.20) has the unique solution which we denote by $u_k$. It follows immediately from (4.18) and (4.19) that functions $u_k$ $(k=1,2,\dots)$ are linearly independent. ■

*If condition* (4.13) *holds and*

$$\int_0^a\int_0^b \|M_0^{-1}(y)\|\Big(\|\mathcal{P}_0(x,y)\|+(1+\|\varphi_0'(x)\|)\|\mathcal{P}_1(x,y)\|+ \\ +\|q(x,y)\|+\|\varphi_1(y)\|\Big)dxdy<+\infty, \tag{4.21}$$

*then problem* (4.1),(4.2) *has one and only one solution.*

*Proof.* According to conditions (4.6),(4.11) and Lemma $2.2_1$,

$$MZ_2^{-1}\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n).$$

If we take this and conditions (4.21) into consideration, then it becomes evident that

$$M_0^{-1}\varphi_1\in L([0,b];\mathbb{R}^n),\quad M_0^{-1}MZ_2^{-1}q\in L(\mathcal{D}_{ab};\mathbb{R}^n),$$
$$M_0^{-1}MZ_2^{-1}\mathcal{P}_0\in L(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$
$$(1+\|\varphi_0'\|)M_0^{-1}MZ_2^{-1}\mathcal{P}_1\in L(\mathcal{D}_{ab};\mathbb{R}^{n\times n}).$$

Thus, the vector and the matrix functions (4.12) are $M_0$-summable. ■

*Remark* 4.1. In the above proven corollary condition (4.21) is essential and we cannot neglect it. To convince ourselves that is so consider the problem

$$\frac{\partial^2u(x,y)}{\partial x\partial y}=y\frac{\partial u(x,y)}{\partial y}+1,$$
$$u(x,0)=0,\quad \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}$$

for which all conditions of Theorem 4.1, except of (4.21), are fulfilled because

$$M_0(y)=\exp(ay)-1.$$

Let us show that this problem has no solution. In fact, otherwise we should have

$$\frac{\partial u(x,y)}{\partial y} = \exp(xy)\frac{\partial u(0,y)}{\partial y} + \frac{\exp(xy)-1}{y} \quad \text{for} \quad 0 < y \le b,$$

from which according to the condition

$$\frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y}$$

it follows that

$$\frac{\partial u(0,y)}{\partial y} = -\frac{1}{y} \quad \text{for} \quad 0 < y \le b$$

and

$$\frac{\partial u(x,y)}{\partial y} = -\frac{1}{y} \quad \text{for} \quad 0 < y \le b.$$

But this contradicts the absolute continuity of $u$.

′ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}([0,b];\mathbb{R}^n)$ *be a linear continuous operator,*

$$\begin{gathered}\mathcal{P}_i \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^n), \\ \varphi_0 \in \mathbb{C}^1([0,a];\mathbb{R}^n), \quad \varphi_1 \in \mathbb{C}([0,b];\mathbb{R}^n).\end{gathered} \tag{4.6$'$}$$

*Moreover, let the vector and the matrix functions* $MZ_2^{-1}q$ *and* $MZ_2^{-1}\mathcal{P}_i\,(i = 0,1)$*, respectively, satisfy the Carathéodory condition with* $M_0$*-weight, and let* $\varphi_1$ *be* $M_0$*-continuous. Then problem* (4.1),(4.2) *has at least one classical solution but for its uniqueness it is necessary and sufficient that*

$$\bar{I}_{M_0} = [0,b]. \tag{4.13$'$}$$

*When condition* (4.13′) *is violated the space of classical solutions of problem* $(4.1_0)$,$(4.2_0)$ *is infinite dimensional.*

*Proof.* According to Lemma $2.3_2$ and the restrictions imposed on $\varphi_1$, $MZ_2^{-1}q$ and $MZ_2^{-1}\mathcal{P}_i$ $(i=0,1)$,

$$M_0 \in \mathbb{C}([0,b];\mathbb{R}^{n\times n}) \tag{4.22}$$

and the representations

$$\begin{gathered}\varphi_1(y) = M_0(y)\varphi_{10}(y), \quad M(x,y)Z_2^{-1}(x,y)q(x,y) = -M_0(y)q_0(x,y), \\ M(x,y)Z_2^{-1}(x,y)\mathcal{P}_i(x,y) = -M_0(y)Q_i(x,y)\ (i=0,1)\end{gathered}$$

are valid, where $\varphi_{10} \in \mathbb{C}([0,b];\mathbb{R}^n)$, while $q_0 \in L(\mathcal{D}_{ab};\mathbb{R}^n)$ and $Q_i \in L(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ $(i=0,1)$ satisfy the Carathéodory condition.

Let $u$ be an arbitrary classical solution of system (4.1). It satisfies condition (4.16) if and only if

$$M_0(y)\frac{\partial u(0,y)}{\partial y} = M_0(y)\Big(\psi(y) +$$

$$+\int_0^a \big[Q_0(s,y)u(s,y)+Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds\Big),$$

where

$$\psi(y)=\varphi_{10}(y)+\int_0^a q_0(s,y)ds,$$

whence, in view of the continuity of $\frac{\partial u(0,y)}{\partial y}$, it becomes clear that for boundary conditions (4.2) to be fulfilled, it is sufficient, and when condition (4.13′) holds, it is necessary that

$$u(x,0)=\varphi_0(x),$$
$$\frac{\partial u(0,y)}{\partial y}=\psi(y)+\int_0^a \big[Q_0(s,y)u(s,y)+Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\big]ds. \tag{4.23}$$

But according to Lemma 3.4′, problem (4.1),(4.23) has the unique solution $u$ and this solution is classical. Thus, problem (4.1),(4.2) has at least one classical solution and when (4.13′) holds the solution is unique.

Suppose that condition (4.13′) is not fulfilled. Then, according to (4.22), there exist an interval $[b_1,b_2]\subset(0,b)$ and a continuous vector function $c_0:[0,b]\to\mathbb{R}^n$ such that

$$\|c_0(y)\|>0 \quad \text{for} \quad b_1<y<b_2, \quad c_0(y)=0 \quad \text{for} \quad y\notin(b_1,b_2)$$

and

$$M_0(y)c_0(y)=0 \quad \text{for} \quad y\in[0,b].$$

Now, repeating the same arguments as in the proof of the second part of Theorem 4.1 and applying Lemma 3.4′ instead of Lemma 3.4, we convince ourselves that homogeneous problem $(4.1_0)$,$(4.2_0)$ has a countable system of classical solutions. ■

*Remark* 4.2. In Theorem 4.1′ the requirements for $\varphi_1$ to be $M_0$-continuous and for the vector and the matrix functions $MZ_2^{-1}q$ and $MZ_2^{-1}\mathcal{P}_i$ ($i=0,1$) to satisfy the Carathéodory conditions with $M_0$ weight are the most essential ones. The examples below show that if at least one of these requirements is not fulfilled, then problem (4.1),(4.2) has no classical solution despite the fact that the coefficients of system (4.1) and the functions given in boundary conditions (4.2) are smooth.

Consider the boundary value problems

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=|y-b_0|^{2k-1}\frac{\partial u(x,y)}{\partial y}-x(y-b_0)^{2k-1}|y-b_0|^{2k-1}, \tag{4.24$_1$}$$
$$u(x,0)=b_0+\frac{x}{2k}b_0^{2k}, \quad \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}+a(y-b_0)^{2k-1}; \tag{4.25$_1$}$$
$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=|y-b_0|^{2k-1}\frac{\partial u(x,y)}{\partial y}+(y-b_0)^{2k-1}, \tag{4.24$_2$}$$
$$u(x,0)=0, \qquad \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y} \tag{4.25$_2$}$$

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = |y-b_0|^{2k-1}\frac{\partial u(x,y)}{\partial y} + \frac{1}{b_0}(y-b_0)^{2k-1}u(x,y) + \\ + \frac{1}{b_0}|y-b_0|(y-b_0)^{2k-1}, \tag{4.24$_3$}$$

$$u(x,0)=0, \qquad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y} \tag{4.25$_3$}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = |y-b_0|^{2k-1}\frac{\partial u}{\partial y} - \exp(-x)(y-b_0)^{2k-1}\frac{\partial u}{\partial x}, \tag{4.24$_4$}$$

$$u(x,0)=\exp(x)+b_0, \qquad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y}, \tag{4.25$_4$}$$

where $b_0 \in (0,b)$ and $k$ is a natural number. For each of these problems (4.24$i$), (4.25$_i$) $(i=1,2,3,4)$ we have

$$h(v)(y) \equiv v(a)-v(0), \quad Z_2(x,y)=\exp(x|y-b_0|^{2k-1}),$$
$$M_0(y)=\exp(a|y-b_0|^{2k-1})-1,$$
$$M(x,y)=\exp(a|y-b_0|^{2k-1}), \quad I_{M_0}=[0,b]\backslash\{b_0\}.$$

Proceeding from this, it is easy to convince ourselves that for these problems all conditions of Theorem 4.1 are fulfilled and therefore they are uniquely solvable. The following functions

$$u_1(x,y)=|y-b_0|+\frac{x}{2k}(y-b_0)^{2k}, \;\; u_2(x,y)=b_0-|y-b_0|,$$
$$u_3(x,y)=b_0-|y-b_0|, \;\; u_4(x,y)=\exp(x)+|y-b_0|$$

are respectively the solutions of problems (4.24$_i$), (4.25$_i$) $(i=1,2,3,4)$. But none of these solutions is classical, because they have no partial derivative in the second argument when $y=b_0$. This case is due to the fact that for each of these problems one (and only one) of the conditions of Theorem 4.1$'$ is violated.

More precisely, for problem (4.24$_1$),(4.25$_1$) the function

$$\varphi_1(y)=a(y-b_0)^{2k-1}$$

is not $M_0$-continuous, for problem (4.24$_2$),(4.25$_2$) the function

$$M(x,y)Z_2^{-1}(x,y)q(x,y)=\exp\big((a-x)|y-b_0|^{2k-1}\big)(y-b_0)^{2k-1}$$

does not satisfy the Carathéodory condition with $M_0$ weight, for problem (4.24$_3$), (4.25$_3$) the function

$$M(x,y)Z_2^{-1}(x,y)\mathcal{P}_0(x,y)=\frac{1}{b_0}\exp\big((a-x)|y-b_0|^{2k-1}\big)(y-b_0)^{2k-1}$$

and for problem (4.24$_4$),(4.25$_4$) the function

$$M(x,y)Z_2^{-1}(x,y)\mathcal{P}_1(x,y)=\exp\big((a-x)|y-b_0|^{2k-1}-x\big)(y-b_0)^{2k-1}$$

behaves similarly.

*Remark* 4.3. Let all conditions of Theorem 4.1$'$ hold. If, in addition, condition (4.13) takes place, then problem (4.1),(4.2) has the unique solution and it is classical. If condition (4.13) is violated, then this problem has the unique classical solution and an infinite set of absolutely continuous solutions.

As an example let us consider the problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = p(y)\frac{\partial u(x,y)}{\partial y} \tag{4.26}$$

$$u(x,0) = 0, \qquad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y}, \tag{4.27}$$

where $p : [0,b] \to [0,+\infty)$ is a continuous function with the set of zeros $J_p$ which is nowhere dense in $[0,b]$ and has a positive measure. In this case

$$M_0(y) = \exp(xp(y)) - 1$$

and

$$I_{M_0} = [0,b] \backslash J_p.$$

Consequently, for problem (4.26),(4.27) all conditions of Theorem 4.1$'$ are fulfilled, but condition (4.13) is violated. Therefore $u_0(x,y) \equiv 0$ is the unique classical solution of the problem under consideration. On the other hand, for every summable function $c : [0,b] \to \mathbb{R}$ the function

$$u_c(x,y) = \int_0^y \delta(t)c(t)\exp(xp(t))dt,$$

where

$$\delta(t) = \begin{cases} 1 & \text{for } t \in J_p \\ 0 & \text{for } t \notin J_p \end{cases}$$

is an absolutely continuous solution of problem (4.26),(4.27).

$'$ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator,* $\mathcal{P}_j$ $(i = 0,1,2)$*,* $q$ *and* $\varphi_i$ $(i = 0,1)$ *satisfy conditions* (4.6$'$) *and*

$$I_{M_0} = [0,b]. \tag{4.13$''$}$$

*Then problem* (4.1),(4.2) *has one and only one solution* $u$ *and it is classical.*

*Proof.* By virtue of Lemmas $2.1_3$ and $2.2_1$,

$$H_0 \in \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^{n\times n}), \quad H \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

$$Z_2 \quad \text{and} \quad \frac{\partial Z_2}{\partial x} \in \mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

whence by equalities (4.10) and conditions (4.6$'$) and (4.13$''$) it follows that $\varphi_1$ is $M_0$-continuous and $MZ_2^{-1}q$ and $MZ_2^{-1}\mathcal{P}_i$ $(i = 0,1)$ satisfy the

Carathéodory condition with $M_0$ weight. Consequently, all conditions of Theorem 4.1$'$ take place. Moreover, instead of (4.13$'$) we have stronger restriction (4.13$''$). Therefore, by virtue of Theorems 4.1 and 4.1$'$, problem (4.1),(4.2) has the unique solution and it is classical. ■

*Remark* 4.4. When the matrix function $\mathcal{P}_2$ satisfies the Lappo-Danilevsky condition in the first argument, i.e. when

$$\mathcal{P}_2(x,y)\Big(\int_0^x \mathcal{P}_2(s,y)ds\Big) = \Big(\int_0^x \mathcal{P}_2(s,y)ds\Big)\mathcal{P}_2(x,y) \quad \text{for } (x,y) \in \mathcal{D}_{ab}, \tag{4.28}$$

we have

$$Z_2(x,y) = \exp\Big(\int_0^x \mathcal{P}_2(s,y)ds\Big).$$

Therefore the matrix function $M_0$ is calculated explicitly and conditions (4.13) and (4.21) may be verified more or less effectively. But if condition (4.28) is violated, then to verify these conditions effectively, we shall have to apply Lemmas 2.7-2.10.

Let us introduce the notation

$$A_0(s,x,y) = \Theta, \quad A_1(s,x,y) = \mathcal{P}_2(s,y),$$
$$A_{j+1}(s,x,y) = \int_s^x \mathcal{P}_2(\xi,y)A_j(s,\xi,y)d\xi \quad (j = 1,2,\dots),$$
$$M_{00}(y) = H_0(y), \quad M_{0j}(y) = H_0(y) +$$
$$+\int_0^a H(s,y)\mathcal{P}_2(s,y)\Big[E + \sum_{i=0}^{j-1}\int_0^s A_i(\xi,s,y)d\xi\Big]ds \quad (j = 1,2,\dots).$$

If the inequality

$$\det M_{0k-1}(y) \neq 0$$

holds for some $y \in [0,b]$ and natural $k$, then for every natural $m$ we assume

$$B_{1m}(s,x,y) = \Big[E + \sum_{i=0}^{m-1}\int_0^x A_i(\xi,x,y)d\xi\Big] \times$$
$$\times H_0^{-1}(y)H(s,y)\mathcal{P}_2(s,y),$$
$$B_{km}(s,x,y) = \Big[E + \sum_{i=0}^{m-1}\int_0^x A_i(\xi,x,y)d\xi\Big]M_{0k-1}^{-1}(y) \times$$
$$\times \int_s^a H(\xi,y)\mathcal{P}_2(\xi,y)A_{k-1}(s,\xi,y)d\xi \quad \text{for } k > 1$$

and

$$B_{km}^0(y) = \max_{0\le x\le a}\Big[\int_0^x |A_m(s,x,y) - B_{km}(s,x,y)|ds +$$

$$+\int_x^a |B_{km}(s,x,y)|ds\Big].$$

From Lemma 2.7 and Corollaries 4.1 and 4.1′ there follow the following assertions.

*Let there exist natural* $k$ *and* $m$ *such that the inequalities*

$$\det M_{0k-1}(y) \neq 1, \quad r(B^0_{km}(y)) < 1 \tag{4.29}$$

*hold almost everywhere in* $[0,b]$ *and*

$$\int_0^a \int_0^b \big\| \big(E - B^0_{km}(y)\big)^{-1} \big\| \, \|M^{-1}_{0k-1}(y)\| \Big( \|\mathcal{P}_0(x,y)\| +$$
$$+(1+\|\varphi_0'(x)\|)\|\mathcal{P}_1(x,y)\| + \|q(x,y)\| + \|\varphi_1(y)\|\Big) dxdy < +\infty.$$

*Then Problem* (4.1),(4.2) *has one and only one solution.*

′ *Let* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator and* $\varphi_i, \mathcal{P}_i$ $(i=0,1)$ *and* $q$ *satisfy conditions* (4.6′). *Moreover, let inequality* (4.29) *hold everywhere in* $[0,b]$. *Then problem* (4.1),(4.2) *has one and only one solution and this solution is classical.*

From Corollaries 4.1 and 4.1′ according to Lemmas 2.8, 2.9 and 2.10 we obtain respectively Corollaries 4.3 and 4.3′, 4.4 and 4.4′ and 4.5 and 4.5′.

*Let*

$$h(v)(y) = \big(v_i(a_i(y))\big)_{i=1}^n, \tag{4.30}$$

*where* $a_i : [0,b] \to [0,a]$ $(i=1,\dots,n)$ *are measurable functions. Moreover, let the matrix function*

$$A(y) = \operatorname*{ess\,sup}_{0\le x\le a} |\mathcal{P}_2(x,y)|$$

*satisfy the inequality*

$$r(A(y)) < \frac{\pi}{2a} \tag{4.31}$$

*almost everywhere in* $[0,b]$, *and*

$$\int_0^a \int_0^b \big\| \big(E - \frac{2a}{\pi} A(y)\big)^{-1} \big\| \Big( \|\mathcal{P}_0(x,y)\| +$$
$$+(1+\|\varphi_0'(x)\|)\|\mathcal{P}_1(x,y)\| + \|q(x,y)\| + \|\varphi_1(y)\|\Big) dxdy < +\infty. \tag{4.32}$$

*Then problem* (4.1),(4.2) *has one and only one solution.*

$'$ *Let conditions* $(4.6')$ *and* $(4.30)$ *hold, where* $a_i : [0,b] \to \to [0,a]$ $(i = 1, \ldots, n)$ *are continuous functions. Let, moreover, the matrix function*

$$A(y) = \max_{0 \le x \le a} |\mathcal{P}(x,y)|$$

*be such that*

$$r\big(A(y)\big) < \frac{\pi}{2a} \quad \textit{for} \quad 0 \le y \le b. \tag{4.33}$$

*Then problem* (4.1),(4.2) *has one and only one solution and this solution is classical.*

*Remark* 4.5. Condition (4.33) is optimal in the sense that it is impossible to replace it by the requirement that inequality (4.31) be fulfilled evrywhere in $[0,b]$ except at some finite number of points and that integral (4.32) be convergent. To convince ourselves that this is so consider the problem

$$\begin{gathered}\frac{\partial^2 u_1(x,y)}{\partial x \partial y} = \frac{\partial u_2(x,y)}{\partial y}, \\ \frac{\partial^2 u_2(x,y)}{\partial x \partial y} = -\omega_\alpha^2(y)\frac{\partial u_1(x,y)}{\partial y} + x\omega_\alpha^2(y),\end{gathered} \tag{4.34}$$

$$u_i(x,0) = 0 \quad (i = 1,2), \qquad \frac{\partial u_1(0,y)}{\partial y} = \frac{\partial u_2(a,y)}{\partial y} = 0, \tag{4.35}$$

where

$$\omega_\alpha(y) = \frac{\pi}{2a}\big[1 - \big(\tfrac{y}{b}\big)^\alpha\big], \quad \alpha \in (0,1].$$

In that case

$$\mathcal{P}_2(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega_\alpha^2(y) & 0 \end{pmatrix}, \quad A(y) = \begin{pmatrix} 0 & 1 \\ \omega_\alpha^2(y) & 0 \end{pmatrix}$$

and

$$r\big(A(y)\big) = \omega_\alpha(y).$$

Consequently, all conditions in Corollary $4.3'$ except (4.33) hold. Instead of condition (4.33) we have

$$r\big(A(y)\big) < \frac{\pi}{2a} \quad \text{for} \quad 0 < y \le b, \quad r\big(A(0)\big) = \frac{\pi}{2a}.$$

In addition,

$$\big(E - \frac{2a}{\pi}A(y)\big)^{-1} = \big(\tfrac{y}{b}\big)^{-\alpha}\Big[2 - \big(\tfrac{y}{b}\big)^\alpha\Big]^{-1}\begin{pmatrix} 1 & \frac{2a}{\pi} \\ \frac{2a}{\pi}\omega_\alpha^2(y) & 1 \end{pmatrix}$$
$$\text{for} \quad 0 < y \le b.$$

From the above it becomes evident that condition (4.32) holds for $\alpha \in (0,1)$ and is not fulfilled for $\alpha = 1$. According to Corollary 4.3, problem

(4.1),(4.2) has the unique solution $(u_i)_{i=1}^2$ for every $\alpha \in (0,1)$. By immediate verification we convince ourselves that

$$u_1(x,y) = -\int_0^y \frac{\sin(\omega_\alpha(t)x)}{\omega_\alpha(t)\cos(a\omega_\alpha(t))}dt + xy,$$
$$u_2(x,y) = -\int_0^y \frac{\cos(x\omega_\alpha(t))}{\cos(a\omega_\alpha(t))}dt + y,$$

and this solution is not classical because

$$\lim_{y\to 0}\frac{\partial u_i(x,y)}{\partial y} = -\infty \quad \text{for} \quad 0 < x < a \quad (i=1,2).$$

*Remark* 4.6. As been admitted above, for problem (4.34),(4.35) all conditions of Corollary 4.3 except (4.32) take place for $\alpha = 1$. Let us show that in this case problem (4.34),(4.35) has no solution. Suppose the converse holds, i.e. the problem does have a solution $(u_i)_{i=1}^2$. Assume

$$z(x,y) = \frac{\partial u_1(x,y)}{\partial y}.$$

Then for every $y \in (0,b)$ we shall have

$$\frac{\partial^2 z(x,y)}{\partial x^2} = -\omega_1^2(y)z(x,y) + x\omega_1^2(y),$$
$$z(0,y) = 0, \quad \frac{\partial z(x,y)}{\partial x}\Big|_{x=a} = 0.$$

Therefore

$$z(x,y) = -\frac{1}{\omega_1(y)\cos(a\omega_1(y))}\sin(\omega_1(y)x) + x$$

and

$$\frac{\partial u_1(x,y)}{\partial y} = -\frac{1}{\omega_1(y)\sin\left(\frac{\pi y}{2b}\right)}\sin(\omega_1(y)x) + x.$$

But this is impossible because for every $x \in (0,a]$ the function in the right-hand side is not summable in the second argument on the interval $[0,b]$. The obtained contradiction proves that problem (4.34),(4.35) has no solution. Thus it is impossible to omit condition (4.32) from Corollary 4.3.

*Remark* 4.7. When $\alpha = 1$ problem (4.34),(4.35) is an example of a problem of type (4.1),(4.2) which is unsolvable although the corresponding homogeneous problem has only the trivial solution.

*Let there exist a diagonal matrix function $A_0 \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$ such that*

$$\det\big(A_0(x,y)\big) \neq 0 \tag{4.36}$$

*almost for every $(x,y) \in \mathcal{D}_{ab}$ and the inequality*

$$r\big(A(y)\big) < 1 \tag{4.37}$$

*holds almost everywhere in* $[0,b]$, *where*

$$A(y)=\operatorname*{ess\,sup}_{0\le x\le a}|A_0^{-1}(x,y)\mathcal{P}_2(x,y)-E|.$$

*Let, moreover, every diagonal element of the matrix* $A_0(\cdot,y)$ *be a function of constant sign almost for every* $y\in[0,b]$ *and*

$$\int_0^a\int_0^b\Big\|\big(E-A(y)\big)^{-1}\Big(\int_0^a A_0(s,y)ds\Big)^{-1}\Big\|\Big(\|\mathcal{P}_0(x,y)\|+$$
$$+(1+\|\varphi_0'(x)\|)\|\mathcal{P}_1(x,y)\|+\|q(x,y)\|+\|\varphi_1(y)\|\Big)dxdy<+\infty. \tag{4.38}$$

*Then problem,* $(4.1),(4.2_1)$ *has one and only one solution.*

′ *Let conditions* $(4.6')$ *hold and there exist a diagonal matrix function* $A_0\in\mathbb{C}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ *such that*

$$\det\big(A_0(x,y)\big)\neq 0\quad \textit{for}\quad (x,y)\in\mathcal{D}_{ab} \tag{4.39}$$

*and*

$$r\big(A(y)\big)<1\quad \textit{for}\quad 0\le y\le b, \tag{4.40}$$

*where*

$$A(y)=\max_{0\le x\le a}|A_0^{-1}(x,y)\mathcal{P}_2(x,y)-E|.$$

*Then problem* (4.1),$(4.2_1)$ *has one and only one solution and this solution is classical.*

*Remark* 4.8. The example considered in Remark 4.1 shows that it is impossible to omit condition (4.38) from Corollary 4.4 and to replace condition (4.39) by the requirement for inequality (4.36) be fulfilled everywhere in $\mathcal{D}_{ab}$ except for one segment $\{(x,y_0):0\le x\le a\}$ for some $y_0\in[0,b]$.

*Remark* 4.9. Condition (4.40) is optimal in the sense that if it is violated even at one point while all other conditions of Corollary 4.4′ and condition (4.38) hold, problem (4.1),$(4.2_1)$ may have no classical solution. As an example verifying this fact, consider the problem

$$\begin{gathered}\frac{\partial^2u_1(x,y)}{\partial x\partial y}=\frac{\partial u_1(x,y)}{\partial y}+(1-\varepsilon_\alpha(y))^2\frac{\partial u_2(x,y)}{\partial y}-\varepsilon_\alpha(y),\\ \frac{\partial^2u_2(x,y)}{\partial x\partial y}=\frac{\partial u_1(x,y)}{\partial y}+\frac{\partial u_2(x,y)}{\partial y}+1,\end{gathered} \tag{4.41}$$

$$u_i(x,0)=0,\quad \frac{\partial u_i(0,y)}{\partial y}=\frac{\partial u_i(a,y)}{\partial y}\quad (i=1,2), \tag{4.42}$$

where

$$\varepsilon_\alpha(y)=\big(1-\frac{y}{b}\big)^\alpha,\quad \alpha\in(0,+\infty).$$

In that case

$$\mathcal{P}_2(x,y)=\begin{pmatrix} 1 & (1-\varepsilon_\alpha(y))^2 \\ 1 & 1 \end{pmatrix}, \quad A_0(x,y)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A(y)=\begin{pmatrix} 0 & (1-\varepsilon_\alpha(y))^2 \\ 1 & 0 \end{pmatrix}$$

and

$$r\big(A(y)\big)=1-\varepsilon_\alpha(y),$$

whence it is clear that all conditions of Corollary 4.4′ except (4.40) take place. Instead of (4.40) we have

$$r\big(A(y)\big)<1 \quad \text{for} \quad 0\le y<b, \quad r\big(A(b)\big)=1.$$

Besides,

$$(E-A(y))^{-1}=\frac{1}{\varepsilon_\alpha(y)(2+\varepsilon_\alpha(y))}\begin{pmatrix} 1 & (1-\varepsilon_\alpha(y))^2 \\ 1 & 1 \end{pmatrix}.$$

Consequently, condition (4.38) holds for $\alpha\in(0,1)$ while this condition is violated for $\alpha\ge 1$. Let us show that for $\alpha\in(0,1)$ the solution of problem (4.41),(4.42), whose existence and uniqueness follow from Corollary 4.4, is not classical and for $\alpha\ge 1$ this problem has no solution whatsoever. Indeed, let $(u_i)_{i=1}^2$ be the solution of problem (4.41),(4.42). Assume

$$z(x,y)=\frac{\partial u_1(x,y)}{\partial y}+(\varepsilon_\alpha(y)-1)\frac{\partial u_2(x,y)}{\partial y}.$$

Then

$$\frac{\partial z(x,y)}{\partial x}=\varepsilon_\alpha(y)z(x,y)-1,$$

$$z(0,y)=z(a,y).$$

Therefore

$$z(x,y)=\frac{1}{\varepsilon_\alpha(y)} \quad \text{for} \quad 0\le x\le a, \ \ 0\le y<b$$

and

$$\lim_{y\to b} z(x,y)=+\infty \quad \text{for} \quad 0\le x\le a.$$

Besides, for $\alpha\ge 1$ the function $z$ is not summable in $\mathcal{D}_{ab}$, which is impossible due to the absolute continuity of the functions $u_i$ $(i=1,2)$.

*Let* $\mathcal{P}_2(x,y)=(p_{2ij}(x,y))_{i,j=1}^n$ *and there exist functions* $\sigma_i:[0,b]\to\{-1,1\}$ $(i=1,\dots,n)$ *such that the real parts of eigenvalues of the matrix*

$$A(y)=(a_{ij}(y))_{i,j=1}^n,$$

*where*

$$a_{ii}(y)=\operatorname*{ess\,sup}_{0\le x\le a}(\sigma_i(y)p_{2ii}(x,y)), \quad a_{ij}(y)=\operatorname*{ess\,sup}_{0\le x\le a}|p_{2ij}(x,y)|$$

$$for \quad (i \neq j; \quad i,j = 1, \dots, n)$$

*are negative almost for all* $y \in [0,b]$ *and*

$$\int_0^b \|A^{-1}(y)\|(1 + \|\varphi_1(y)\|)dy < +\infty. \tag{4.43}$$

*Then problem* (4.1),(4.$2_1$) *has one and only one solution.*

$'$ *Let conditions* (4.6$'$) *hold,* $\mathcal{P}_2(x,y) = (p_{2ij}(x,y))_{i,j=1}^n$ *and there exist numbers* $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$ *such that the real parts of eigenvalues of the matrix*

$$A(y) = (a_{ij}(y))_{i,j}^n, \tag{4.44}$$

*where*

$$a_{ii}(y) = \max_{0 \le x \le a} (\sigma_i p_{2ii}(x,y)), \quad a_{ij}(y) = \max_{0 \le x \le a} |p_{2ij}(x,y)|$$
$$for \quad (i \neq j; \quad i,j = 1, \dots, n)$$

*are negative for all* $y \in [0,b]$. *Then problem* (4.1),(4.$2_1$) *has one and only one solution and this solution is classical.*

*Remark* 4.10. The restriction in Corollary 4.5$'$ imposed on the eigenvalues of matrix (4.44) is optimal and cannot be weakened. As an example let us consider problem (4.41),(4.42). If $\sigma_1 = \sigma_2 = -1$, then matrix (4.44) for system (4.41) has the form

$$A(y) = \begin{pmatrix} -1 & (1 - \varepsilon_\alpha(y))^2 \\ 1 & -1 \end{pmatrix}$$

and its eigenvalues

$$\lambda_1(y) = \varepsilon_\alpha(y) - 2, \quad \lambda_2(y) = -\varepsilon_\alpha(y)$$

satisfy the conditions

$$\lambda_1(y) < 0 \text{ for } 0 \le y \le b, \;\; \lambda_2(y) = -\varepsilon_\alpha(y) \;\; 0 \le y < b, \;\; \lambda_2(b) = 0$$

and condition (4.43) also holds for $\alpha \in (0,1)$. Nevertheless, problem (4.41), (4.42) has no classical solution for any $\alpha > 0$.

$$M_0(y) \equiv \Theta$$

*Let*

$$\mathcal{P}_i \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \quad (i=0,1,2), \quad q \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^n), \tag{4.45}$$

$$\varphi_0 \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad \varphi_1 \in \widetilde{\mathbb{C}}([0,b];\mathbb{R}^n), \tag{4.46}$$

*and* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator such that*

$$M_0(y) = \Theta \quad \textit{for} \quad 0 \le y \le b \tag{4.47}$$

*and*

$$\det\Big(\int_0^a M(s,y)Z_2^{-1}(s,y)[\mathcal{P}_0(s,y)+\mathcal{P}_1(s,y)\mathcal{P}_2(s,y)]Z_2(s,y)ds\Big) \neq 0 \quad \textit{for} \quad 0 \le y \le b. \tag{4.48}$$

*Then for the unique solvability of problem* (4.1),(4.2) *it is necessary and sufficient that*

$$\int_0^a M(s,0)Z_2^{-1}(s,0)[\mathcal{P}_0(s,0)\varphi_0(s)+\mathcal{P}_1(s,0)\varphi_0'(s)+ +q(s,0)]ds = \varphi_1(0). \tag{4.49}$$

*Proof.* Let $u$ be an arbitrary solution of system (4.1). Then by Lemma 3.3,

$$u(x,y) = Z_2(x,y)\Big[u(0,y)+\int_0^x Z_2^{-1}(s,y)Z_1(s,y)\varphi(s)ds\Big] + Z_2(x,y)\times \times \int_0^x\int_0^y Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+q(s,t)\big)ds\,dt \tag{4.50}$$

and

$$\frac{\partial}{\partial x}\big(Z_2^{-1}(x,y)u(x,y)\big) = Z_2^{-1}(x,y)Z_1(x,y)\times \times\Big[\varphi(x)+\int_0^y Z_1^{-1}(x,t)\big(\mathcal{P}(x,t)u(x,t)+q(x,t)\big)dt\Big], \tag{4.51}$$

where

$$\varphi(x) = \frac{\partial u(x,0)}{\partial x} - \mathcal{P}_2(x,0)u(x,0) \tag{4.52}$$

and

$$\mathcal{P}(x,y) = \mathcal{P}_0(x,y)+\mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_2(x,y)}{\partial y}.$$

In view of (4.50), from the obvious equality

$$Z_2^{-1}(x,y)\frac{\partial u(x,y)}{\partial y} = \frac{\partial}{\partial y}\big(Z_2^{-1}(x,y)u(x,y)\big) - \frac{\partial Z_2^{-1}(x,y)}{\partial y}u(x,y)$$

we obtain

$$Z_2^{-1}(x,y)\frac{\partial u(x,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}+\int_0^x Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)\varphi(s)ds+$$
$$+\int_0^x Z_2^{-1}(s,y)\big(\bar{\mathcal{P}}(s,y)u(s,y)+q(s,y)\big)ds+\int_0^x\int_0^y Z_2^{-1}(s,y)\mathcal{P}_1(s,y)\times$$
$$\times Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+q(s,t)\big)ds\,dt+v(x,y), \tag{4.53}$$

where

$$\bar{\mathcal{P}}(x,y)=\mathcal{P}_0(x,y)+\mathcal{P}_1(x,y)\mathcal{P}_2(x,y) \tag{4.54}$$

and

$$v(x,y)=-\int_0^x Z_2^{-1}(s,y)\frac{\partial\mathcal{P}_2(s,y)}{\partial y}u(s,y)ds+\int_0^x\frac{\partial Z_2^{-1}(s,y)}{\partial y}Z_1(s,y)\times$$
$$\times\Big[\varphi(s)+\int_0^y Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+q(s,t)\big)dt\Big]ds-\frac{\partial Z_2^{-1}(x,y)}{\partial y}u(x,y).$$

But

$$Z_2^{-1}(x,y)\frac{\partial\mathcal{P}_2(x,y)}{\partial y}=$$
$$=Z_2^{-1}(x,y)\Big[\frac{\partial}{\partial y}\big(\mathcal{P}_2(x,y)Z_2(x,y)\big)-\mathcal{P}_2(x,y)\frac{\partial Z_2(x,y)}{\partial y}\Big]Z_2^{-1}(x,y)=$$
$$=Z_2^{-1}(x,y)\Big[\frac{\partial^2 Z_2(x,y)}{\partial x\partial y}-\frac{\partial Z_2(x,y)}{\partial x}Z_2^{-1}(x,y)\frac{\partial Z_2(x,y)}{\partial y}\Big]Z_2^{-1}(x,y)=$$
$$=\Big[Z_2^{-1}(x,y)\frac{\partial}{\partial x}\big(\frac{\partial Z_2(x,y)}{\partial y}\big)+\frac{\partial Z_2^{-1}(x,y)}{\partial x}\cdot\frac{\partial Z_2(x,y)}{\partial y}\Big]Z_2^{-1}(x,y)=$$
$$=\frac{\partial}{\partial x}\Big(Z_2^{-1}(x,y)\frac{\partial Z_2(x,y)}{\partial y}\Big)Z_2^{-1}(x,y)=$$
$$=-\frac{\partial}{\partial x}\Big(\frac{\partial Z_2^{-1}(x,y)}{\partial y}Z_2(x,y)\Big)Z_2^{-1}(x,y).$$

If we take into account the above equality and identity (4.51), we shall have

$$v(x,y)=\int_0^x\frac{\partial}{\partial s}\Big(\frac{\partial Z_2^{-1}(s,y)}{\partial y}Z_2(s,y)\Big)Z_2^{-1}(s,y)u(s,y)ds+$$
$$+\int_0^x\frac{\partial Z_2^{-1}(s,y)}{\partial y}Z_2(s,y)\frac{\partial}{\partial s}\big(Z_2^{-1}(s,y)u(s,y)\big)ds-\frac{\partial Z_2^{-1}(x,y)}{\partial y}u(x,y)=$$
$$=\int_0^x\frac{\partial}{\partial s}\Big(\frac{\partial Z_2^{-1}(s,y)}{\partial y}u(s,y)\Big)ds-\frac{\partial Z_2^{-1}(x,y)}{\partial y}u(x,y)=$$
$$=\frac{\partial Z_2^{-1}(0,y)}{\partial y}u(0,y)=0. \tag{4.55}$$

Therefore from (4.53) we get

$$Z_2^{-1}(x,y)\frac{\partial u(x,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y} + \int_0^x Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)\varphi(s)ds +$$
$$+ \int_0^x Z_2^{-1}(s,y)\big(\bar{\mathcal{P}}(s,y)u(s,y) + q(s,y)\big)ds + \int_0^x \int_0^y Z_2^{-1}(s,y) \times$$
$$\times \mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t) + q(s,t)\big)ds\,dt. \qquad (4.56)$$

According to representations (4.8) and (4.56) and condition (4.47), $u$ satisfies boundary condition (4.2) if and only if

$$\varphi(x) = \varphi_0'(x) - \mathcal{P}_2(x,0)\varphi_0(x), \qquad (4.57)$$
$$u(0,0) = \varphi_0(0) \qquad (4.58)$$

and

$$\varphi_1(y) = \int_0^a M(s,y)Z_2^{-1}(s,y)\big(\mathcal{P}_1(s,y)Z_1(s,y)\varphi(s) + q(s,y)\big)ds +$$
$$+ \int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)u(s,y)ds +$$
$$+ \int_0^y \int_0^a M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y) \times$$
$$\times Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t) + q(s,t)\big)ds\,dt \quad \text{for} \quad 0 \le y \le b. \qquad (4.59)$$

On the other hand, if conditions (4.57) and (4.59) hold, then by (4.52) we have

$$u(x,0) = \varphi_0(x) + Z_2(x,0)(u(0,0) - \varphi_0(0))$$

and

$$\varphi_1(0) = \int_0^a M(s,0)Z_2^{-1}(s,0)[\mathcal{P}_0(s,0)\varphi_0(s) + \mathcal{P}_1(s,0)\varphi_0'(s) + q(s,0)]ds +$$
$$+ \Big(\int_0^a M(s,0)Z_2^{-1}(s,0)\bar{\mathcal{P}}(s,0)Z_2(s,0)ds\Big)(u(0,0) - \varphi_0(0)),$$

whence it is clear that condition (4.49) is necessary and sufficient for equality (4.58) to be fulfilled.

Thus we have proved that condition (4.49) is necessary for problem (4.1),(4.2) to be solvable. Moreover, if this condition is fulfilled, then the solution $u$ of system (4.1) satisfies boundary conditions (4.2) if and only if

$$\frac{\partial u(x,0)}{\partial x} - \mathcal{P}_2(x,0)u(x,0) = \varphi(x) \quad \text{for} \quad 0 \le x \le a \qquad (4.60_1)$$

and equality (4.59) holds, where $\varphi$ is the vector function given by (4.57).

Thus to complete the proof we have to show that system (4.1) has one and only one solution satisfying (4.59) and $(4.60_1)$.

According to (4.50),

$$\int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)u(s,y)ds = Q_0(y)u(0,y) +$$

$$+\int_0^y \Big(\int_0^a \int_0^x Q_1(x,y,s)Z_1^{-1}(s,t)\mathcal{P}(s,t)u(s,t)ds\,dx\Big)dt + \varphi_2(y),$$

where

$$Q_0(y) = \int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)Z_2(s,y)ds,$$

$$Q_1(x,y,s) = M(x,y)Z_2^{-1}(x,y)\bar{\mathcal{P}}(x,y)Z_2(x,y)Z_2^{-1}(s,y)Z_1(s,y)$$

and

$$\varphi_2(y) = \int_0^a \int_0^x Q_1(x,y,s)\Big[\varphi(s) + \int_0^y Z_1^{-1}(s,t)q(s,t)dt\Big]ds\,dx.$$

But

$$\int_0^a \int_0^x Q_1(x,y,s)Z_1^{-1}(s,t)\mathcal{P}(s,t)u(s,t)ds\,dx = \int_0^a Q_2(y,s,t)u(s,t)ds,$$

where

$$Q_2(y,s,t) = \Big(\int_s^a Q_1(x,y,s)dx\Big)Z_1^{-1}(s,t)\mathcal{P}(s,t).$$

Therefore

$$\int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)u(s,y)ds =$$

$$= Q_0(y)u(0,y) + \int_0^y \int_0^a Q_2(y,s,t)u(s,t)ds\,dt + \varphi_2(y).$$

If we take onto account the formulas above and condition (4.48), then equality (4.59) will take the form

$$u(0,y) = \psi(y) + \int_0^y \int_0^a Q(y,s,t)u(s,t)ds\,dt \quad \text{for} \quad 0 \le y \le b, \qquad (4.60_2)$$

where

$$\psi(y) = Q_0^{-1}(y)\Big[\varphi_1(y) - \varphi_2(y) -$$

$$-\int_0^a M(s,y)Z_2^{-1}(s,y)\big(\mathcal{P}_1(s,y)Z_1(s,y)\varphi(s) + q(s,y)\big)ds\Big] -$$

$$-Q_0^{-1}(y)\int_0^y \int_0^a M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)q(s,t)ds\,dt$$

and

$$Q(y,s,t) = -Q_0^{-1}(y)\big[Q_2(y,s,t) +$$

$$+M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}(s,t)\big].$$

By conditions (4.45),(4.46) and (4.48) and Lemmas $2.2_2$ and $2.3_2$, the vector and matrix functions $\varphi$, $\psi$ and $Q$ satisfy the conditions of Lemma 3.5′. Therefore problem (4.1),($4.60_1$),($4.60_2$) has one and only one solution. ■

′ *Let $\mathcal{P}_i$ $(i = 0, 1, 2)$ and $q$ be continuous and have a continuous partial derivative in the second argument, $\varphi_0$ and $\varphi_1$ be continuously differentiable and $h : \widetilde{\mathbb{C}}([0, a]; \mathbb{R}^n) \to \mathbb{C}^1([0, b]; \mathbb{R}^n)$ be a linear continuous operator satisfying conditions* (4.47) *and* (4.48). *Then the fulfilment of* (4.49) *is the necessary and sufficient condition for problem* (4.1),(4.2) *to be uniquely solvable and to have a classical solution.*

This theorem is proved using the same arguments as in proving Theorem 4.2, but instead of Lemmas $2.3_3$ and 3.5′ we apply Lemmas $2.3_4$ and 3.5″.

Consider the case when boundary conditions (4.2) have the form

$$u(x, 0) = \varphi_0(x), \quad \sum_{k=1}^{m} \Phi_k(y) \frac{\partial u(a_k, y)}{\partial y} = \varphi_1(y). \tag{4.61}$$

In that case

$$M_0(y) = \sum_{k=1}^{m} \Phi_k(y) Z_2(a_k, y)$$

and

$$M(x, y) = \sum_{k=1}^{m} \Phi_k(y) Z_2(a_k, y) \chi_k(x),$$

where

$$\chi_k(x) = \begin{cases} 1 & \text{for } 0 \le x \le a_k \\ 0 & \text{for } a_k < x \le a \end{cases}.$$

Therefore Theorems 4.2 and 4.2′ imply several assertions.

*Let conditions* (4.45) *and* (4.46) *be fulfilled and* $\Phi_k \in \widetilde{\mathbb{C}}_\infty([0, b]; \mathbb{R}^{n\times n})$ *and* $a_k \in [0, a]$ $(k = 1, \dots, m)$ *be such that*

$$\sum_{k=1}^{m} \Phi_k(y) Z_2(a_k, y) = \Theta \quad \textit{for} \quad 0 \le y \le b \tag{4.62}$$

*and*

$$\det \Big( \sum_{k=1}^{m} \Phi_k(y) Z_2(a_k, y) \int_0^{a_k} Z_2^{-1}(s, y) [\mathcal{P}_0(s, y) +$$

$$+ \mathcal{P}_1(s, y) \mathcal{P}_2(s, y)] Z_2(s, y) ds \Big) \ne 0 \quad \textit{for} \quad 0 \le y \le b. \tag{4.63}$$

*Then problem* (4.1), (4.61) *is uniquely soluvable if and only if*

$$\sum_{k=1}^{m} \Phi_k(0) Z_2(a_k, 0) \int_0^{a_k} Z_2^{-1}(s, 0) \times$$

$$\times[\mathcal{P}_0(s,0)\varphi_0(s)+\mathcal{P}_1(s,0)\varphi_0'(s)+q(s,0)]ds=\varphi_1(0). \tag{4.64}$$

′ *Let* $\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\varphi_0$, $\varphi_1$ *and* $\Phi_k$ $(k==1,\dots,m)$ *be continuously differentiable,* $a_k\in[0,a]$ $(k=1,\dots,m)$, *and conditions* (4.62) *and* (4.63) *hold. Then then fulfilment of equality* (4.64) *is necessary and sufficient for problem* (4.1),(4.61) *to be uniquely solvable and to have the classical solution.*

Corollaries 4.6 and 4.6′ for problem (4.1),($4.2_1$) take the form of

*Let conditions* (4.45),(4.46) *hold,*

$$Z_2(a,y)=E \quad for \quad 0\le y\le b \tag{4.65}$$

*and*

$$\det\Big(\int_0^a Z_2^{-1}(s,y)[\mathcal{P}_0(s,y)+\mathcal{P}_1(s,y)\mathcal{P}_2(s,y)]Z_2(s,y)ds\Big)\neq 0 \quad for \quad 0\le y\le b. \tag{4.66}$$

*Then problem* (4.1),($4.2_1$) *is uniquely solvable if and only if*

$$\int_0^a Z_2^{-1}(s,0)[\mathcal{P}_0(s,0)\varphi_0(s)+\mathcal{P}_1(s,0)\varphi_0'(s)+q(s,0)]ds=\varphi_1(0). \tag{4.67}$$

′ *Let* $\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\varphi_0$ *and* $\varphi_1$ *be continuously differentiable and conditions* (4.65) *and* (4.66) *hold. Then problem* (4.1),($4.2_1$) *is uniquely solvable and its solution is classical if and only if equality* (4.64) *takes place.*

Problem (4.1),(4.2) is ill-posed under conditions (4.47) and (4.48) due to the fact that its solvability may be violated at arbitrarily small perturbations either of coefficients of the system under consideration or of boundary conditions. But problem (4.1),(4.3) is free from such a deficiency. Namely, the following theorem is valid.

*Let*

$$\mathcal{P}_i\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2),\quad q\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n}),$$

$\psi_0$ *be summable,* $\psi_1$ *absolutely continuous and* $h:\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)\to\widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator satisfying conditions* (4.47) *and* (4.48). *Then problem* (4.1),(4.3) *has one and only one solution.*

*Proof.* First we assume that problem (4.1),(4.3) has a solution $u$. Then $u$ is a solution of problem (4.1),(4.2), where

$$\varphi_0(x)=Z_2(x,y)u(0,0)+z_0(x),\quad \varphi_1(y)=\psi_1(y)$$

and $z_0$ is the solution of the system of ordinary differential equations

$$\frac{dz(x)}{dx} = \mathcal{P}_2(x,0)z(x) + \psi_0(x)$$

with the initial condition

$$z(0) = 0.$$

According to Theorem 4.2,

$$Q_0(0)u(0,0) = c_0,$$

where

$$Q_0(0) = \int_0^a M(s,0)Z_2^{-1}(s,0)[\mathcal{P}_0(s,0) + \mathcal{P}_1(s,0)\mathcal{P}_2(s,0)]Z_2(s,0)ds,$$

$$c_0 = \psi_1(0) - \int_0^a M(s,0)Z_2^{-1}(s,0)[\mathcal{P}_0(s,0)z_0(s) + \mathcal{P}_1(s,0)z_0'(s) + q(s,0)]ds.$$

However, by condition (4.48),

$$\det Q_0(0) \neq 0.$$

Therefore

$$u(0,0) = Q_0^{-1}(0)c_0.$$

Thus we have proved that every solution of problem (4.1),(4.3) is a solution of problem (4.1),(4.2), where

$$\varphi_0(x) = Z_2(x,0)Q_0^{-1}(0)c_0 + z_0(x), \quad \varphi_1(y) = \psi_1(y). \tag{4.68}$$

The converse statement can be easily verified. If equalities (4.68) take place, then every solution of problem (4.1),(4.2) is a solution of problem (4.1),(4.3). Consequently, problem (4.1),(4.3) is equivalent to problem (4.1),(4.2), where $\varphi_0$ and $\varphi_1$ are given by equalities (4.68). But from (4.68) there follows equality (4.49). Therefore by Theorem 4.2 we conclude that in that case problem (4.1),(4.2) has one and only one solution. ■

In the same way we can prove

′ *Let* $\mathcal{P}_i$ $(i = 0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\psi_0$ *be continuous,* $\psi_1$ *be continuously differentiable and* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}^1([0,b];\mathbb{R}^n)$ *be a linear continuous operator satisfying conditions* (4.47) *and* (4.48). *Then problem* (4.1),(4.3) *has one and only one solution and this solution is classical.*

When boundary conditions (4.3) have the form

$$\frac{\partial u(x,0)}{\partial x} - \mathcal{P}_0(x,0)u(x,0) = \psi_0(x), \quad \sum_{k=1}^m \Phi_k(y)\frac{\partial u(a_k,y)}{\partial y} = \varphi_1(y), \tag{4.69}$$

Theorems 4.3 and 4.3′ result in the following statements.

*Let*

$$\mathcal{P}_i \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n}),$$

*$\psi_0$ be summable, $\psi_1$ absolutely continuous and $\Phi_k \in \widetilde{\mathbb{C}}_{\infty}([0,b];\mathbb{R}^{n\times n})$ and $a_k \in [0,a]$ $(k=1,\dots,m)$ be such that conditions* (4.62) *and* (4.63) *hold. Then problem* (4.1),(4.69) *has one and only one solution.*

$'$ *Let $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ be continuous and have a continuous partial derivative in the second argument. Let $\psi_0$ be continuous, $\psi_1$ continuously differentiable and $\Phi_k \in \mathbb{C}^1([0,b];\mathbb{R}^{n\times n})$ and $a_k \in [0,a]$ $(k = = 1,\dots,m)$ be such that conditions* (4.62) *and* (4.63) *take place. Then problem* (4.1),(4.69) *has one and only one solution and this solution is classical.*

*Let*

$$\mathcal{P}_i \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n}),$$

*$\psi_0$ be summable, $\psi_1$ be absolutely continuous and conditions* (4.65) *and* (4.66) *hold. Then problem* (4.1),($4.3_1$) *has one and only one solution.*

$'$ *Let $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ be continuous and have a continuous partial derivative in the second argument, $\psi_0$ be continuous, $\psi_1$ continuously differentiable and conditions* (4.65),(4.66) *take place. Then problem* (4.1),($4.3_1$) *has one and only one solution and this solution is calssical.*

Condition (4.48), appearing in Theorem 4.3 and its corollaries, is optimal in the sense that it cannot be weakened. As an example, let us consider problem (4.1),($4.3_1$) for $\mathcal{P}_2(x,y)\equiv 0$, i.e. when it has the form

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + q(x,y), \tag{4.70}$$

$$\frac{\partial u(x,0)}{\partial x} = \psi_0(x), \qquad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y} + \psi_1(y) \tag{4.71}$$

For this problem we have

*If*

$$\mathcal{P}_0 \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

*then fulfilment of the inequality*

$$\det\Big(\int_0^a \mathcal{P}_0(s,y)ds\Big) \neq 0 \quad \textit{for} \quad 0 \le y \le b$$

*is necessary and sufficient for problem* (4.70),(4.71) *to be uniquely solvable for every $\mathcal{P}_1, q, \psi_0$ and $\psi_1$ satisfying the conditions*

$$\begin{gathered}\mathcal{P}_1 \in \widetilde{\mathbb{C}}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \quad q \in \widetilde{\mathbb{C}}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n}),\\ \psi_0 \in L([0,a];\mathbb{R}^n), \ \ \psi_1 \in \widetilde{\mathbb{C}}([0,b];\mathbb{R}^n).\end{gathered} \tag{4.72}$$

*Proof.* The sufficiency follows from Corollary 4.9. Thus we have to show that if for any $y_0 \in [0,b]$

$$\det\Big(\int_0^a \mathcal{P}_0(s,y_0)ds\Big) = 0,$$

then there exist matrix and vector functions which satisfy conditions (4.72) and for which problem (4.70),(4.71) has no solution. Indeed, choose $c \in \mathbb{R}^n$ such that the system of algebraic equations

$$\Big(\int_0^a \mathcal{P}_0(s,y_0)ds\Big)z = c \tag{4.73}$$

is unsolvable and assume

$$\mathcal{P}_1(x,y) = \int_0^x \mathcal{P}_0(s,y)ds, \;\; q(x,y) = -\frac{c}{a}, \;\; \psi_0(x) = 0, \;\; \psi_1(y) = 0. \tag{4.74}$$

Suppose that for such $\mathcal{P}_1, q, \psi_0$, and $\psi_1$ problem (4.70),(4.71) has the solution $u$. Then

$$\frac{\partial}{\partial x}\Big(\frac{\partial u(x,y)}{\partial y}\Big) = \frac{\partial}{\partial x}\Big(\int_0^x \mathcal{P}_0(s,y)ds\Big)u(x,y)\Big) - \frac{c}{a}.$$

Integrating this identity with respect to $x$ from 0 to $a$ and taking into account conditions (4.71) and (4.74), we obtain

$$\Big(\int_0^a \mathcal{P}_0(s,y_0)ds\Big)u(a,y_0) = c.$$

But this is impossible because of the fact that system (4.73) is unsolvable. The obtained contradiction shows that if $\mathcal{P}_1, q, \psi_0$ and $\psi_1$ are given by equalities (4.74), then problem (4.70),(4.71) has no solution despite the fact that conditions (4.72) hold. ■

*Let*

$$\mathcal{P}_2 \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$

$$\operatorname{mes} I_{M_0} = b, \quad \int_0^b (1 + \|\psi_1(y)\|)\|M_0^{-1}(y)\|dy < +\infty. \tag{4.75}$$

*Then problem* (4.1),(4.4) *has one and only one generalized solution.*

*Proof.* Let $u$ be an arbitrary generalized solution of system (4.1). Then, according to Lemma 3.3,

$$u(x,y) = Z_2(x,y)\Big[u(0,y) + \int_0^x Z_2^{-1}(s,y)Z_1(s,y)v_1(s)ds\Big] + Z_2(x,y) \times$$
$$\times \int_0^x\int_0^y Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t) + q(s,t)\big)dsdt, \tag{4.76}$$

where

$$\mathcal{P}(x,y)=\mathcal{P}_0(x,y)+\mathcal{P}_1(x,y)\mathcal{P}_2(x,y)-\frac{\partial\mathcal{P}_2(x,y)}{\partial y},$$
$$v_1(x)=\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x}-\mathcal{P}_2(x,y)u(x,y)\Big).$$

From this and representation (4.8) it becomes clear that boundary conditions (4.4) are fulfilled if and only if

$$v_1(x)=\psi_0(x)$$

and

$$\psi_1(y)=M_0(y)u(0,y)+\psi_2(y)+\int_0^y\int_0^a Q_1(y,s,t)u(s,t)ds\,dt,$$
$$\psi_2(y)=\int_0^a M(s,y)Z_2^{-1}(s,y)Z_1(s,y)\Big[\psi_0(s)+\int_0^y Z_1^{-1}(s,t)q(s,t)dt\Big]ds,$$
$$Q_1(y,s,t)=M(s,y)Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}(s,t).$$

Moreover, as follows from Lemmas $2.2_2$ and $2.3_1$,

$$\psi_2\in L_\infty([0,b];\mathbb{R}^{n\times n}),\ \ Q_1\in L_\infty([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n}). \tag{4.77}$$

Obviously, problem (4.1),(4.4) is equivalent to the problem of finding the generalized solution of system (4.1), satisfying the boundary conditions

$$\begin{aligned}&\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial y}-\mathcal{P}_2(x,y)u(x,y)\Big)=\psi_0(x),\\ &u(0,y)=\psi(y)+\int_0^y\int_0^a Q(y,s,t)u(s,t)ds\,dt,\end{aligned} \tag{4.78}$$

where

$$\psi(y)=M_0^{-1}(y)(\psi_1(y)-\psi_2(y)),$$
$$Q(y,s,t)=-M_0^{-1}(y)Q_1(y,s,t).$$

However, by conditions (4.75) and (4.77),

$$\psi\in L([0,b];\mathbb{R}^n),\quad Q\in L([0,b]\times\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$
$$\|Q(y,s,t)\|\le\eta(y)\quad\text{for}\ \ y\in[0,b],\ (s,t)\in\mathcal{D}_{ab}$$

and

$$\eta\in L([0,b];\mathbb{R}).$$

Therefore, by Lemma 3.5 problem (4.1),(4.78) has one and only one generalized solution. ■

*′ Let*

$$\mathcal{P}_2 \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}), \quad \psi_1 \in \widetilde{\mathbb{C}}([0,b]; \mathbb{R}^n),$$

*and* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \widetilde{\mathbb{C}}_{\infty}([0,b]; \mathbb{R}^n)$ *be a linear continuous operator such that*

$$I_{M_0} = [0,b]. \tag{4.79}$$

*Then problem* (4.1),(4.4) *has the unique generalized solution and this solution is absolutely continuous.*

*″ Let* $\mathcal{P}_i$ $(i = 0,1,2)$ *and* $q$ *be continuous,* $\mathcal{P}_2$ *have a continuous partial derivative in the second argument,* $\psi_0$ *be continuous,* $\psi_1$ *be continuously differentiable and* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \mathbb{C}^1([0,b]; \mathbb{R}^n)$ *be a linear continuous operator, satisfying condition* (4.79). *Then problem* (4.1),(4.4) *has a unique generalized solution which is classical.*

Theorem 4.4′ (Theorem 4.4″) can be proved similarly to Theorem 4.4 but in that case instead of Lemmas $2.3_1$ and 3.5 we apply Lemmas $2.3_3$ and 3.5′ (Lemmas $2.3_4$ and 3.5″).

*Remark* 4.11. The effective conditions guaranteeing the fulfilment of condition (4.75) (Condition (4.79)) are given in the above proven Corollaries 4.2-4.5 (Corollaries 4.2′-4.5′).

*Remark* 4.12. The restrictions imposed on the operator $h$ in Theorem 4.4′ are optimal in the sense that they cannot be weakened. As an example, consider the problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0, \tag{4.80}$$

$$\lim_{y\to 0} \frac{\partial u(x,y)}{\partial x} = \frac{1}{a}, \quad u(a,y) = (1+y^{\alpha})u(0,y), \tag{4.81}$$

where $\alpha \in (0,1)$. In that case $h(v)(y) = v(a) - (1+y^{\alpha})v(0)$ and

$$M_0(y) = -y^{\alpha}.$$

Consequently, the operator $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \widetilde{\mathbb{C}}([0,b]; \mathbb{R}^n)$ is continuous and satisfies condition (4.75). On the other hand, problem (4.80),(4.81) has the unique generalized solution

$$u(x,y) = y^{-\alpha} + \frac{x}{a},$$

which is not absolutely continuous.

Finally, let us pass to the investigation of problem (4.1),(4.4) when $M_0(y) \equiv \Theta$ and $h$ acts from $\widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n)$ to $\widetilde{\mathbb{C}}_{\infty}([0,b]; \mathbb{R}^n)$. Let us introduce the matrix functions

$$\bar{\mathcal{P}}(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y)$$

and

$$\bar{M}(y) = H_0'(y) + \int_0^a \Big[\frac{\partial H(s,y)}{\partial y} \cdot \frac{\partial Z_2(s,y)}{\partial s} +$$
$$+ M(s,y) Z_2^{-1}(s,y) \bar{\mathcal{P}}(s,y) Z_2(s,y)\Big] ds. \tag{4.82}$$

*Let*

$$\mathcal{P}_2 \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}), \quad \psi_1 \in \widetilde{\mathbb{C}}([0,b]; \mathbb{R}^n)$$

*and* $h : \widetilde{\mathbb{C}}([0,a]; \mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b]; \mathbb{R}^n)$ *be a linear continuous operator such that*

$$M_0(y) = \Theta \quad \textit{for} \quad 0 \le y \le b,$$
$$\operatorname{mes} I_{\bar{M}} = b, \quad \int_0^b (1 + \|\psi_1'(y)\|) \|\bar{M}^{-1}(y)\| dy < +\infty. \tag{4.83}$$

*Then for the existence and uniqueness of a generalized solution of problem* (4.1), (4.4) *it is necessary and sufficient that*

$$\psi_1(0) = \int_0^a M(s,0) Z_2^{-1}(s,0) \psi_0(s) ds. \tag{4.84}$$

*Proof.* In view of representation (4.10) and condition $M_0(y) \equiv \Theta$, we have

$$H_0(y) + \int_0^a H(s,y) \frac{\partial Z_2(s,y)}{\partial s} ds = \Theta \quad \text{for} \quad 0 \le y \le b. \tag{4.85}$$

If we differentiate this identity, then by virtue of Lemmas $2.1_3$ and $2.2_2$ we obtain

$$H_0'(y) + \int_0^a \frac{\partial H(s,y)}{\partial y} \cdot \frac{\partial Z_2(s,y)}{\partial s} da = - \int_0^a H(s,y) \frac{\partial^2 Z_2(s,y)}{\partial s \partial y} ds. \tag{4.86}$$

Let $u$ be an arbitrary generalized solution of system (4.1) and

$$\bar{u}(x,y) = u(x,y) - Z_2(x,y) u(0,y).$$

Then, according to Lemma 3.3, representation (4.76) is valid and, consequently,

$$Z_2^{-1}(x,y) \bar{u}(x,y) = \int_0^x Z_2^{-1}(s,y) Z_1(s,y) v_1(s) ds +$$
$$+ \int_0^x \int_0^y Z_2^{-1}(s,y) Z_1(s,y) Z_1^{-1}(s,t) \big(\mathcal{P}(s,t) u(s,t) + q(s,t)\big) ds\, dt, \tag{4.87}$$

where

$$v_1(x) = \lim_{y \to 0} \Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y) u(x,y)\Big),$$
$$\mathcal{P}(x,y) = \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y) \mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_2(x,y)}{\partial y}.$$

Proceeding from this representation in a way which is similar to that of proving equality (4.56), we can show that

$$Z_2^{-1}(x,y)\frac{\partial\bar u(x,y)}{\partial y}=\int_0^x Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)v_1(s)ds+$$
$$+\int_0^x Z_2^{-1}(s,y)\big(\bar{\mathcal{P}}(s,y)u(s,y)+q(s,y)\big)ds+$$
$$+\int_0^x\int_0^y Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+$$
$$+q(s,t)\big)ds\,dt-Z_2^{-1}(x,y)\frac{\partial Z_2(x,y)}{\partial y}u(0,y). \tag{4.88}$$

According to representations (4.7),(4.10) and condition (4.85), we obtain

$$h(u(\cdot,y))(y)=h(\bar u(\cdot,y))(y)=\int_0^a H(s,y)\frac{\partial\bar u(s,y)}{\partial s}ds=$$
$$=\int_0^a M(s,y)\frac{\partial}{\partial s}\big(Z_2^{-1}(s,y)u(s,y)\big)ds.$$

Therefore

$$h(u(\cdot,0))(0)=\int_0^a M(s,0)\frac{\partial}{\partial s}\big(Z_2^{-1}(s,0)u(s,0)\big)ds=$$
$$=\int_0^a M(s,0)Z_2^{-1}(s,0)v_1(s)ds$$

and

$$\frac{d}{dy}h(u(\cdot,y))(y)=\int_0^a\frac{\partial H(s,y)}{\partial y}\cdot\frac{\partial\bar u(s,y)}{\partial s}ds+h\Big(\frac{\partial\bar u(\cdot,y)}{\partial y}\Big)(y),$$

whence it is clear that boundary conditions (4.4) are fulfilled if and only if

$$v_1(x)=\psi_0(x), \tag{4.89}$$

$$\int_0^a\frac{\partial H(s,y)}{\partial y}\cdot\frac{\partial\bar u(s,y)}{\partial s}ds+h\Big(\frac{\partial\bar u(\cdot,y)}{\partial y}\Big)(y)=\psi_1'(y) \tag{4.90}$$

and equality (4.84) holds.

In view of (4.8),(4.86) and (4.89) from (4.88) we obtain

$$h\Big(\frac{\partial\bar u(\cdot,y)}{\partial y}\Big)(y)=\int_0^a M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)\psi_0(s)ds+$$
$$+\int_0^a M(s,y)Z_2^{-1}(s,y)\big(\bar{\mathcal{P}}(s,y)u(s,y)+q(s,y)\big)ds+$$
$$+\int_0^y\int_0^a M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)\big(\mathcal{P}(s,t)u(s,t)+$$
$$+q(s,t)\big)ds\,dt-\Big(\int_0^a H(s,y)\frac{\partial^2 Z_2(s,y)}{\partial s\partial y}ds\Big)u(0,y)=\psi_2(y)+$$

$$+\int_0^y\int_0^a Q_1(y,s,t)u(s,t)ds\,dt+\int_0^a M(s,y)Z_2^{-1}(s,y)\times\bar{\mathcal{P}}(s,y)\times$$
$$\times u(s,y)ds+\Big[H_0'(y)+\int_0^a\frac{\partial H(s,y)}{\partial y}\cdot\frac{\partial Z_2(s,y)}{\partial s}ds\Big]u(0,y),\quad (4.91)$$

where

$$\psi_2(y)=\int_0^a M(s,y)Z_2^{-1}(s,y)[\mathcal{P}_1(s,y)Z_1(s,y)\psi_0(s)+q(s,y)]ds+$$
$$+\int_0^y\int_0^a M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)q(s,t)ds\,dt,$$
$$Q_1(y,s,t)=M(s,y)Z_2^{-1}(s,y)\mathcal{P}_1(s,y)Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}(s,t).$$

But according to (4.76),

$$\int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)u(s,y)ds=$$
$$=\Big[\int_0^a M(s,y)Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)Z_2(s,y)ds\Big]u(0,y)+\psi_3(y)+$$
$$+\int_0^y\int_0^a\Big(\int_s^a Q_2(x,y,s)dx\Big)Z_1^{-1}(s,t)\mathcal{P}(s,t)u(s,t)ds\,dt,\quad (4.92)$$

where

$$Q_2(x,y,s)=M(x,y)Z_2^{-1}(x,y)\bar{\mathcal{P}}(x,y)Z_2(x,y)Z_2^{-1}(s,y)Z_1(s,y),$$
$$\psi_3(y)=\int_0^a\int_0^x Q_2(x,y,s)\Big[\psi_0(s)+\int_0^y Z_1^{-1}(s,t)q(s,t)dt\Big]ds\,dx.$$

On the other hand, in view of (4.87),

$$\int_0^a\frac{\partial H(s,y)}{\partial y}\cdot\frac{\partial\bar{u}(s,y)}{\partial s}ds=\psi_4(y)+\int_0^y\int_0^a Q_3(y,s,t)u(s,t)ds\,dt,\quad (4.93)$$

where

$$\psi_4(y)=\int_0^a\frac{\partial H(x,y)}{\partial y}\Big[Z_1(x,y)\psi_0(x)+$$
$$+\frac{\partial Z_2(x,y)}{\partial x}\int_0^x Z_2^{-1}(s,y)Z_1(s,y)\psi_0(s)ds\Big]dx+$$
$$+\int_0^y\int_0^a\frac{\partial H(x,y)}{\partial y}\Big[Z_1(x,y)Z_1^{-1}(x,t)q(x,t)+$$
$$+\frac{\partial Z_2(x,y)}{\partial x}\int_0^x Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)q(s,t)ds\Big]dx\,dt,$$
$$Q_3(y,s,t)=\Big[\int_s^a\frac{\partial H(x,y)}{\partial y}\cdot\frac{\partial Z_2(x,y)}{\partial x}Z_2^{-1}(s,t)dx+\frac{\partial H(s,y)}{\partial y}\Big]\times$$
$$\times Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}(s,t).$$

From (4.89)-(4.93) we obtain equalities (4.78), where

$$\psi(y) = \bar{M}^{-1}(y)[\psi_1'(y) - \psi_2(y) - \psi_3(y) - \psi_4(y)],$$
$$Q(y,s,t) = -\bar{M}^{-1}(y)\Big[Q_1(y,s,t) +$$
$$+\Big(\int_s^a Q_2(x,y,s)dx\Big) Z_1^{-1}(s,t)\mathcal{P}(s,t) + Q_3(y,s,t)\Big].$$

Thus we have proved that condition (4.84) is necessary for the solvability of problem (4.1),(4.4). Moreover, if it is fulfilled, then problem (4.1),(4.4) is equivalent to problem (4.1),(4.78). On the other hand, by virtue of Lemmas $2.1_3$, $2.2_2$ and $2.3_3$ and condition (4.83), the vector and matrix functins, $\psi$ and $Q$ respectively, satisfy conditions of Lemma 3.5. Therefore problem (4.1),(4.78) has one and only one generalized solution. ■

If instead of Lemma 3.5 we apply Lemma 3.5′ (Lemma 3.5″), we shall be able to convince ourselves that under specific additional restrictions on $\mathcal{P}_i\,(i=0,1,2)$, $q, h, \psi_0$ and $\psi_1$ the generalized solution of problem (4.1),(4.4) is absolutely continuous (classical). Namely, the following assertions are valid.

′ *Let*

$$\mathcal{P}_i \in \widetilde{\mathbb{C}}^{(-1,0)}_{\infty}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in \widetilde{\mathbb{C}}^{(-1,0)}_{\infty}(\mathcal{D}_{ab};\mathbb{R}^{n}),$$
$$\psi_1 \in \widetilde{\mathbb{C}}^{1}([0,b];\mathbb{R}^n)$$

*and* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}^1_{\infty}([0,b];\mathbb{R}^n)$ *be a linear continuous operator such that*

$$M_0(y) = \Theta \quad for \quad 0 \le y \le b, \quad I_{\bar{M}} = [0,b]. \tag{4.94}$$

*Then the fulfilment of equality* (4.84) *is necessary and sufficient for problem* (4.1),(4.4) *to have the unique generalized solution and for this solution to be absolutely continuous.*

″ *Let* $\mathcal{P}_i\,(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\psi_0$ *be continuous,* $\psi_1$ *twice continuously differentiable and* $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \mathbb{C}^2([0,b];\mathbb{R}^n)$ *be a linear continuous operator satsfying conditions* (4.94)*. Then the fulfilment of equality* (4.84) *is necessary and sufficient for problem* (4.1),(4.4) *to have the unique generalized solution and for this solution to be classical.*

*Remark* 4.13. The restrictions imposed on $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ in Theorems 4.5′ and 4.5″ are optimal and they cannot be weakened. As an example, consider the problems

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = p_0(y)u(x,y) - 1,$$

$$\lim_{y\to 0}\frac{\partial u(x,y)}{\partial x}=0,\quad u(a,y)=u(0,y),$$
$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=u(x,y)-p_1(y)\frac{\partial u}{\partial x}-x,$$
$$\lim_{y\to 0}\frac{\partial u(x,y)}{\partial x}=1,\quad u(a,y)=u(0,y)+a$$

and

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=u(x,y)-q(y),$$
$$\lim_{y\to 0}\frac{\partial u(x,y)}{\partial x}=0,\quad u(a,y)=u(0,y),$$

where $p_0:[0,b]\to(0,+\infty)$, $p_1:[0,b]\to\mathbb{R}$ and $q:[0,b]\to\mathbb{R}$ are continuous functions. All conditions of Theorem 4.5 for these problems are fulfilled. Therefore each of these problems has the unique generalized solution

$$u(x,y)=\frac{1}{p_0(y)},\quad u(x,y)=x+p_1(y)\quad\text{and}\quad u(x,y)=q(y).$$

Moreover these solutions are absolutely continuous (classical) if and only if $p_0, p_1$ and $q$ are absolutely continuous (classical).

Similarly to Theorem 4.5 we can prove

*Let*

$$\mathcal{P}_2\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}),$$
$$M_0(y)=\Theta\quad for\quad 0\le y\le b$$

*and*

$$\operatorname{mes} I_{\bar M}=b,\quad \int_0^b (1+\|\psi_1(y)\|)\|\bar M^{-1}(y)\|dy<+\infty.$$

*Then problem* (4.1),(4.5) *has one and only one generalized solution.*

$'$ *Let*

$$\mathcal{P}_i\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2),$$
$$q\in\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n}),\quad \psi_1\in\widetilde{bc}([0,b];\mathbb{R}^n)$$

*and* $h:\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)\to\widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ *be a linear continuous operator satisfying conditions* (4.94). *Then problem* (4.1),(4.5) *has one and only one generalized solution and this solution is absolutely continuous.*

$''$ *Let* $\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\psi_0$ *be continuous,* $\psi_1$ *continuously differentiable and* $h:\widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)\to\mathbb{C}^1([0,b];\mathbb{R}^n)$ *be a linear continuous operator satisfying conditions* (4.94). *Then problem* (4.1),(4.5) *has one and only one generalized solution and this solution is classical.*

In conclusion, let us admit that the conditions of unique solvability of problems (4.1), $(4.4_1)$ and (4.1),$(4.5_1)$ are sufficiently transparent because for these problems

$$M_0(y) = Z_2(a,y) - E$$

and

$$\bar{M}(y) = \int_0^a Z_2^{-1}(s,y)\bar{\mathcal{P}}(s,y)Z_2(s,y)ds.$$

## § 5.

Consider the boundary value problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y), \tag{5.1}$$

$$u(x,0) = \varphi_0(x), \qquad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \varphi_1(y), \tag{5.2}$$

where

$$\mathcal{P}_i \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \ \ (i=0,1,2), \quad q \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n), \\ \varphi_0 \in \widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^n), \quad \varphi_1 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$$

and $h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to L_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator.

As in Section 4, by $Z_2$ we shall mean the solution of the matrix differential equation

$$\frac{\partial Z_2(x,y)}{\partial x} = \mathcal{P}_2(x,y)Z_2(x,y)$$

with the initial condition

$$Z_2(0,y) = E.$$

According to Lemma $2.3_1$, the operator $h$ admits the representation

$$h(Z_2(\cdot,y)v(\cdot))(y) = M_0(y)v(0) + \\ + \int_0^a M(s,y)v'(s)ds \quad \text{for} \quad v \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n),$$

where

$$M_0 \in L_\infty([0,b];\mathbb{R}^n \times n), \quad M \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}).$$

Analyzing the proof of Theorem 4.1, we can see that the following statement is valid.

*Let*

$$\operatorname*{ess\,inf}_{0\le y\le b} |\det M_0(y)| > 0. \tag{5.3}$$

*Then problem* (5.1),(5.2) *has the unique solution* $u$ *and*

$$\begin{aligned} &u_m(x,y) \rightrightarrows u(x,y), \quad \frac{\partial u_m(x,y)}{\partial x} \rightrightarrows \frac{\partial u(x,y)}{\partial x}, \\ &\frac{\partial u_m(x,y)}{\partial y} \rightrightarrows \frac{\partial u(x,y)}{\partial y} \quad \textit{for} \quad m \to +\infty, \end{aligned} \tag{5.4}$$

*where* $u_0(x,y) \equiv 0$ *and for an arbitrary* $m$ *the vector function* $u_m$ *is the solution of the problem*

$$\begin{aligned} \frac{\partial^2 u_m(x,y)}{\partial x \partial y} &= \mathcal{P}_2(x,y)\frac{\partial u_m(x,y)}{\partial y} + \mathcal{P}_0(x,y)u_{m-1}(x,y) + \\ &+\mathcal{P}_1(x,y)\frac{\partial u_{m-1}(x,y)}{\partial x} + q(x,y), \end{aligned} \tag{5.5}$$

$$u_m(x,0) = \varphi_0(x), \qquad h\Big(\frac{\partial u_m(\cdot,y)}{\partial y}\Big)(y) = \varphi_1(y). \tag{5.6}$$

To construct $u_m$ almost for all $y \in [0,b]$ we have to solve the system of ordinary differential equations

$$\begin{aligned} \frac{\partial z(x,y)}{\partial x} &= \mathcal{P}_2(x,y)z(x,y) + \mathcal{P}_0(x,y)u_{m-1}(x,y) + \\ &+\mathcal{P}_1(x,y)\frac{\partial u_{m-1}(x,y)}{\partial x} + q(x,y) \end{aligned} \tag{5.7}$$

with the boundary condition

$$h(z)(y) = \varphi_1(y). \tag{5.8}$$

In view of (5.3) problem (5.7),(5.8) has the unique solution $z_m(\cdot,y)$ and

$$u_m(x,y) = \varphi_0(x) + \int_0^y z_m(x,t)dt.$$

However, if $n > 1$, then the solution $z_m(\cdot,y)$ can be effectively constructed only in exceptional cases. Consequently, the method of constructing the solution of problem (5.1),(5.2) described above fails in the general case.

We shall consider the case when $\mathcal{P}_2$ and $h$ admit the representation

$$\mathcal{P}_2(x,y) = \mathcal{P}_{20}(x,y) + \mathcal{P}_{21}(x,y)$$

and

$$h(v)(y) = h_0(v)(y) - \sum_{j=1}^{\nu} H_j v(s_j),$$

where

$$\mathcal{P}_{20} \quad \text{and} \quad \mathcal{P}_{21} \in L_\infty(\mathcal{D}_{ab}; \mathbb{R}^{n\times n}),$$

$$s_j \in [0,a], \quad H_j \in \mathbb{R}^{n\times n} \ (j=1,\dots,\nu),$$

and the problem

$$\frac{\partial z(x,y)}{\partial x} = \mathcal{P}_{20}(x,y)z(x,y), \quad h_0(z)(y) = 0 \tag{5.9}$$

has only the trivial solution almost for all $y \in [0,b]$.

In that case problem (5.1),(5.2) takes the form

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} +$$
$$+[\mathcal{P}_{20}(x,y) + \mathcal{P}_{21}(x,y)]\frac{\partial u(x,y)}{\partial y} + q(x,y), \tag{5.10}$$
$$u(x,0) = \varphi_0(x), \quad h_0\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \sum_{j=1}^{\nu} H_j \frac{\partial u(s_j,y)}{\partial y} + \varphi_1(y). \tag{5.11}$$

Let $u_0(x,y) \equiv 0$. For every natural $m$ by $u_m$ we denote the solution of the problem

$$\frac{\partial^2 u_m(x,y)}{\partial x \partial y} = \mathcal{P}_{20}(x,y)\frac{\partial u_m(x,y)}{\partial y} + \mathcal{P}_0(x,y)u_{m-1}(x,y) +$$
$$+\mathcal{P}_1(x,y)\frac{\partial u_{m-1}(x,y)}{\partial x} + \mathcal{P}_{21}(x,y)\frac{\partial u_{m-1}(x,y)}{\partial y} + q(x,y), \tag{5.12}$$
$$u_m(x,0) = \varphi_0(x),$$
$$h_0\Big(\frac{\partial u_m(\cdot,y)}{\partial y}\Big)(y) = \sum_{j=1}^{\nu} H_j \frac{\partial u_{m-1}(s_j,y)}{\partial y} + \varphi_1(y). \tag{5.13}$$

Below we determine the conditions whose fulfilment guarantees the fulfilment of conditions (5.4).

By $Z_0$ will be meant the solution of the matrix differential equation

$$\frac{\partial Z_0(x,y)}{\partial x} = \mathcal{P}_{20}(x,y)Z_0(x,y)$$

with the initial condition

$$Z_0(0,y) = E,$$

and by $\mathcal{G}(\cdot,\cdot,y)$ the Green's matrix of problem (5.9).

By Lemma $2.3_1$ operator $h_0$ admits the representation

$$h_0\big(Z_0(\cdot,y)v(\cdot)\big)(y) = M_{10}(y)v(0) +$$
$$+\int_0^a M_1(s,y)v'(s)ds \quad \text{for} \quad v \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \tag{5.14}$$

where

$$M_{10} \in L_\infty([0,b];\mathbb{R}^{n\times n}), \quad M_1 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n \times n).$$

Therefore

$$\mathcal{G}(x,s,y)=\begin{cases} Z_0(x,y)[E-M_{10}^{-1}(y)M_1(s,y)]Z_0^{-1}(s,y) & \text{for } s\le x\\ -Z_0(x,y)M_{10}^{-1}(y)M_1(s,y)Z_0^{-1}(s,y) & \text{for } s>x. \end{cases} \tag{5.15}$$

We have

*Let*

$$\operatorname*{ess\,inf}_{0\le y\le b}|\det M_{10}(y)|>0 \tag{5.16}$$

*and there exist a matrix function* $A\in L_\infty([0,b];\mathbb{R}^{n\times n})$ *such that*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\big(A(y)\big)<1 \tag{5.17}$$

*and the inequality*

$$\sum_{j=1}^{\nu}|Z_0(x,y)M_{10}^{-1}(y)H_j|+\int_0^a|\mathcal{G}(x,s,y)\mathcal{P}_{21}(s,y)|ds\le A(y) \tag{5.18}$$

*holds almost everywhere in* $\mathcal{D}_{ab}$. *Then problem* (5.10),(5.11) *has the unique solution* $u$ *and condition* (5.4) *takes place, where* $u_0(x,y)\equiv 0$ *and for every natural* $m$ *the vector function* $u_m$ *is a solution of problem* (5.12),(5.13).

*Proof.* Let $n_1=2n$, $n_2=n$. For arbitrary $z_1=(z_1^i)_{i=0}^1\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})$, where $z_1^i\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ $(i=0,1)$ and $z_2\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$, assume

$$\begin{gathered} g_1^0(z_1,z_2)(x,y)=\varphi_0(x)+\\ +\int_0^y Z_0(x,t)M_{10}^{-1}(t)\Big[\sum_{j=1}^{\nu}H_jz_2(s_j,t)+\varphi_1(t)\Big]dt+\\ +\int_0^y\int_0^a\mathcal{G}(x,s,t)\Big[\mathcal{P}_0(s,t)z_1^0(s,t)+\mathcal{P}_1(s,t)z_1^1(s,t)+\\ +\mathcal{P}_{21}(s,t)z_2(s,t)+q(s,t)\Big]ds\,dt, \end{gathered} \tag{5.19}$$

$$\begin{gathered} g_1^1(z_1,z_2)(x,y)=\frac{\partial g_1^0(z_1,z_2)(x,y)}{\partial x},\quad g_1(z_1,z_2)(x,y)=\\ =\big(g_1^i(z_1,z_2)(x,y)\big)_{i=0}^1,\quad g_2(z_1,z_2)(x,y)=\frac{\partial g_1^0(z_1,z_2)(x,y)}{\partial y} \end{gathered} \tag{5.20}$$

and show that problem (5.10),(5.11) is equivalent to the system of operator equations

$$z_i(x,y)=g_i(z_1,z_2)(x,y)\ \ (i=1,2). \tag{5.21}$$

Indeed, let problem (5.10),(5.11) have a solution $u$. Put

$$z_1^0(x,y)=u(x,y),\quad z_1^1(x,y)=\frac{\partial u(x,y)}{\partial x},\quad z_2(x,y)=\frac{\partial u(x,y)}{\partial y}.$$

By Lemma 3.1 and equalities (5.19) and (5.20)

$$(z_i)_{i=1}^2 \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1}) \times L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$$

is the solution of system (5.21). Taking into account (5.15) and (5.16), we can easily show that if $z_1^i \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ $(i = 0,1)$, $z_1 = (z_1^i)_{i=0}^1$, $z_2 \in$ $\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n)$ and $(z_i)_{i=1}^2$ is a solution of system (5.21), then $u(\cdot,\cdot) =$ $= z_1^0(\cdot,\cdot)$ is a solution of problem (5.10),(5.11) and the equalities

$$z_1^1(x,y) = \frac{\partial u(x,y)}{\partial x}, \quad z_2(x,y) = \frac{\partial u(x,y)}{\partial y}$$

are valid.

In view of (5.15)-(5.20), for any $\zeta_i$ and $\bar\zeta \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_i})$ $(i = 1,2)$ the operators $g_i : L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1}) \times L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2}) \to L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_i})$ $(i = 1,2)$ satisfy inequalities (1.3) and (1.21) almost everywhere in $\mathcal{D}_{ab}$, where $I =$ $= [0,a]$, $g_0(t) \equiv \text{const}$, $A_{01}$ is a non-negative constant $n_2 \times n_1$ matrix and $A_{02}$ is the zero matrix. The validity of the theorem becomes evident by applying Lemma 1.3. ■

*Let inequality* (5.16) *hold,*

$$\operatorname*{ess\,sup}_{0\le y\le b} r\Big(\sum_{j=1}^{\nu} |H_j|\,|Z_0(s_j,y)M_{10}^{-1}(y)|\Big) < 1 \tag{5.22}$$

*and there exist a matrix function* $A \in L_\infty([0,b];\mathbb{R}^{n\times n})$ *satisfying condition* (5.17) *such that the inequality*

$$|Z_0(x,y)M_{10}^{-1}(y)|\Big[E - \sum_{j=1}^{\nu} |H_j|\,|Z_0(s_j,y)M_{10}^{-1}(y)|\Big]^{-1} \times$$
$$\times \sum_{j=1}^{\nu}\int_0^a |H_j|\,|\mathcal{G}(s_j,s,y)\mathcal{P}_{21}(s,y)|ds +$$
$$+ \int_0^a |\mathcal{G}(x,s,y)\mathcal{P}_{21}(s,y)|ds \le A(y) \tag{5.23}$$

*holds almost everywhere in* $\mathcal{D}_{ab}$. *Then problem* (5.10),(5.11) *has the unique solution* $u$ *and condition* (5.4) *takes place, where* $u_0(x,y) \equiv 0$ *and for any natural* $m$ *the vector function* $u_m$ *is a solution of problem* (5.12),(5.13).

*Proof.* Let $n_1 = 2n$, $n_2 = n$, $\Lambda_1 = L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_1})$, $\Lambda_2$ be a set of all $\zeta \in$ $\in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n_2})$ such that $\zeta(\cdot,y) \in \mathbb{C}([0,a];\mathbb{R}^n)$ almost for all $y \in [0,b]$ and $\Lambda_0 = \mathbb{C}([0,a];\mathbb{R}^n)$, while $g_i : \Lambda_1 \times \Lambda_2 \to \Lambda_i$ $(i = 1,2)$ be the operators given by equalities (5.19),(5.20).

As shown above, problem (5.1),(5.2) is equivalent to the system of operator equations (5.21) due to condition (5.16).

In view of (5.15),(5.16),(5.19) and (5.20) for any $\zeta_i$ and $\bar\zeta_i \in \Lambda_i$ $(i = 1,2)$ the operators $g_i$ $(i = 1,2)$ satisfy inequalities (1.3),(1.4) almost everywhere in $\mathcal{D}_{ab}$, where $I = [0,a]$, $g_0(t) \equiv \text{const}$, $A_{01}$ is a non-negative constant $n_2 \times n_1$ matrix, $A_{02}$ is the zero matrix,

$$A_1(x,y) = \int_0^a |\mathcal{G}(x,s,y)\mathcal{P}_{21}(s,y)|ds, \quad A_2(x,y) = |Z_0(x,y)M_{10}^{-1}(y)|$$

and $l : \Lambda_0 \to \mathbb{R}^{n_2}$ is a non-negative linear operator given by the equality

$$l(v) = \sum_{j=1}^{\nu} |H_j| v(s_j).$$

In view of (5.22), condition (1.5) takes place and in view of (5.23) we may assume without loss of generality that equality (1.7) holds. Besides, since condition (1.6) is also fulfilled, by virtue of Lemma 1.1 system (5.21) has the unique solution $(z_i)_{i=1}^2$ and conditions (1.8) take place. The validity of Theorem 5.3 immediately follows from the above arguments. ■

Let us give two corollaries of the above proven theorems for the periodic boundary value problem. For the convenience we rewrite system (5.1) and the boundary condition in the scalar form

$$\frac{\partial^2 u_i(x,y)}{\partial x \partial y} = \sum_{k=1}^{n} \Big( p_{0ik}(x,y)u_k(x,y) + p_{1ik}(x,y)\frac{\partial u_k(x,y)}{\partial x} + $$
$$+ p_{2ik}(x,y)\frac{\partial u_k(x,y)}{\partial y}\Big) + q_i(x,y) \ \ (i = 1,\dots,n), \tag{5.24}$$

$$u_i(x,0) = \varphi_{0i}(x), \quad \frac{\partial u_i(a,y)}{\partial y} = \frac{\partial u_i(0,y)}{\partial y} + \varphi_{1i}(y) \ \ (i = 1,\dots,n). \tag{5.25}$$

Moreover, as above we assume that

$$p_{0ik}, p_{1ik}, p_{2ik} \ \text{ and } \ q_i \in L_\infty(\mathcal{D}_{ab};\mathbb{R}) \ (i,k = 1,\dots,n),$$
$$\varphi_{0i} \in \widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}), \quad \varphi_{1i} \in L_\infty([0,b];\mathbb{R}) \ (i = 1,\dots,n).$$

*Let there exist $\sigma_i \in \{-1,1\}$ $(i = 1,\dots,n)$ and $\delta > 0$ such that the real parts of eigenvalues of matrix $S(y) = \big(s_{ij}(y)\big)_{i,j=1}^n$, where*

$$\begin{aligned} s_{ii}(y) &= \operatorname*{ess\,sup}_{0 \le x \le a}\{\sigma_i p_{2ii}(x,y)\}, \\ s_{ij}(y) &= \operatorname*{ess\,sup}_{0 \le x \le a}|p_{2ij}(x,y)| \quad \textit{for} \quad (i \ne j; i,j = 1,\cdots,n) \end{aligned} \tag{5.26}$$

*are less than $-\delta$ almost for all $y \in [0,b]$. Then problem (5.24),(5.25) has the unique solution $(u_i)_{i=1}^n$ and*

$$\begin{aligned} u_{im}(x,y) \rightrightarrows u_i(x,y), \quad & \frac{\partial u_{im}(x,y)}{\partial x} \rightrightarrows \frac{\partial u_i(x,y)}{\partial x}, \\ \frac{\partial u_{im}(x,y)}{\partial y} \rightrightarrows \frac{\partial u_i(x,y)}{\partial y} \quad & \textit{for} \quad m \to +\infty \ (i = 1,\dots,n), \end{aligned} \tag{5.27}$$

*where* $u_{i0}(x,y) \equiv 0 \, (i=1,\dots,n)$ *and for any natural* $m$ *and* $i \in \{1,\dots,n\}$ *the function* $u_{im}$ *is a solution of the equation*

$$\begin{gathered}
\frac{\partial^2 u_{im}(x,y)}{\partial x \partial y} = p_{2ii}(x,y)\frac{\partial u_{im}(x,y)}{\partial y} + \\
+ \sum_{i=1}^{n}(1-\delta_{ij})p_{2ij}(x,y)\frac{\partial u_{jm-1}(x,y)}{\partial y} + \\
+ \sum_{j=1}^{n}\Big(p_{0ij}(x,y)u_{jm-1}(x,y) + p_{1ij}(x,y)\frac{\partial u_{jm-1}(x,y)}{\partial x}\Big) + q_i(x,y) \quad (5.28)
\end{gathered}$$

*with the boundary conditions*

$$u_{im}(x,0) = \varphi_0(x), \qquad \frac{\partial u_{im}(a,y)}{\partial y} = \frac{\partial u_{im}(0,y)}{\partial y} + \varphi_{1i}(y). \qquad (5.29)$$

*Proof.* From the restrictions imposed on the matrix function $S$ it follows that

$$s_{ii}(y) \le -\delta \quad \text{almost for all } \; y \in [0,b] \qquad (5.30)$$

and

$$A(y) = \Big((1-\delta_{ij})\frac{s_{ij}(y)}{|s_{ii}(y)|}\Big)_{i,j=1}^{n} \qquad (5.31)$$

satisfies condition (5.17).

Assume

$$\begin{gathered}
\mathcal{P}_{20}(x,y) = \big(\delta_{ij}p_{2ij}(x,y)\big)_{i,j=1}^{n}, \\
\mathcal{P}_{21}(x,y) = \big((1-\delta_{ij})p_{2ij}(x,y)\big)_{i,j=1}^{n}, \qquad (5.32) \\
h_0(v)(y) = \big(v_i(a) - v_i(0)\big)_{i=1}^{n}, \quad H_j = \Theta \;\; (j=1,\dots,\nu).
\end{gathered}$$

Then

$$M_{10}(y) = \operatorname{diag}\big(\gamma_1(y),\dots,\gamma_n(y)\big),$$

where

$$\gamma_i(y) = \exp\Big(\int_0^a p_{2ii}(s,y)ds\Big) - 1 \;\; (i=1,\dots,n).$$

In view of (5.26) and (5.30)

$$|\gamma_i(y)| \ge |1 - \exp(s_{ii}(y)a)| \ge |1-\exp(-a\delta)| > 0 \, (i=1,\dots,n), \quad (5.33)$$

whence it is clear that $M_{10}$ satisfies condition (5.16). According to Theorem 5.2, to complete the proof it suffices to show that inequality (5.18) holds almost everywhere in $\mathcal{D}_{ab}$.

In view of (5.32) and (5.33) Green's matrix of problem (5.9) has the form

$$\mathcal{G}(x,s,y) = \operatorname{diag}(g_1(x,s,y),\dots,g_n(x,s,y)),$$

where

$$g_i(x,s,y)=\begin{cases}\gamma_i^{-1}(y)\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big) & \text{for } s\le x\\ \gamma_i^{-1}(y)\exp\Big(\int_0^a p_{2ii}(\xi,y)d\xi+\int_s^x p_{2ii}(\xi,y)d\xi\Big) & \text{for } s>x\end{cases}.$$

Therefore

$$\sum_{j=1}^{\nu}|Z_0(x,y)M_{10}^{-1}(y)H_j|+\int_0^a|\mathcal{G}(x,s,y)\mathcal{P}_{21}(s,y)|ds=$$
$$=\Big((1-\delta_{ij})\int_0^a|g_i(x,s,y)p_{2ij}(s,y)|ds\Big)_{i,j=1}^n. \tag{5.34}$$

According to (5.26) and (5.30s), for $\sigma_i=1$ we have

$$\int_0^a|g_i(x,s,y)p_{2ij}(s,y)|ds\le|\gamma_i^{-1}(y)|\frac{s_{ij}(y)}{|s_{ii}(y)|}\times$$
$$\times\int_0^x\exp\Big(-\int_s^x|p_{2ii}(\xi,y)|d\xi\Big)|p_{2ii}(s,y)|ds+$$
$$+|\gamma_i^{-1}(y)|\frac{s_{ij}(y)}{|s_{ii}(y)|}\exp\Big(-\int_0^x|p_{2ii}(\xi,y)|d\xi\Big)\times$$
$$\times\int_x^a\exp\Big(-\int_s^a|p_{2ii}(\xi,y)|d\xi\Big)|p_{2ii}(s,y)|ds=\frac{s_{ij}(y)}{|s_{ii}(y)|}.$$

Similarly, we can show that for $\sigma_i=-1$, the estimate

$$\int_0^a|g_i(x,s,y)p_{2ij}(s,y)|ds\le\frac{s_{ij}(y)}{|s_{ii}(y)|}$$

is also valid.

By virtue of this estimate, inequality (5.18) follows from equalities (5.31) and (5.34). ■

*Let the conditions of Corollary* 5.1 *take place. Then problem* (5.24), (5.25) *has the unique solution* $(u_i)_{i=1}^n$ *and condition* (5.27) *holds, where* $u_{i0}(x,y)\equiv 0$ $(i=1,\dots,n)$ *and for any natural* $m$ *and* $i\in\{1,\dots,n\}$ *the function* $u_{im}$ *is a solution of equation* (5.28) *with the boundary conditions*

$$u_{im}(x,0)=\varphi_{0i}(x),\quad \frac{\partial u_{im}(x_i,y)}{\partial y}=\frac{\partial u_{im-1}(s_i,y)}{\partial y}+\varphi_{1i}(y), \tag{5.35}$$

*where* $s_i=\frac{1+\sigma_i}{2}a$ *and* $x_i=\frac{1-\sigma_i}{2}a$.

*Proof.* To prove the corollary it suffices to find that if $\nu=n$,

$$\mathcal{P}_{20}(x,y)=\big(\delta_{ik}p_{2ik}(x,y)\big)_{i,k=1}^n,\quad \mathcal{P}_{21}(x,y)=\big((1-\delta_{ik})p_{2ik}(x,y)\big)_{i,k=1}^n,$$
$$h_0(v)(y)=(v_i(x_i))_{i=1}^n,\quad H_j=(\delta_{ij}\delta_{ik})_{i,k=1}^n,$$

then all conditions of Theorem 5.3 are fulfilled.

In our case

$$Z_0(x,y)=\operatorname{diag}\Big(\exp\Big(\int_0^x p_{211}(s,y)ds\Big),\dots,\exp\Big(\int_0^x p_{2nn}(s,y)ds\Big)\Big),$$

$$M_{10}=\operatorname{diag}\Big(\exp\Big(\int_0^{x_1} p_{211}(s,y)ds\Big),\dots,\exp\Big(\int_0^{x_n} p_{2nn}(s,y)ds\Big)\Big)$$

and Green's matrix of problem (5.9) has the form

$$\mathcal{G}(x,s,y)=\operatorname{diag}(g_1(x,s,y),\dots,g_n(x,s,y)),$$

where

$$g_i(x,s,y)=\begin{cases}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)\operatorname{sign}(x-x_i) \text{ for } (x-x_i)(s-x_i)\geq 0,\\ \qquad\qquad\qquad\qquad\qquad\qquad |s-x_i|\leq|x-x_i|\\ 0 \quad\text{for}\quad (x-x_i)(s-x_i)\geq 0,\quad |s-x_i|>|x-x_i|\\ 0 \quad\text{for}\quad (x-x_i)(s-x_i)<0\end{cases},$$

whence in view of (5.26) and (5.30) we obtain

$$M_{10}(y)\geq E,$$

$$\sum_{j=1}^{\nu}|H_j|\,|Z_0(s_j,y)M_{10}^{-1}(y)|=$$
$$=\operatorname{diag}\Big(\exp\Big(\int_{x_1}^{s_1} p_{211}(s,y)ds\Big),\dots,\exp\Big(\int_{x_n}^{s_n} p_{2nn}(s,y)ds\Big)\Big)=$$
$$=\operatorname{diag}\Big(\exp\Big(-\int_0^a |p_{211}(s,y)|ds\Big),\dots,\exp\Big(-\int_0^a |p_{2nn}(s,y)|ds\Big)\Big)\leq$$
$$\leq\operatorname{diag}(\exp(-\delta a),\dots,\exp(-\delta a))$$

and

$$|Z_0(x,y)M_{10}^{-1}(y)|\Big[E-\sum_{j=1}^{\nu}|H_j|\,|Z_0(s_j,y)M_{10}^{-1}(y)|\Big]^{-1}\times$$
$$\times\sum_{j=1}^{\nu}\int_0^a|H_j|\,|\mathcal{G}(s_j,s,y)\mathcal{P}_{21}(s,y)|ds+\int_0^a|\mathcal{G}(x,s,y)\mathcal{P}_{21}(s,y)|ds=$$
$$=\Big((1-\delta_{ij})\Big[\gamma_i^{-1}(y)\exp\Big(\int_{x_i}^x p_{2ii}(s,y)ds\Big)\int_0^a|g_i(s_i,s,y)|\times$$
$$\times|p_{2ij}(s,y)|ds+\int_0^a|g_i(x,s,y)||p_{2ij}(s,y)|ds\Big]\Big)_{i,j=1}^{n},\tag{5.36}$$

where

$$\overline{\gamma}_i(y)=1-\exp\Big(-\int_0^a|p_{2ii}(s,y)|ds\Big)\quad(i=1,\dots,n).$$

Consequently, inequalities (5.16) and (5.22) are fulfilled. On the other hand,

$$\int_0^a g_i(s_i,s,y)|p_{2ij}(s,y)|ds =$$
$$= \Big|\int_{x_i}^{s_i} \exp\Big(\int_s^{s_i} p_{2ii}(\xi,y)d\xi\Big)|p_{2ij}(s,y)|ds\Big| \le$$
$$\le \frac{s_{ij}(y)}{|s_{ii}(y)|}\Big|\int_0^a \exp\Big(\int_s^{s_i} p_{2ii}(\xi,y)d\xi\Big)p_{2ii}(s,y)ds\Big| =$$
$$= \overline{\gamma}_i(y)\frac{s_{ij}(y)}{|s_{ii}(y)|} \quad (i=1,\dots,n)$$

and

$$\int_0^a g_i(x,s,y)|p_{2ij}(s,y)|ds \le$$
$$\le \frac{s_{ij}(y)}{|s_{ii}(y)|}\Big|\int_{x_i}^{x} \exp\Big(\int_s^{x} p_{2ii}(\xi,y)d\xi\Big)p_{2ii}(s,y)ds\Big| =$$
$$= \frac{s_{ij}(y)}{|s_{ii}(y)|}\Big|1-\exp\Big(\int_{x_i}^{x} p_{2ii}(\xi,y)d\xi\Big)\Big|.$$

Therefore from equalities (5.31) and (5.36) there follows estimate (5.23) and, as admitted above, $A$ satisfies condition (5.17). ■

*Remark* 5.1. If in boundary condition (5.35) we put $s_i = \frac{1-\sigma_i}{2}a$ and $x_i = \frac{1+\sigma_i}{2}a$ for any $i \in \{1,\dots,n\}$, then the conclusion of Corollary 5.2 becomes invalid. Indeed, consider the problem

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \frac{\partial u(x,y)}{\partial y} + 1, \tag{5.37}$$
$$u(x,0)=0, \qquad \frac{\partial u(a,y)}{\partial y} = \frac{\partial u(0,y)}{\partial y}$$

for which all conditions of Corollary 5.2 are fulfilled. In that case $n=1$ and $\sigma_1=-1$. If we assume that $x_1=\frac{1+\sigma_1}{2}=0$ and $s_1=a$, then (5.35) takes the form

$$u_m(x,0)=0, \qquad \frac{\partial u_m(0,y)}{\partial y} = \frac{\partial u_{m-1}(a,y)}{\partial y}. \tag{5.38}$$

For any natural $m$ problem (5.37),(5.38) has the unique solution $u_m$ and

$$\frac{\partial u_m(x,y)}{\partial y} = \Big(\frac{\partial u_{m-1}(a,y)}{\partial y}+1\Big)\exp(x)-1.$$

Since $u_1(a,y)=0$, from the last equality we obtain by induction

$$\frac{\partial u_m(x,y)}{\partial y} = \Big(\exp((m-1)a)+1\Big)\exp(x)-1 \to +\infty \quad \text{for} \quad m\to+\infty.$$

§ 6.

In this section for the hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ +\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y) \tag{6.1}$$

we consider the boundary value problems

$$u(x,0) = \varphi_0(x), \quad h\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \varphi_1(y); \tag{6.2}$$

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x), \quad h(u(\cdot,y))(y) = \psi_1(y) \tag{6.3}$$

and establish the conditions for the stability of their solutions with respect both to small perturbations of coefficients of the system and boundary data. As above, unless otherwise stated, it is assumed that

$$\mathcal{P}_i \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ (i=0,1,2), \quad q \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n),$$
$$\varphi_0 \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad \varphi_1 \in L([0,b];\mathbb{R}^n),$$
$$\psi_0 \in L([0,a];\mathbb{R}^n), \quad \psi_1 \in L([0,b];\mathbb{R}^n),$$

$h : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator and $Z_2$ is the solution of the matrix differential equation

$$\frac{\partial Z(x,y)}{\partial x} = \mathcal{P}_2(x,y)Z(x,y)$$

satisfying the initial condition $Z(0,y) = E$.

Alongside with (6.1),(6.2), for any natural $k$, consider the problem

$$\frac{\partial^2 u(x,y)}{\partial x\partial y} = \mathcal{P}_{0k}(x,y)u(x,y) + \mathcal{P}_{1k}(x,y)\frac{\partial u(x,y)}{\partial x} + \\ +\mathcal{P}_{2k}(x,y)\frac{\partial u(x,y)}{\partial y} + q_k(x,y), \tag{6.4}$$

$$u(x,0) = \varphi_{0k}(x), \quad h_k\Big(\frac{\partial u(\cdot,y)}{\partial y}\Big)(y) = \varphi_{1k}(x), \tag{6.5}$$

where

$$\mathcal{P}_{ik} \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n})\ \ (i=0,1,2), \quad q_k \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^n),$$
$$\varphi_{0k} \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n), \quad \varphi_{1k} \in L([0,b];\mathbb{R}^n),$$

and $h_k : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to L_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator.

By Lemmas $2.1_1$ and $2.3_1$ for an arbitrary $v \in \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n)$ we have

$$h(v)(y) = H_0(y)v(0) + \int_0^a H(s,y)v'(s)ds, \tag{6.6}$$

$$h_k(v)(y) = H_{0k}(y)v(0) + \int_0^a H_k(s,y)v'(s)ds, \tag{6.7}$$

where

$$H_0 \in \widetilde{\mathbb{C}}_\infty([0,b];\mathbb{R}^n \times n) \quad \text{and} \quad H_{0k} \in L_\infty([0,b];\mathbb{R}^{n\times n}),$$
$$H \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}) \quad \text{and} \quad H_k \in L_\infty(\mathcal{D}_{ab};\mathbb{R}^{n\times n}).$$

Put

$$\begin{aligned} M_0(y) &= H_0(y) + \int_0^a H(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds, \\ M(x,y) &= H(x,y)Z_2(x,y) + \int_x^a H(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} M_{0k}(y) &= H_{0k}(y) + \int_0^a H_k(s,y)\frac{\partial Z_{2k}(s,y)}{\partial s}ds, \\ M_k(x,y) &= H_k(x,y)Z_{2k}(x,y) + \int_x^a H_k(s,y)\frac{\partial Z_{2k}(s,y)}{\partial s}ds, \end{aligned} \tag{6.9}$$

where $Z_{2k}$ is the solution of the matrix differential equation

$$\frac{\partial Z(x,y)}{\partial x} = \mathcal{P}_{2k}(x,y)Z(x,y)$$

satisfying the initial condition $Z(0,y) = E$.

*Let*

$$\operatorname*{ess\,inf}_{0\le y\le b} |\det(M_0(y))| > 0, \tag{6.10}$$

$$\begin{gathered} \sup \|\mathcal{P}_{ik}\|_{L_\infty} < +\infty \quad (i=1,2), \\ \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_{ab}} \left\| \int_0^x \big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]ds \right\| \to 0 \;\; \textit{for} \;\; k \to +\infty, \end{gathered} \tag{6.11}$$

$$\lim_{k\to+\infty} \|\mathcal{P}_{ik} - \mathcal{P}_i\|_L = 0 \;\; (i=0,1,2), \quad \lim_{k\to+\infty} \|q_k - q\|_L = 0 \tag{6.12}$$

*and*

$$\begin{gathered} \lim_{k\to+\infty} \|\varphi_{0k} - \varphi_0\|_{\widetilde{\mathbb{C}}} = 0, \quad \lim_{k\to+\infty} \|\varphi_{1k} - \varphi_1\|_L = 0, \\ \lim_{k\to+\infty} \|h - h_k\|^{10} = 0 \end{gathered} \tag{6.13}$$

*Then, starting from some* $k_0$, *problem* (6.4), (6.5) *has the unique solution* $u_k$ *and*

$$\lim_{k\to+\infty} \|u_k - u_0\|_{\widetilde{\mathbb{C}}} = 0. \tag{6.14}$$

*where* $u_0$ *is the solution of problem* (6.1), (6.2).

*Proof.* By Lemmas $2.2_1$ and $2.3_3$

$$Z_2 \ \text{ and } \ Z_2^{-1} \in \widetilde{\mathbb{C}}_{\infty}^{(0,-1)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}), \tag{6.15}$$

$$M_0 \in \widetilde{\mathbb{C}}_{\infty}([0,b];\mathbb{R}^{n\times n}), \quad M \in \widetilde{\mathbb{C}}_{\infty}^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n}). \tag{6.16}$$

Let us show that

$$\lim_{k\to+\infty} \|Z_{2k} - Z_2\|_{L_\infty} = 0, \quad \lim_{k\to+\infty} \|Z_{2k}^{-1} - Z_2^{-1}\|_{L_\infty} = 0. \tag{6.17}$$

Put

$$Z_{0k}(x,y) = Z_{2k}(x,y) - Z_2(x,y).$$

Then

$$Z_{0k}(x,y) = \int_0^x \big[\mathcal{P}_{2k}(s,y)Z_{2k}(s,y) - \mathcal{P}_2(s,y)Z_2(s,y)\big]ds =$$

$$= \int_0^x \mathcal{P}_{2k}(s,y)Z_{0k}(s,y)ds + \int_0^x \big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]Z_2(s,y)ds.$$

Therefore

$$\|Z_{0k}(x,y)\| \le \alpha \int_0^x \|Z_{0k}(s,y)\|ds + \varepsilon_k,$$

where

$$\varepsilon_k = \operatorname*{ess\,sup}_{(x,y)\in D_{ab}} \bigg\| \int_0^x \big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]Z_2(s,y)ds\bigg\|,$$

$$\alpha = \sup_{k\ge 1} \|\mathcal{P}_{2k}\|_{L_\infty},$$

whence by Gronwall's lemma we have

$$\|Z_{0k}\|_{L_\infty} \le \varepsilon_k \exp(a\alpha).$$

However, according to conditions (6.11) and (6.15)

$$\lim_{k\to+\infty} \varepsilon_k = 0.$$ [11]

Therefore

$$\lim_{k\to+\infty} \|Z_{2k} - Z_2\|_{L_\infty} = 0.$$

---

[10] Here and everywhere below by $\|h\|$ is understood the norm of operator $h$.

[11] See the proof of Lemma 3.14.

According to representations (6.6) and (6.7), it follows from the convergence by the norm of a sequence of operators $h_k$ $(k = 1, 2, \dots)$ to the operator $h$ that

$$\lim_{k\to+\infty} \|H_{0k} - H_0\|_{L_\infty} = 0, \lim_{k\to+\infty} \|H_k - H\|_{L_\infty} = 0. \tag{6.18}$$

In view of (6.8) and (6.9),

$$M_{0k}(y) - M_0(y) = H_{0k}(y) - H_0(y) + \int_0^a \big[H_k(s,y) -$$
$$-H(s,y)\big]\mathcal{P}_{2k}(s,y)Z_{2k}(s,y)ds + \int_0^a H(s,y)\mathcal{P}_{2k}(s,y)\big[Z_{2k}(s,y) -$$
$$-Z_2(s,y)\big]ds + \int_0^a H(s,y)\big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]Z_2(s,y)ds,$$
$$M_k(x,y) - M(x,y) = H_k(x,y)Z_{2k}(x,y) - H(x,y)Z_2(x,y) +$$
$$+\int_x^a \big[H_k(s,y) - H(s,y)\big]\mathcal{P}_{2k}(s,y)Z_{2k}(s,y)ds + \int_x^a H(s,y)\mathcal{P}_{2k}(s,y) \times$$
$$\times\big[Z_{2k}(s,y) - Z_2(s,y)\big]ds + \int_x^a H(s,y)\big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]Z_2(s,y)ds.$$

Therefore

$$\|M_{0k} - M_0\|_{L_\infty} \le \|H_{0k} - H_0\|_{L_\infty} +$$
$$+\alpha_0(\|H_k - H\|_{L_\infty} + \|Z_{2k} - Z_2\|_{L_\infty}) + \varepsilon_{0k}, \tag{6.19}$$
$$\|M_k - M\|_{L_\infty} \le \|H_k Z_{2k} - HZ_2\|_{L_\infty} +$$
$$+\alpha_0(\|H_k - H\|_{L_\infty} + \|Z_{2k} - Z_2\|_{L_\infty}) + \varepsilon_{0k}, \tag{6.20}$$

where

$$\alpha_0 = \sup_{k\ge1}(a\|\mathcal{P}_{2k}Z_{2k}\|_{L_\infty} + a\|H\mathcal{P}_{2k}\|_{L_\infty}),$$
$$\varepsilon_{0k} = \operatorname*{ess\,sup}_{(x,y)\in D_{ab}} \Big\|\int_x^a H(s,y)\big[\mathcal{P}_{2k}(s,y) - \mathcal{P}_2(s,y)\big]Z_2(s,y)ds\Big\|.$$

Because of the fact that $H \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab}; \mathbb{R}^{n\times n})$ and $Z_2 \in \widetilde{\mathbb{C}}_\infty^{(0,-1)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ and taking into account condition (6.11),

$$\lim_{k\to+\infty} \varepsilon_{0k} = 0.$$

If together with this we take into consideration conditions (6.17) and (6.18), then from (6.19) and (6.20) we get

$$\lim_{k\to+\infty} \|M_{0k} - M_0\|_{L_\infty} = 0, \quad \lim_{k\to+\infty} \|M_k - M\|_{L_\infty} = 0. \tag{6.21}$$

According to conditions (6.10) and (6.21), there exist a natural $k_0$ and a positive number $\delta$ such that

$$\operatorname*{ess\,inf}_{0\le y\le b} |\det(M_{0k}(y))| > \delta \quad \text{for} \quad k \ge k_0 \tag{6.22}$$

and

$$\lim_{k\to+\infty} \|M_{0k}^{-1} - M_0^{-1}\|_{L_\infty} = 0. \tag{6.23}$$

By Theorem 4.1 condition (6.10) guarantees the unique solvability of problem (6.1),(6.2), while condition (6.22) ensures the unique solvability of problem (6.4),(6.5) for an arbitrary $k \ge k_0$. Let us denote the solutions of these problems by $u_0$ and $u_k$. In proving Theorem 4.1 we showed that $u = u_0$ and $u = u_k$ satisfy, respectively, the boundary conditions

$$\begin{gathered} u(x,0) = \varphi_0(x), \quad \frac{\partial u(0,y)}{\partial y} = \psi(y) + \\ + \int_0^a \Big[Q_0(s,y)u(s,y) + \gamma_1^{-1}(s)Q_1(s,y)\frac{\partial u(s,y)}{\partial s}\Big]ds \end{gathered} \tag{6.24}$$

and

$$\begin{gathered} u(x,0) = \varphi_{0k}(x), \quad \frac{\partial u(0,y)}{\partial y} = \psi_k(y) + \\ + \int_0^a \Big[Q_{0k}(s,y)u(s,y) + \gamma_{1k}^{-1}(s)Q_{1k}(s,y)\frac{\partial u(s,y)}{\partial s}\Big]ds, \end{gathered} \tag{6.25}$$

where

$$\begin{gathered} \gamma_1(x) = 1 + \|\varphi_0'(x)\|, \\ \psi(y) = M_0^{-1}(y)\Big[\varphi_1(y) - \int_0^a M(s,y)Z_2^{-1}(s,y)q(s,y)ds\Big], \\ Q_0(x,y) = -M_0^{-1}(y)M(x,y)Z_2^{-1}(x,y)\mathcal{P}_0(x,y), \\ Q_1(x,y) = -\gamma_1(x)M_0^{-1}(y)M(x,y)Z_2^{-1}(x,y)\mathcal{P}_1(x,y) \end{gathered} \tag{6.26}$$

and

$$\begin{gathered} \gamma_{1k}(x) = 1 + \|\varphi_{0k}'(x)\|, \\ \psi_k(y) = M_{0k-1}^{-1}(y)\Big[\varphi_{1k}(y) - \int_0^a M_k(s,y)Z_{2k}^{-1}(s,y)q_k(s,y)ds\Big], \\ Q_{0k}(x,y) = -M_{0k}^{-1}(y)M_k(x,y)Z_{2k}^{-1}(x,y)\mathcal{P}_{0k}(x,y), \\ Q_{1k}(x,y) = -\gamma_{1k}(x)M_{0k}^{-1}(y)M_k(x,y)Z_{2k}^{-1}(x,y)\mathcal{P}_{1k}(x,y). \end{gathered} \tag{6.27}$$

More exactly, problem (6.1),(6.2) is equivalent to problem (6.4),(6.24) and for any $k \ge k_0$ problem (6.4),(6.5) is equivalent to problem (6.4),(6.25).

By conditions (6.15)-(6.17) and (6.21)-(6.23) we have

$$\varepsilon_{1k} = \|M_{0k}^{-1} - M_0^{-1}\|_{L_\infty} +$$

$$+\|M_{0k}^{-1}M_kZ_{2k}^{-1}-M_0^{-1}MZ_2^{-1}\|_{L_\infty}\to 0 \quad\text{for}\quad k\to+\infty$$

and

$$\beta=\|M_0^{-1}\|_{L_\infty}+\|M_0^{-1}MZ_2^{-1}\|_{L_\infty}<+\infty.$$

If along with this we take into account conditions (6.11)-(6.13), then from equalities (6.26) and (6.27) we find

$$\sup_{k\geq 1}\|\gamma_{1k}^{-1}Q_{1k}\|_{L_\infty}\leq\beta\sup_{k\geq 1}\|\mathcal{P}_{1k}\|_{L_\infty}<+\infty,$$

$$\|Q_{0k}-Q_0\|_L\leq\|M_{0k}^{-1}M_kZ_{2k}^{-1}-M_0^{-1}MZ_2^{-1}\|_{L_\infty}\|\mathcal{P}_{0k}\|_L+$$
$$+\|M_0^{-1}MZ_2^{-1}\|_{L_\infty}\|\mathcal{P}_{0k}-\mathcal{P}_0\|_L\leq$$
$$\leq\varepsilon_{1k}\|\mathcal{P}_{0k}\|_L+\beta\|\mathcal{P}_{0k}-\mathcal{P}_0\|_L\to 0\quad\text{for}\quad k\to+\infty,$$

$$\|\gamma_1\gamma_{1k}^{-1}Q_{1k}-Q_1\|_L\leq\varepsilon_{1k}\|\gamma_1\mathcal{P}_{1k}\|_L+\beta\|\gamma_1(\mathcal{P}_{1k}-\mathcal{P}_1)\|_L\to 0$$
$$\text{for}\quad k\to+\infty,$$

$$\|\psi_k-\psi\|\leq\varepsilon_{1k}\|\varphi_{1k}\|_L+\beta(\|\varphi_{1k}-\varphi_1\|_L+\|q_k-q\|_L)\to\quad\text{for}\quad k\to+\infty.$$

Consequently, all conditions of Lemma 3.9 are fulfilled and by virtue of this lemma equality (6.14) is valid. ■

Based on Lemmas 3.10 and 3.11, similarly to the above reasoning, we prove the following theorems.

*Let conditions* (6.10) *and* (6.13) *hold,*

$$\sup\|\mathcal{P}_{ik}\|_{L_\infty}<+\infty\quad(i=0,1,2),\quad\sup_{k\geq 1}\|q_k\|_{L_\infty}<+\infty,\tag{6.28}$$

$$\operatorname*{ess\,sup}_{(x,y)\in D_{ab}}\left\|\int_0^x\big[\mathcal{P}_{2k}(s,y)-\mathcal{P}_2(s,y)\big]ds\right\|\to 0\quad\textit{for}\quad k\to+\infty$$

*and*

$$\lim_{k\to+\infty}\|\mathcal{P}_{ik}-\mathcal{P}_i\|_L^{(1)}=0\ \ (i=0,1,2),\quad\lim_{k\to+\infty}\|q_k-q\|_L^{(1)}=0.$$

*Then, starting from some* $k_0$*, problem* (6.4),(6.5) *has the unique solution* $u_k$ *and*

$$\lim_{k\to+\infty}\|u_k-u_0\|_{\widetilde{\mathbb{C}}}^{(1)}=0,\tag{6.29}$$

*where* $u_0$ *is the solution of problem* (6.1),(6.2).

*Let conditions* (6.10) *and* (6.28) *hold and, besides,* $\varphi_0$ *and* $\varphi_{0k}\in\widetilde{\mathbb{C}}_\infty([0,a];\mathbb{R}^n)$*,* $\varphi_1$ *and* $\varphi_{1k}\in L_\infty([0,b];\mathbb{R}^n)$*,*

$$\lim_{k\to+\infty}\|\mathcal{P}_{ik}-\mathcal{P}_i\|_L^{(2)}=0\ \ (i=0,1,2),\quad\lim_{k\to+\infty}\|q_k-q\|_L^{(2)}=0$$

*and*

$$\lim_{k\to+\infty}\|\varphi_{0k}-\varphi_0\|_{\widetilde{\mathbb{C}}_\infty}=0,\quad\lim_{k\to+\infty}\|\varphi_{1k}-\varphi_1\|_{L_\infty}=0,\quad\lim_{k\to+\infty}\|h-h_k\|=0.$$

*Then, starting from some $k_0$, problem* (6.4),(6.5) *has the unique solution $u_k$ and*

$$\lim_{k\to+\infty} \|u_k - u_0\|_{\widetilde{\mathbb{C}}}^{(2)} = 0, \tag{6.30}$$

*where $u_0$ is the solution of problem* (6.1),(6.2).

*Remark* 6.1. In Theorems 6.2 and 6.3 the condition

$$\sup_{k\geq 1} \|\mathcal{P}_{1k}\|_{L_\infty} < +\infty, \quad \sup_{k\geq 1} \|q_k\| < +\infty \tag{6.31}$$

are essential and cannot be neglected. As an example, consider the problems

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial u(x,y)}{\partial y}, \tag{6.32}$$

$$u(x,0) = 0, \quad u(0,y) - u(a,y) = 0 \tag{6.33}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = -[k\cos k^2(x+y)]\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} + k\sin k^2(x+y), \tag{6.34}$$

$$u(x,0) = 0, \quad u(0,y) - u(a,y) = 0, \tag{6.35}$$

for which all conditions of Theorems 6.2 and 6.3, except (6.31), are fulfilled. Since $M_{0k}(y) \equiv M_0(y) = \exp(ay) - 1$, by Corollary 4.1 problem (6.32), (6.33) has only the trivial solution $u_0(x,y) \equiv 0$ and problem (6.34), (6.35) has the unique solution $u_k$ for any natural $k$. Moreover,

$$\frac{\partial u_k(x,y)}{\partial x} = \exp\Big(-\frac{\sin k^2(x+y)}{k}\Big) \times$$
$$\times \int_0^y \exp\Big(\frac{\sin k^2(x+t)}{k}\Big)\Big(\frac{\partial u_k(x,t)}{\partial t} + k\sin k^2(x+t)\Big)dt.$$

The assumption for conditions (6.29) or (6.30) to be valid leads us to the false equality

$$\lim_{k\to+\infty} k\int_0^x \Big|\int_0^y \exp\big(\frac{\sin k^2(s+t)}{k}\big)\sin k^2(s+t)dt\Big|ds = 0,$$

because

$$k\int_0^y \exp\big(\frac{\sin k^2 t}{k}\big)\sin k^2 t dt = \frac{1}{k} - \frac{\cos k^2 y}{k}\exp\big(\frac{\sin k^2 y}{k}\big) +$$
$$+\frac{1}{2}\int_0^y \exp\big(\frac{\sin k^2 t}{k}\big)(1+\cos 2k^2 t)dt \rightrightarrows \frac{y}{2} \quad \text{for} \quad k\to+\infty.$$

For any natural $k$ consider the hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_{0k}(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q_k(x,y) \tag{6.36}$$

with the boundary conditions

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_{0k}(x), \quad h_k(u(\cdot,y))(y) = \psi_{1k}(y), \tag{6.37}$$

where

$$\mathcal{P}_{0k} \in L_\infty(D_{ab};\mathbb{R}^{n\times n}), \quad q_k \in L_\infty(D_{ab};\mathbb{R}^n),$$
$$\psi_{0k} \in L([0,a];\mathbb{R}^n), \quad \psi_{1k} \in L_\infty([0,b];\mathbb{R}^n)$$

and $h_k : \widetilde{\mathbb{C}}([0,a];\mathbb{R}^n) \to L_\infty([0,b];\mathbb{R}^n)$ is a linear continuous operator.

As above, we use representations (6.6)-(6.8) and by $M_{0k}$ and $M_k$ are meant the matrix functions given by the equalities

$$M_{0k}(y) = H_{0k}(y) + \int_0^a H_k(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds,$$
$$M_k(x,y) = H_k(x,y)Z_2(x,y) + \int_x^a H_k(s,y)\frac{\partial Z_2(s,y)}{\partial s}ds.$$

*Let* $\mathcal{P}_1 \in \widetilde{\mathbb{C}}_\infty^{(0,-1)}(D_{ab};\mathbb{R}^{n\times n})$, $\mathcal{P}_2 \in \widetilde{\mathbb{C}}_\infty^{(-1,0)}(D_{ab};\mathbb{R}^{n\times n})$,

$$\operatorname*{ess\,inf}_{0\le y\le b} |\det(M_0(y))| > 0,$$
$$\sup_{k\ge 1}\|\mathcal{P}_{0k}\|_{L_\infty} < +\infty, \quad \sup_{k\ge 1}\|q_k\|_{L_\infty} < +\infty, \tag{6.38}$$
$$\lim_{k\to+\infty}\|\mathcal{P}_{0k} - \mathcal{P}_0\|_L^{(0)} = 0, \quad \lim_{k\to+\infty}\|q_k - q\|_L^{(0)} = 0 \tag{6.39}$$

*and*

$$\lim_{k\to+\infty}\|\psi_{0k} - \psi_0\|_L = 0, \quad \lim_{k\to+\infty}\|\psi_{1k} - \psi_1\|_{L_\infty} = 0, \\ \lim_{k\to+\infty}\|\!|h_k - h|\!\| = 0. \tag{6.40}$$

*Then, starting from some* $k_0$, *problem* (6.36),(6.37) *has the unique generalized solution* $u_k$ *and*

$$\lim_{k\to+\infty}\|u_k - u_0\|_{L_\infty} = 0, \tag{6.41}$$

*where* $u_0$ *is the generalized solution of problem* (6.1),(6.3).

*Proof.* As shown above, conditions (6.21)-(6.23) follow from the convergence by the norm of the sequence of operators $h_k$ $(k = 1, 2, \cdots)$ to $h$ and the restrictions imposed on $\det(M_0(y))$, where $k_0$ is a natural number and $\delta$ is a positive constant independent of $k$.

The unique solvability of problem (6.1),(6.3) as well as of problem (6.36), (6.37) for any $k \geq k_0$ follows from Theorem 4.4. In proving this theorem it was admitted that $u = u_0$ and $u = u_k$ satisfy, respectively, the boundary conditions

$$\begin{aligned} &\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_0(x),\\ &u(0,y) = \psi(y) + \int_0^y\int_0^a Q(y,s,t)u(s,t)dsdt \end{aligned} \tag{6.42}$$

and

$$\begin{aligned} &\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial y} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi_{0k}(x),\\ &u(0,y) = \psi_k(y) + \int_0^y\int_0^a Q_k(y,s,t)u(s,t)dsdt, \end{aligned} \tag{6.43}$$

where

$$\begin{aligned} \psi(y) &= M_0^{-1}(y)\psi_1(y) - M_0^{-1}(y)\int_0^a M(x,y)Z_2^{-1}(x,y)Z_1(x,y)\,\times\\ &\quad\times\Big[\psi_0(x) + \int_0^y Z_1^{-1}(x,t)q(x,t)dt\Big]dx,\\ Q(y,s,t) &= -M_0^{-1}(y)M(s,y)Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}(s,t),\\ \mathcal{P}(x,y) &= \mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial\mathcal{P}_2(x,y)}{\partial y} \end{aligned}$$

and

$$\begin{aligned} \psi_k(y) &= M_{0k}^{-1}(y)\psi_{1k}(y) - M_{0k}^{-1}(y)\int_0^a M_k(x,y)Z_2^{-1}(x,y)Z_1(x,y)\,\times\\ &\quad\times\Big[\psi_{0k}(x) + \int_0^y Z_1^{-1}(x,t)q_k(x,t)dt\Big]dx,\\ Q_k(y,s,t) &= -M_{0k}^{-1}(y)M_k(s,y)Z_2^{-1}(s,y)Z_1(s,y)Z_1^{-1}(s,t)\mathcal{P}_k(s,t),\\ \mathcal{P}_k(x,y) &= \mathcal{P}_{0k}(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial\mathcal{P}_2(x,y)}{\partial y}. \end{aligned}$$

More exactly, problem (6.1),(6.3) is equivalent to problem (6.1),(6.42) and problem (6.36),(6.37) to problem (6.36),(6.43).

It follows from conditions (6.21)-(6.23) and (6.38)-(6.40) that

$$\sup_{k\geq 1}\|Q_k\|_{L_\infty} < +\infty, \quad \lim_{k\to+\infty}\|\psi_k - \psi\|_{L_\infty} = 0$$

and

$$\operatorname*{ess\,sup}_{\substack{0\le y_1<y_2\le b\\ (x,y)\in\mathcal{D}_{ab}}}\left\|\int_{y_1}^{y_2}\int_0^x\big[Q_k(y,s,t)-Q(y,s,t)\big]dsdt\right\|\to 0\quad\text{for}\ \ k\to+\infty.$$

Consequently, all conditions of Lemma 3.15 hold and by virtue of this lemma equality (6.41) is valid. ■

*Remark* 6.2. If $h_k = h\ \ (k=1,2,\cdots)$, the operator $h$ being such that

$$H\in\widetilde{\mathbb{C}}_{\infty}^{(0,-1)}(D_{ab};\mathbb{R}^{n\times n}), \tag{6.44}$$

then the assumption

$$\sup_{k\ge1}\|q_k\|_{L_\infty}<+\infty \tag{6.45}$$

in Theorem 6.4 becomes unnecessary. If, however, (6.44) is violated, then restriction (6.45) is essential and cannot be neglected. As an example, consider the problems

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=0 \tag{6.46}$$

$$\frac{\partial u(x,0)}{\partial x}=0,\quad u(0,y)+\int_0^a H(s)\frac{\partial u(s,y)}{\partial s}ds=0 \tag{6.47}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=-k^3\sin k^8x, \tag{6.48}$$

$$\frac{\partial u(x,0)}{\partial x}=0,\quad u(0,y)+\int_0^a H(s)\frac{\partial u(s,y)}{\partial s}ds=0, \tag{6.49}$$

where

$$H(x)=\sum_{m=1}^{+\infty}\frac{2}{m^2}\sin m^8x.$$

For these problems all conditions of Theorem 6.4, except (6.44), are fulfilled. On the other hand, problem (6.46),(6.47) has the unique solution $u_0(x,y)\equiv$ $\equiv 0$ and problem (6.48),(6.49) has the unique solution

$$u_k(x,y)=k^3y\int_0^a H(s)\sin k^8sds+\frac{y}{k^5}(\cos k^8x-1)$$

for any natural $k$.

However,

$$k^3\int_0^a H(s)\sin k^8sds=2k\int_0^a\sin^2k^8sds+$$

$$+\sum_{m\ne k,m=1}^{+\infty}\frac{2k^3}{m^2}\int_0^a\sin m^8s\sin k^8sds=k\int_0^a(1-\cos 2k^8s)ds+$$

$$+\sum_{m\neq k,m=1}^{+\infty}\frac{k^3}{m^2}\int_0^a[\cos(k^8-m^8)s-\cos(k^8+m^8)s]ds\geq$$

$$\geq ak-\frac{1}{2k^7}-\sum_{m\neq k,m=1}^{+\infty}\frac{k^3}{m^2}\Big(\frac{1}{|k^8-m^8|}+\frac{1}{k^8+m^8}\Big)>ak-\alpha,$$

where

$$\alpha=\sum_{m=1}^{+\infty}\frac{1}{m^2}.$$

Therefore

$$\|u_k-u_0\|_{L_\infty}>abk-\alpha b-2b\to+\infty \quad \text{for} \quad k\to+\infty,$$

i.e. (6.41) is violated. ■

CHAPTER III

## § 7.

This section deals with the problem on the existence and uniqueness in a strip $\mathcal{D}_b$ of a solution $u$ of the linear hyperbolic system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + $$
$$+\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y), \tag{7.1}$$

satisfying the conditions

$$u(x,0) = \varphi(x), \qquad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \Big( \Big\| \frac{\partial u(x,y)}{\partial x} \Big\| + \Big\| \frac{\partial u(x,y)}{\partial y} \Big\| \Big) < +\infty, \tag{7.2}$$

where

$$\begin{gathered} \mathcal{P}_j = \big(p_{jik}\big)_{j,k=1}^n \in L_\infty(\mathcal{D}_b; \mathbb{R}^{n\times n}) \ (j=0,1,2), \\ q = (q_i)_{i=1}^n \in L_\infty(\mathcal{D}_b;\mathbb{R}^n), \quad \varphi = (\varphi_i)_{i=1}^n \in \widetilde{\mathbb{C}}_\infty(\mathbb{R},\mathbb{R}^n). \end{gathered} \tag{7.3}$$

For an arbitrary function $z \in L_\infty(\mathcal{D}_b;\mathbb{R})$ the following notation will be used:

$$I_+(z) = \{y \in [0,b] : \sup_{x\geq 0} \int_0^x z(s,y)ds = +\infty\},$$

$$I_-(z) = \{y \in [0,b] : \sup_{x\geq 0} \int_0^x z(s,y)ds < +\infty \text{ and } \sup_{x\leq 0} \int_0^x z(s,y)ds = +\infty\},$$

$$I_0(z) = \{y \in [0,b] : \sup_{x\in\mathbb{R}} \int_0^x z(s,y)ds < +\infty\},$$

$$\chi(z)(y) = \begin{cases} +\infty & \text{for } y \in I_+(z) \\ 0 & \text{for } y \in I_0(z) \\ -\infty & \text{for } y \in I_-(z) \end{cases}.$$

For an arbitrary set $I \subset \mathbb{R}$ we denote by $\operatorname{mes} I$ its Lebesque measure and by $\bar{I}$ its closure.

*Let there exist constants $\alpha \in (0,1)$, $\beta > 0$ and essentially bounded measurable functions $a_{ik} : [0,b] \to [0,+\infty)$ $(i \neq k; i,k = 1,\dots,n)$ such that the spectral radius of the matrix $A(y) = (a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y) \equiv 0$ $(i = 1,\dots,n)$, is less than $\alpha$ almost for all $y \in [0,b]$ and the inequalities*

$$\Big| \int_x^{\chi(p_{2ii})(y)} \exp\Big( \int_s^x p_{2ii}(\xi,y)d\xi \Big) \times$$

$$\times \big(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| + |q_i(s,y)|\big)ds\Big| \le \beta \ \ (i,k = 1,\dots,n) \quad (7.4)$$

*and*

$$\Big| \int\limits_{x}^{\chi(p_{2ii})(y)} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \le a_{ik}(y) \qquad (7.5)$$
$$(i \ne k;\ i,k = 1,\dots,n)$$

*hold almost everywhere in* $\mathcal{D}_b$. *Then problem* (7.1),(7.2) *is solvable; moreover, the solution is unique if and only if*

$$\operatorname{mes} I_0(p_{2ii}) = 0 \quad (i = 1,\dots,n). \qquad (7.6)$$

*Proof.* In view of (7.3), without loss of generality it can be assumed that the inequality

$$\|\mathcal{P}_0(x,y)\| + b\|\mathcal{P}_1(x,y)\| + \|\mathcal{P}_2(x,y)\| \le \beta \qquad (7.7)$$

is fulfilled in $\mathcal{D}_b$. First we prove the solvability of the problem under consideration on the basis of Lemma 1.3.

Let

$$\psi_i(y) = \begin{cases} 1 & \text{for } y \in I_0(p_{2ii}) \\ 0 & \text{for } y \notin I_0(p_{2ii}) \end{cases}, \qquad (7.8)$$
$$\psi(y) = \big(\psi_i(y)\big)_{i=1}^n,$$

and let $c : [0,b] \to \mathbb{R}$ be an arbitrary continuous function. For any $z_j = (z_{ji})_{i=1}^n \in L_\infty(\mathcal{D}_b;\mathbb{R}^n)$ $(j = 1,2)$ and $i \in \{1,\dots,n\}$ we put

$$f_i(z_1,z_2)(x,y) = \sum_{k=1}^n p_{0ik}(x,y)\Big[\varphi_i(x) + \int_0^y z_{2i}(x,t)dt\Big] +$$
$$+ \sum_{k=1,\,k\ne i}^n p_{2ik}(x,y)z_{2k}(x,y) + \sum_{k=1}^n p_{1ik}(x,y)z_{1k}(x,y) + q_i(x,y), \quad (7.9)$$

$$g_{1i}(z_1,z_2)(x,y) = \varphi_i'(x) +$$
$$+ \int_0^y \big[p_{2ii}(x,t)z_{2i}(x,t) + f_i(z_1,z_2)(x,t)\big]dt, \qquad (7.10)$$

$$g_{2i}(z_1,z_2)(x,y) = \exp\Big(\int_0^x p_{2ii}(\xi,y)d\xi\Big)c(y)\psi_i(y) +$$
$$+ \int\limits_{\chi(p_{2ii})(y)}^{x} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big) f_i(z_1,z_2)(s,y)ds \qquad (7.11)$$

and

$$\begin{aligned} g_1(z_1,z_2)(x,y) &= \big(g_{1i}(z_1,z_2)(x,y)\big)_{i=1}^n, \\ g_2(z_1,z_2)(x,y) &= \big(g_{2i}(z_1,z_2)(x,y)\big)_{i=1}^n. \end{aligned} \tag{7.12}$$

By virtue of conditions (7.3)-(7.5) and (7.7)-(7.12) the operators $g_1$ and $g_2$ transform the space $L_\infty(\mathcal{D}_b;\mathbb{R}^n)\times L_\infty(\mathcal{D}_b;\mathbb{R}^n)$ into $L_\infty(\mathcal{D}_b;\mathbb{R}^n)$ and satisfy conditions (1.3) and (1.21), where $g_0(t)\equiv n\beta$, while $A_{01}$ and $A_{02}$ are constant $n\times n$ matrices whose all elements equal to $\beta$.

Thus, all conditions of Lemma 1.3 are satisfied and therefore system (1.1) has the unique solution $(z_1,z_2)$. Assume

$$u(x,y)=\varphi(x)+\int_0^y z_2(x,t)dt.$$

On account of conditions (7.3)-(7.5) equalities (7.9)-(7.11) yield that $u$ is locally absolutely continuous,

$$\begin{aligned} \frac{\partial^2u(x,y)}{\partial x\partial y}&=\frac{\partial z_2(x,y)}{\partial x}=\mathcal{P}_0(x,y)u(x,y)+\mathcal{P}_1(x,y)z_1(x,y)+\\ &\qquad +\mathcal{P}_2(x,y)z_2(x,y)+q(x,y),\\ \frac{\partial u(x,y)}{\partial x}&=\varphi'(x)+\int_0^y\frac{\partial z_2(x,t)}{\partial x}dt=\varphi'(x)+\\ &\quad+\int_0^y[\mathcal{P}_0(x,t)u(x,t)+\mathcal{P}_1(x,t)z_1(x,t)+\\ &\quad+\mathcal{P}_2(x,t)z_2(x,t)+q(x,t)]dt=z_1(x,y) \end{aligned}$$

and

$$\frac{\partial u(0,y)}{\partial y}=c(y)\psi(y). \tag{7.13}$$

Consequently, $u$ is the solution of problem (7.1),(7.2). On the other hand, if $\operatorname{mes} I_0(p_{2ii})>0$ for any $i\in\{1,\dots,n\}$, then in view of equalities (7.8) and (7.13) and an arbitrary choice of $c$ it is evident that the problem under consideration has an infinite dimensional set of solutions.

To complete the proof we have to show that (7.1),(7.2) is uniquely solvable when (7.6) is fulfilled.

In view of (7.6) we get from (7.8) and (7.11) that

$$g_{2i}(z_1,z_2)(x,y)=\int\limits_{\chi(p_{2ii})(y)}^{x}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)f_i(z_1,z_2)(s,y)ds \qquad (i=1,\dots,n). \tag{7.14}$$

As proved above, system (1.1) has the unique solution $(z_1^0,z_2^0)$; moreover, $u^0(x,y)=\varphi(x)+\int_0^y z_2^0(x,t)dt$ is the solution of problem (7.1),(7.2). Let $u$

be an arbitrary solution of this problem. Assume

$$z_1(x,y)=\frac{\partial u(x,y)}{\partial x}, \quad z_2(x,y)=\frac{\partial u(x,y)}{\partial y}.$$

Then

$$\operatorname*{ess\,sup}_{y\in[0,b]}\Big(\sup_{x\in\mathbb{R}}\|z_2(x,y)\|\Big)<+\infty, \tag{7.15}$$

and for any $x_{0i}\in\mathbb{R}$ $(i=1,\dots,n)$ we have

$$\begin{gathered} z_1(x,y)=g_1(z_1,z_2)(x,y), \\ z_{2i}(x,y)=\exp\Big(\int_{x_{0i}}^{x}p_{2ii}(\xi,y)d\xi\Big)z_{2i}(x_{0i},y)+ \\ +\int_{x_{0i}}^{x}\exp\Big(\int_{s}^{x}p_{2ii}(\xi,y)d\xi\Big)f_i(z_1,z_2)(s,y)ds \quad (i=1,\dots,n). \end{gathered} \tag{7.16}$$

On the other hand, in view of (7.6), we have

$$\liminf_{x_{0i}\to\chi(p_{2ii})(y)}\int_{x_{0i}}^{x}p_{2ii}(\xi,y)d\xi=-\infty \quad \text{for} \quad x\in\mathbb{R} \tag{7.17}$$

almost for every $y\in[0,b]$. By virtue of conditions (7.14),(7.15) and (7.17) it follows from (7.16) that $(z_1,z_2)$ is a solution of system (1.1). Thus $z_i(x,y)\equiv$ $\equiv z_i^0(x,y)$ $(i=1,2)$ and, consequently,

$$\frac{\partial u(x,y)}{\partial x}\equiv z_1^0(x,y), \quad \frac{\partial u(x,y)}{\partial y}\equiv z_2^0(x,y).$$

Then with regard to (7.2) we get

$$u(x,y)\equiv u^0(x,y). \quad \blacksquare$$

The following theorem can be proved similarly to Theorem 7.1.

$'$ *Let $\mathcal{P}_j$ $(j=0,1,2)$ and $q$ be continuous and bounded, $\varphi$ be continuously differentiable and bounded together with its derivative and for every $i\in\{1,\dots,n\}$ either $\bar I_+(p_{2ii})=[0,b]$ or $\bar I_-(p_{2ii})=[0,b]$. Moreover,let the integrals*

$$\begin{gathered} \int_0^{\chi_i}\exp\Big(\int_s^0 p_{2ii}(\xi,y)d\xi\Big)\big(|p_{0ik}(s,y)|+|p_{1ik}(s,y)|+ \\ +(1-\delta_{ik})|p_{2ik}(s,y)|+|q_i(s,y)|\big)ds \quad (k=1,\dots,n) \end{gathered} \tag{7.18}$$

*where $\chi_i=+\infty$ and $\chi_i=-\infty$ for $\bar I_+(p_{2ii})=[0,b]$ and $\bar I_-(p_{2ii})=[0,b]$, respectively, converge uniformly with respect to $y\in[0,b]$. Let, besides, there exist a constant $\beta>0$ and continuous functions $a_{ik}:[0,b]\to[0,+\infty)$ $(i\neq k;\ i,k=1,\dots,n)$ such that the spectral radius of the matrix $A(y)=$*

$= (a_{ik}(y))_{i,k=1}^n$, *where* $a_{ii}(y) \equiv 0$ $(i = 1, \dots, n)$, *is less than unity for every* $y \in [0, b]$ *and the inequalities*

$$\Big|\int_x^{\chi_i} \exp\Big(\int_s^x p_{2ii}(\xi, y)d\xi\Big)\big(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| + \\ +|q_i(s,y)|\big)ds\Big| \leq \beta \quad (k = 1, \dots, n) \tag{7.19}$$

*and*

$$\Big|\int_x^{\chi_i} \exp\Big(\int_s^x p_{2ii}(\xi, y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \leq a_{ik}(y) \\ (i \neq k;\ i, k = 1, \dots, n) \tag{7.20}$$

*hold in* $\mathcal{D}_b$. *Then problem* (7.1), (7.2) *has the unique classical solution.*

*Remark* 7.1. If for any $i \in \{1, \dots, n\}$ integral (7.18) does not converge uniformly, then problem (7.1),(7.2) may have no classical solutions. Indeed in $\widetilde{\mathbb{C}}_{loc}(\mathcal{D}_b; \mathbb{R})$ the problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \Big|y - \frac{b}{2}\Big|^3 \frac{\partial u(x,y)}{\partial y} + \Big(y - \frac{b}{2}\Big)^3, \\ u(x,0) = 0, \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \Big(\Big|\frac{\partial u(x,y)}{\partial x}\Big| + \Big|\frac{\partial u(x,y)}{\partial y}\Big|\Big) < +\infty \tag{7.21}$$

has the unique solution

$$u(x,y) = \frac{b}{2} - \Big|y - \frac{b}{2}\Big|,$$

which is not classical despite the fact that all conditions of Theorem 7.1$'$ except those of the uniform convergence of integral (7.18) which in this case takes the form

$$\int_0^{+\infty} \exp\Big(-\Big|y - \frac{b}{2}\Big|^3 s\Big)\Big|y - \frac{b}{2}\Big|^3 ds,$$

are fulfilled.

Remark 7.2. The conditions of Theorem 7.1$'$ ensure the uniqueness only of a classical but not of an absolutely continuous solution. In fact, let $p_2 : [0, b] \to [0, +\infty)$ be a continuous function with a nowhere dense set of zeros of positive measure. It is clear that

$$I_0(p_2) = \{y \in [0,b] : p_2(y) = 0\}, \\ I_+(p_2) = [0,b] \backslash I_0(p_2), \quad \bar{I}_+(p_2) = [0,b].$$

For an arbitrary continuous $c : [0, b] \to \mathbb{R}$ the function

$$u_c(x,y) = \int_0^y \exp(p_2(t)x)c(t)\psi(t)dt,$$

where

$$\psi(y)=\begin{cases}1 & \text{for } y\in I_0(p_2)\\ 0 & \text{for } y\in I_+(p_2)\end{cases}$$

belongs to $\widetilde{\mathbb{C}}_{loc}(\mathcal{D}_b;\mathbb{R})$ and this function is a solution of the equation

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=p_2(y)\frac{\partial u(x,y)}{\partial y}, \tag{7.22}$$

satisfying condition (7.21). On the other hand, problem (7.22),(7.21), for which all conditions of Theorem 7.1$'$ hold, has the unique classical solution $u_0(x,y)\equiv 0$.

*Let there exist constants $a>0$, $\alpha\in(0,1)$, $\beta_0>0$ and essentially bounded measurable functions $a_{ik}:[0,b]\to\mathbb{R}_+$ $(i\neq k,\ i,k==1,\dots,n)$, $\sigma_i:[0,b]\to\{-1,1\}$ and $\gamma_i:[0,b]\to[0,+\infty)$ $(i=1,\dots,n)$ such that the spectral radius of the matrix $A(y)=(a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y)\equiv 0$ $(i=1,\dots,n)$, is less than $\alpha$ almost for all $y\in[0,b]$ and the inequalities*

$$\sigma_i(y)\int\limits_x^{x+a} p_{2ii}(\xi,y)d\xi\leq-\gamma_i(y)\quad(i=1,\dots,n), \tag{7.23}$$

$$\int\limits_x^{x+a}\big(|p_{0ik}(s,y)|+|p_{1ik}(s,y)|+|q_i(s,y)|\big)ds\leq$$
$$\leq\beta_0\gamma_i(y)\quad(i,k=1,\dots,n) \tag{7.24}$$

*and*

$$\Big|\int\limits_{x-\sigma_i(y)a}^{x}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big|\leq$$
$$\leq a_{ik}(y)\big(1-\exp(-\gamma_i(y))\big)\quad(i\neq k;\ i,k=1,\dots,n) \tag{7.25}$$

*hold almost everywhere in $\mathcal{D}_b$. Then problem (7.1),(7.2) is solvable, and for the uniqueness of a solution it is sufficient that*

$$\mathrm{mes}\{y\in[0,b]:\gamma_i(y)=0\}=0\ \ (i=1,\dots,n). \tag{7.26}$$

*Proof.* Put

$$\beta_1=\|\mathcal{P}_2\|_{L_\infty}<+\infty,\quad \beta=\beta_0\exp(a\beta_1),$$
$$I_i^0=\{y\in[0,b]:\gamma_i(y)=0\},\quad I_i^+=\{y\in[0,b]:\gamma_i(y)>0,\ \sigma_i(y)=-1\},$$
$$I_i^-=\{y\in[0,b]:\gamma_i(y)>0,\ \sigma_i(y)=1\},\ \ (i=1,\dots,n).$$

Then by virtue of (7.23),

$$[0,b]=I_i^+\cup I_i^0\cup I_i^-,\quad I_+(p_{2ii})\supset I_i^+,\ \ I_-(p_{2ii})\supset I_i^-$$

and

$$I_i^0 \supset I_0(p_{2ii}) \quad (i=1,\dots,n). \tag{7.27}$$

If $y \in I_i^0$, then in view of (7.24) without loss of generality we may assume that

$$p_{0ik}(x,y) \equiv p_{1ik}(x,y) \equiv q_i(x,y) \equiv 0 \quad \text{for} \ \ x \in \mathbb{R}, \ k \in \{1,\dots,n\}.$$

Therefore it is obvious that inequalities (7.4) and (7.5) are fulfilled in $\mathbb{R} \times I_i^0$ for every $i \in \{1,\dots,n\}$.

If $y \in I_i^+$, then $\chi(p_{2ii})(y) = +\infty$ and in view of inequalities (7.23)–(7.25) we have

$$\int_x^{+\infty} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)\big(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| + |q_i(s,y)|\big)ds =$$
$$= \sum_{m=1}^{+\infty} \exp\Big(-\sum_{l=1}^{m} \int\limits_{x+(l-1)a}^{x+la} p_{2ii}(\xi,y)d\xi\Big) \times$$
$$\times \int\limits_{x+(m-1)a}^{x+ma} \exp\Big(\int\limits_{s}^{x+ma} p_{2ii}(\xi,y)d\xi\Big)\big(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| +$$
$$+|q_i(s,y)|\big)ds \le \beta_0\gamma_i(y)\exp(a\beta_1)\sum_{m=1}^{+\infty}\exp(-m\gamma_i(y)) =$$
$$= \beta\gamma_i(y)\big(\exp(\gamma_i(y)) - 1\big)^{-1} \le \beta \quad (k=1,\dots,n)$$

and

$$\int_x^{+\infty} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds =$$
$$= \sum_{m=1}^{+\infty} \exp\Big(\int\limits_{x}^{x+(m-1)a} p_{2ii}(\xi,y)d\xi\Big) \times$$
$$\times \int\limits_{x+(m-1)a}^{x+ma} \exp\Big(\int\limits_{s}^{x+(m-1)a} p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds \le$$
$$\le a_{ik}(y)(1-\exp(-\gamma_i(y)))\sum_{m=1}^{+\infty}\exp(-(m-1)\gamma_i(y)) = a_{ik}(y)$$
$$(k \ne i; \ i, \, k = 1,\dots,n).$$

Therefore, inequalities (7.4) and (7.5) are fulfilled in $\mathbb{R} \times I_i^+$. Similarly, we can prove that these inequalities are also fulfilled in $\mathbb{R} \times I_i^-$ $(i=1,\dots,n)$.

As for (7.6), these equalities are fulfilled by virtue of (7.27) if (7.26) holds. The validity of Corollary 7.1 becomes evident by applying Theorem 7.1. ■

*′ Let $\mathcal{P}_j$ $(j = 0,1,2)$ and $q$ be continuous and bounded, $\varphi$ be continuously differentiable and bounded together with its derivative. Let, besides, there exist constants $a > 0$, $\beta_0 > 0$, $\varepsilon > 0$, $\sigma_i \in \{-1,1\}$ $(i = 1,\dots,n)$ and continuous functions $\gamma_i : [0,b] \to [0,+\infty)$ $(i = 1,\dots,n)$ and $a_{ik} : [0,b] \to [0,+\infty)$ such that the set of zeros of $\gamma_i$ is nowhere dense in $\mathcal{D}_b$ $(i = 1,\dots,n)$, the spectral radius of the matrix $A(y) = (a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y) \equiv 0$ $(i = 1,\dots,n)$, is less than unity for every $y \in [0,b]$ and the inequalities*

$$\sigma_i \int\limits_x^{x+a} p_{2ii}(\xi,y)d\xi \le -\gamma_i(y) \quad (i = 1,\dots,n), \tag{7.28}$$

$$\int\limits_x^{x+a} \big(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| + (1-\delta_{ik})|p_{2ik}(s,y)| + |q_i(s,y)|\big)ds \le$$
$$\le \beta_0 \gamma_i^{1+\varepsilon}(y) \quad (i = 1,\dots,n) \tag{7.29}$$

*and*

$$\Big| \int\limits_{x-\sigma_i a}^{x} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \le$$
$$\le a_{ik}(y)\big(1-\exp(-\gamma_i(y))\big) \quad (i \ne k;\ i,k = 1,\dots,n) \tag{7.30}$$

*holds in $\mathcal{D}_b$. Then problem (7.1), (7.2) has one and only one classical solution.*

*Proof.* In view of (7.28),

$$\sigma_i\Big(\int_0^x p_{2ii}(\xi,y)d\xi\Big)\operatorname{sign} x \le -\Big(\frac{|x|}{a}+1\Big)\gamma_i(y) + 2a\beta_1 \\ (i = 1,\dots,n), \tag{7.31}$$

where

$$\beta_1 = \sup_{(x,y)\in\mathcal{D}_b} \|\mathcal{P}_2(x,y)\|.$$

Taking into account the fact that the set of zeros of $\gamma_i$ is nowhere dense in $[0,b]$ $(i = 1,\dots,n)$, we obtain from (7.31) that

$$\bar{I}_+(p_{2ii}) = [0,b] \quad \text{for} \quad \sigma_i = -1, \quad \bar{I}_-(p_{2ii}) = [0,b] \quad \text{for} \quad \sigma_i = 1. \tag{7.32}$$

Let us show that for every $i \in \{1,\dots,n\}$ integrals (7.18) converge uniformly with respect to $y \in [0,b]$. In view of (7.32) we have $\chi_i = +\infty$ and $\chi_i = -\infty$ for $\sigma_i = -1$ and $\sigma_i = 1$, respectively. First we consider the case when $\sigma_i = -1$. By (7.29) and (7.31) for any $x > 0$ and $y \in [0,b]$ we have

$$\int_x^{+\infty} \exp\Big(\int_s^0 p_{2ii}(\xi,y)d\xi\Big)(|p_{0ik}(s,y)| + |p_{1ik}(s,y)| +$$

$$+(1-\delta_{ik})|p_{2ik}(s,y)|+|q_i(s,y)|\big)ds=$$

$$=\sum_{m=1}^{+\infty}\int\limits_{x+(m-1)a}^{x+ma}\exp\Big(-\int_s^0 p_{2ii}(\xi,y)d\xi\Big)\big(|p_{0ik}(s,y)|+|p_{1ik}(s,y)|+$$

$$+(1-\delta_{ik})|p_{2ik}(s,y)|+|q_i(s,y)|\big)ds\le$$

$$\le\beta_0\exp(a\beta_1)\gamma_i^{1+\varepsilon}(y)\sum_{m=1}^{+\infty}\exp\Big(-\big(\frac{x}{a}+m\big)\gamma_i(y)\Big)=$$

$$=\beta_0\exp(a\beta_1)\gamma_i^{1+\varepsilon}(y)\big(\exp(\gamma_i(y))-1\big)^{-1}\exp\Big(-\frac{x}{a}\gamma_i(y)\Big)\le$$

$$\le\beta_0\exp(a\beta_1)\gamma_i^{\varepsilon}(y)\exp\Big(-\frac{x}{a}\gamma_i(y)\Big)\le\beta_2x^{-\varepsilon}\quad(k=1,\dots,n),$$

where $\beta_2=\beta_0\exp(a\beta_1-\varepsilon)(a\varepsilon)^{\varepsilon}$. Consequently, for $\sigma_i=-1$ integrals (7.18) converge uniformly. Analogously we can prove that integrals (7.18) converge uniformly for $\sigma_i=1$ as well.

Similarly to the prove of Corollary 7.1, we can show that conditions (7.28)-(7.30) imply conditions (7.19) and (7.20).

Applying now Theorem 7.1′, the validity of Corollary 7.1′ becomes obvious. ■

*Let the inequalities*

$$\begin{aligned}&\sigma_i(y)p_{2ii}(x,y)\le l_{ii}\quad(i=1,\dots,n),\\&|p_{2ik}(x,y)|\le l_{ik}\quad(i\ne k;\ i,k=1,\dots,n),\end{aligned}\tag{7.33}$$

*where* $\sigma_i:[0,b]\to\{-1,1\}$ $(i=1,\dots,n)$ *are measurable functions and* $l_{ik}$ $(i,k=1,\dots,n)$ *are constants such that the real parts of eigenvalues of the matrix* $(l_{ik})_{i,k=1}^n$ *are negative. Then problem* (7.1),(7.2) *has one and only one solution.*

*Proof.* Since $l_{ik}$ $(i\ne k;\,i,k=1,\dots,n)$ are non-negative and eigenvalues of the matrix $(l_{ik})_{i,k=1}^n$ are negative, we have

$$l_{ii}<0\quad(i=1,\dots,n)\tag{7.34}$$

and the spectral radius of the matrix $(a_{ik})_{i,k=1}^n$, where

$$a_{ii}=0\ \ (i=1,\dots,n),\quad a_{ik}=\frac{l_{ik}}{|l_{ii}|}\ \ (i\ne k;\ i,k=1,\dots,n),$$

is less than unity.

In view of (7.3),(7.33) and (7.34), inequalities (7.23) and (7.24) hold almost everywhere in $\mathcal{D}_b$, where $a=1$, $\gamma_i(y)\equiv|l_{ii}|$,

$$\beta_0=\big(\|\mathcal{P}_0\|_{L_\infty}+\|\mathcal{P}_1\|_{L_\infty}+\|q\|_{L_\infty}\big)/|l_{ii}|.$$

Besides,

$$\Big| \int\limits_{x-\sigma_i(y)}^{x} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \le$$

$$\le l_{ik}\Big| \int\limits_{x-\sigma_i(y)}^{x} \exp\big(\sigma_i(y)(s-x)|l_{ii}|\big)ds\Big| =$$

$$= \frac{l_{ik}}{|l_{ii}|}\big(1-\exp(-|l_{ii}|)\big) \quad (i \ne k;\ \ i,k = 1,\dots,n),$$

i.e. inequalities (7.25) also hold. Now applying Corollary 7.1, the unique solvability of problem (7.1),(7.2) becomes obvious. ■

Based on Corollary 7.1′, similarly to Corollary 7.2, we prove the validity of

′ *Let $\mathcal{P}_j$ $(j = 0,1,2)$ and $q$ be continuous and bounded, $\varphi$ be continuously differentiable and bounded together with its derivative and the inequalities*

$$\sigma_i p_{2ii}(x,y) \le l_{ii} \quad (i = 1,\dots,n),$$
$$|p_{2ik}(x,y)| \le l_{ik} \quad (i \ne k;\ \ i,k = 1,\dots,n),$$

*hold in $\mathcal{D}_b$, where $\sigma_i \in \{-1,1\}$ and $l_{ik}$ $(i,k = 1,\dots)$ are constants and the real parts of eigenvalues of matrix $(l_{ik})_{i,k=1}^n$ are negative. Then problem* (7.1), (7.2) *has one and only one solution and this solution is classical.*

## § 8.

In this section for the system

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} +$$
$$+\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y) \tag{8.1}$$

the conditions of the existence and uniqueness of a solution (generalized, absolutely continuous and classical) defined in the strip $\mathcal{D}_b$ and satisfying one of the following two boundary conditions

$$u(x,0) = \varphi(x), \quad u(x+a,y) = u(x,y) \tag{8.2}$$

and

$$\lim_{y\to 0}\Big(\frac{\partial u(x,y)}{\partial x} - \mathcal{P}_2(x,y)u(x,y)\Big) = \psi(x), \quad u(x+a,y) = u(x,y) \tag{8.3}$$

are established. Besides, we investigate the problem of relation between problem (8.1),(8.2) and the problem of bounded solutions

$$u(x,0)=\varphi(x), \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b}\Big(\Big\|\frac{\partial u(x,y)}{\partial x}\Big\|+\Big\|\frac{\partial u(x,y)}{\partial y}\Big\|\Big). \tag{8.4}$$

Throughout the remainder of this section it is assumed that $a$ is a positive number, $\mathcal{P}_j=\big(p_{jik}\big)_{i,k=1}^{n}:\mathcal{D}_b\to\mathbb{R}^{n\times n}$ $(j=0,1,2)$ and $q=(q_i)_{i=1}^{n}:\mathcal{D}_b\to\mathbb{R}^n$ are measurable and essentially bounded, $\varphi:\mathbb{R}\to\mathbb{R}^n$ is absolutely continuous and $\psi:\mathbb{R}\to\mathbb{R}^n$ is locally summable,

$$\begin{gathered}\mathcal{P}_i(x+a,y)\equiv\mathcal{P}_i(x,y)\ (i=0,1,2), \quad q(x+a,y)\equiv q(x,y),\\ \varphi(x+a)\equiv\varphi(x), \quad \psi(x+a)\equiv\psi(x).\end{gathered} \tag{8.5}$$

Moreover, use is made of the notation

$$N_0(y)=Z_2(a,y)-E,$$
$$N(y)=\int_0^a Z_2^{-1}(s,y)\big[\mathcal{P}_0(s,y)+\mathcal{P}_1(s,y)\mathcal{P}_2(s,y)\big]Z_2(s,y)ds.$$

Let $u$ be a solution of problem (8.1),(8.2). Then its restriction on $\mathcal{D}_{ab}$ satisfies the boundary conditions

$$u(x,0)=\varphi(x), \quad \frac{\partial u(a,y)}{\partial y}=\frac{\partial u(0,y)}{\partial y}. \tag{8.6}$$

Assume now that $u$ is a solution of problem (8.1),(8.6). Then in view of the periodicity of $\varphi$ we shall have $u(a,y)=u(0,y)$. Let $\overline{u}$ be the extension of $u$ in $\mathcal{D}_b$ satisfying the condition

$$\overline{u}(x+a,y)\equiv\overline{u}(x,y).$$

In view of (8.5) it is evident that $\overline{u}$ is a solution of problem (8.1),(8.2). Consequently, problems (8.1),(8.2) and (8.1),(8.6) are equivalent when (8.5) holds. But (8.1),(8.6) is the special case of problem (4.1),(4.2) for $h(v)(y)=$ $=v(a)-v(0)$ and $\varphi_1(y)\equiv 0$. Now from representations (4.7) and (4.10) it is clear that $H_0\equiv\Theta$, $H(x,y)\equiv E$,

$$M_0(y)=N_0(y), \quad M(x,y)=Z_2(a,y).$$

Therefore Theorem 4.1 and Corollaries 4.1 and 4.1′ yield the following assertions.

*Let the vector and the matrix functions $Z_2^{-1}q$, $Z_2^{-1}\mathcal{P}_0$, $(1+\|\varphi_0'\|)\times Z_2^{-1}\mathcal{P}_1$ be $N_0$-summable in $\mathcal{D}_{ab}$. Then problem* (8.1),(8.2) *is solvable and the non-singularity of $N_0$ almost everywhere in* $[0,b]$ *is necessary and sufficient for the solution to be unique. If $N_0$ is singular in the set with a positive measure, then the homogeneous problem corresponding to* (8.1), (8.2) *has an infinite dimensional set of solutions.*

*If $N_0$ is non-singular almost everywhere in $[0, b]$ and*

$$\int_0^a \int_0^b \|N_0^{-1}(y)\| \big( \|\mathcal{P}_0(x,y)\| + \\ +(1 + \|\varphi'(x)\|)\|\mathcal{P}_1(x,y)\| + \|q(x,y)\| \big) dxdy < +\infty,$$

*then problem* (8.1), (8.2) *has one and only one solution.*

$'$ *If $\mathcal{P}_i$ $(i = 0, 1, 2)$ and $q$ are continuous, $\varphi_0$ is continuously differentiable and $N_0$ is non-singular in $[0, b]$, then problem* (8.1), (8.2) *has the unique solution and this solution is classical.*

Assume

$$B_0(y) = \int_0^a \mathcal{P}_2(s,y)ds,$$

and when $B_0(y)$ is nonsingular

$$B(y) = \max_{0 \le x \le a} \Big[ \int_0^x \Big| B_0^{-1}(y) \int_0^s \mathcal{P}_2(\xi,y)d\xi \mathcal{P}_2(s,y) \Big| ds + \\ + \int_x^a \Big| B_0^{-1}(y) \int_s^a \mathcal{P}_2(\xi,y)d\xi \mathcal{P}_2(s,y) \Big| ds \Big].$$

For $k = 2$ and $m = 1$ the following assertions follow from Corollaries 4.2 and 4.2$'$.

*Let the inequalities*

$$\det(B_0(y)) \neq 0, \quad r(B(y)) < 1 \tag{8.7}$$

*hold almost everywhere in $[0, b]$ and*

$$\int_0^b \big\| \big(E - B(y)\big)^{-1} \big\| \big\| B_0^{-1}(y) \big\| dy < +\infty.$$

. *Then problem* (8.1),(8.2) *has one and only one solution.*

$'$ *If $\mathcal{P}_i$ $(i = 0, 1, 2)$ and $q$ are continuous, $\varphi$ is continuously differentiable and inequalities* (8.7) *hold in $[0, b]$, then problem* (8.1),(8.2) *has one and only one solution and this solution is classical.*

The conditions for the unique solvability of problem (8.1),(8.2) are given also in [2,4,7,42], where the periodic boundary value problem is investigsted for quasilinear hyperbolic equations and systems with continuous right sides.

For example, L.Cesari [7] proved the solvability of problem (8.1),(8.2) under the assumptions that $\det(N_0(0)) \neq 0$, $b$ is sufficiently small and

$$2a\|\mathcal{P}_2(x,0)\| < 1 \quad \text{for} \quad x \in \mathbb{R}. \tag{8.8}$$

It is obvious that this result follows from Corollary 8.1$'$; moreover, condition (8.8) is unnecessary.

A.K.Aziz and S.L.Brodsky [2] considered a system with a small parameter $\varepsilon > 0$

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \varepsilon\Big[\mathcal{P}_0(x,y)u(x,y)+\mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x}+\mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y}+q(x,y)\Big]$$

and proved the unique solvability of problem (8.1),(8.2) for sufficiently small $b$ and $\varepsilon$ assuming that

$$\det\Big(\int_0^a \mathcal{P}_2(s,0)ds\Big) \neq 0.$$

This result also follows from Corollary 8.1′.

A.K.Aziz and A.M.Meyers [4] considered problem for $n = 1$ and proved its unique solvability under assumptions that $\mathcal{P}_1$ has a continuous partial derivative in the first argument,

$$\mathcal{P}_2(x,y) \neq 0 \quad \text{for} \quad (x,y) \in \mathcal{D}_b \tag{8.9}$$

and

$$l\Big[\frac{1}{l_*} + \frac{\exp(l^* a) - 1}{l^*(\exp(l_* a) - 1)}\Big] < 1, \tag{8.10}$$

where

$$l_* = \min_{(x,y)\in\mathcal{D}_b} |\mathcal{P}_2(x,y)|, \quad l^* = \max_{(x,y)\in\mathcal{D}_b} |\mathcal{P}_2(x,y)|,$$

$$l = \max_{(x,y)\in\mathcal{D}_b} \Big|\mathcal{P}_0(x,y) + \mathcal{P}_1(x,y)\mathcal{P}_2(x,y) - \frac{\partial \mathcal{P}_1(x,y)}{\partial x}\Big|.$$

But, in view of Corollary 8.1′, for $n = 1$ it is sufficient to have instead of condition (8.9) a more weaker condition

$$\int_0^a \mathcal{P}_2(s,y)ds \neq 0 \quad \text{for} \quad 0 \le y \le b.$$

As for condition (8.10) and the requirement for $\frac{\partial \mathcal{P}_1}{\partial x}$ to exist, they are unnecessary.

B.P.Tkach [42] proved the unique solvability of problem (8.1),(8.2) under the assumptions that

$$\det(B_0(y)) \neq 0, \quad r(\overline{B}(y)) < 1 \quad \text{for} \quad 0 \le y \le b,$$

where

$$\overline{B}(y) = \frac{a^2}{2}|B_0^{-1}(y)|\big[\max_{(x,y)\in\mathcal{D}_b} |\mathcal{P}_2(x,y)|\big]^2.$$

This result is the special case of Corollary 8.2 because

$$B(y) \le \overline{B}(y) \quad \text{for} \quad 0 \le y \le b.$$

The following assertions follow from Theorems 4.2, 4.2′, 4.5 and 4.5′.

*Let the restrictions of* $\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *on* $\mathcal{D}_{ab}$ *belong, respectively, to* $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ *and* $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n})$ $(\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument),* $\varphi_0$ *be absolutely continuous* (*continuously differentiable*) *and*

$$N_0(y)=\Theta,\quad \det(N(y))\neq 0 \quad \textit{for} \quad 0\leq y\leq b.$$

*Then problem* (8.1),(8.2) *is uniquely solvable* (*and its solution is classical*) *if and only if*

$$\int_0^a Z_2^{-1}(s,0)\big[\mathcal{P}_0(s,0)\varphi(s)+\mathcal{P}_1(s,0)\varphi'(s)+q(s,0)\big]ds=0.$$

*Let the restriction of* $\mathcal{P}_2$ *on* $\mathcal{D}_{ab}$ *belong to* $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$,

$$N_0(y)=\Theta,\quad \det(N(y))\neq 0 \ \ \textit{almost everywhere in} \ \ [0,b]$$

*and*

$$\int_0^b \|N^{-1}(y)\|dy<+\infty.$$

*Then problem* (8.1),(8.3) *has one and only one generalized solution if and only if*

$$\int_0^a Z_2(s,0)\varphi(s)ds=0. \tag{8.11}$$

′ *Let the restriction of* $\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *on* $\mathcal{D}_{ab}$ *belong, respectively, to* $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ *and* $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n})$ $(\mathcal{P}_i$ $(i=0,1,2)$ *and* $q$ *be continuous and have a continuous partial derivative in the second argument,* $\psi$ *be continuous) and*

$$N_0(y)=\Theta,\quad \det(N(y))\neq 0 \quad \textit{for} \quad 0\leq y\leq b.$$

*Then the fulfilment of condition* (8.11) *is necessary and sufficient for problem* (8.1), (8.3) *to have the unique generalized solution which is absolutely continuous* (*classical*).

For $\mathcal{P}_2(x,y)\equiv\Theta$ the result similar to Theorem 8.3 was obtained by S.V.Žestkov [51].

In Theorem 8.3′ the requirement for $N$ to be non-singular in $[0,b]$ is optimal and cannot be weakened. The following corollary concerning to the problem

$$\frac{\partial^2 u(x,y)}{\partial x\partial y}=\mathcal{P}_0(x,y)u(x,y)+\mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x}+q(x,y), \tag{8.12}$$

$$\lim_{y\to 0}\frac{\partial u(x,y)}{\partial x}=0,\quad u(x+a,y)=u(x,y) \tag{8.13}$$

verifies this assertion.

*Let the restriction of $\mathcal{P}_0$ in $\mathcal{D}_{ab}$ belong to $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ ($\mathcal{P}_0$ be continuous and have a continuous partial derivative in the second argument). Then for an arbitrary $\mathcal{P}_1$ and $q$ whose restrictions in $\mathcal{D}_{ab}$ belong repectively, to $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n\times n})$ and $\widetilde{\mathbb{C}}_\infty^{(-1,0)}(\mathcal{D}_{ab};\mathbb{R}^{n})$ (are continuous and have a continuous partial derivative in the second argument), problem (8.12), (8.13) has the unique generalized solution and this solution is absolutely continuous (classical) if and only if*

$$\det\Big(\int_0^a \mathcal{P}_0(s,y)ds\Big)\neq 0 \quad \textit{for } 0\le y\le b.$$

This corollary follows from Corollary 4.10 and Theorem 8.3′.

Now consider the problem of relation between problems (8.1),(8.2) and (8.1),(8.4).

*Let ${\varphi_0}'$ be essentially bounded and almost for every $y\in$ $\in[0,b]$*

$$\gamma_{0i}(y)\equiv\int_0^a p_{2ii}(s,y)ds\neq 0 \quad (i=1,\dots,n). \tag{8.14}$$

*Let, besides, there exist constants $\alpha\in(0,1)$,$\beta_0>0$ and essentially bounded measurable functions $a_{ik}:[0,b]\to\mathbb{R}_+$ $(i\neq k,\quad i,k=1,\dots,n)$, such that the spectral radius of the matrix $A(y)=(a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y)\equiv 0$ $(i=1,\dots,n)$, is less than $\alpha$ almost for all $y\in[0,b]$ and the inequalities*

$$\begin{gathered}\int_0^a \big(|p_{0ik}(s,y)|+|p_{1ik}(s,y)|+|q_i(s,y)|\big)ds\le\beta_0|\gamma_{0i}(y)|\\ (i,k=1,\dots,n)\end{gathered} \tag{8.15}$$

*and*

$$\begin{aligned}&\int_x^{x+a}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\le\\ &\le a_{ik}(y)\big(1-\exp(-\gamma_{0i}(y))\big)\quad(i\neq k;\ i,k=1,\dots,n)\end{aligned} \tag{8.16}$$

*hold almost everywhere in $\mathcal{D}_{ab}$. Then problems (8.1),(8.2) and (8.1),(8.4) are uniquely solvable and their solutions coincide.*

*Proof.* In view of (8.5), conditions (8.14)-(8.16) yield conditions (7.23)-(7.26), where

$$\gamma_i(y)=|\gamma_{0i}(y)|,\quad \sigma_i(y)=\begin{cases}-1 & \text{for } \gamma_{0i}(y)\ge 0\\ 1 & \text{for } \gamma_{0i}(y)<0\end{cases}\quad (i=1,\dots,n).$$

Therefore, by virtue of Corollary 7.1 problem (8.1),(8.4) has the unique solution $u_0$. Consider the vector function

$$\overline{u}(x,y)=u_0(x+a,y).$$

According to (8.5), $\overline{u}$ is also a solution of problem (8.1),(8.4) and in view of the unique solvability of the latter, we have $\overline{u}(x,y) \equiv u_0(x,y)$. Consequently, $u_0$ is a solution of problem (8.1),(8.2). Theorem will be proved if we show that an arbitrary solution $u$ of problem (8.1),(8.2) is also a solution of problem (8.1),(8.4), i.e. $u$ satisfies the condition

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b}\Big(\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial u(x,y)}{\partial y}\Big\|\Big) < +\infty.$$

By virtue of Lemma 3.1 and the essential boundedness of $\varphi'$, we have

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b}\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| < +\infty. \tag{8.17}$$

Consequently, it remains to show that

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b}\Big\|\frac{\partial u(x,y)}{\partial y}\Big\| < +\infty. \tag{8.18}$$

Put

$$\frac{\partial u(x,y)}{\partial y} = \big(z_i(x,y)\big)_{i=1}^n,$$

$$\mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + q(x,y) = \big(q_{0i}(x,y)\big)_{i=1}^n.$$

Then

$$\frac{\partial z_i(x,y)}{\partial x} = \sum_{k=1}^n p_{2ik}(x,y)z_k(x,y) + q_{0i}(x,y) \quad (i=1,\dots,n) \tag{8.19}$$

$$z_i(0,y) = z_i(a,y) \quad (i=1,\dots,n). \tag{8.20}$$

On the other hand, in view of conditions (8.5),(8.15),(8.17) and the essential boundedness of $\gamma_i$ $(i=1,\dots,n)$, there exists a constant $\beta$ such that the inequalities

$$\int_x^{x+a} |q_{0i}(s,y)|ds \le \beta|1-\exp(-\gamma_{0i}(y))| \quad (i=1,\dots,n) \tag{8.21}$$

hold almost everywhere in $\mathcal{D}_{ab}$. In view of (8.14) from (8.19) and (8.20), we have

$$\begin{aligned} z_i(x,y) = \big[\exp(-\gamma_{0i}(y)) - 1\big]^{-1}\sum_{k=1}^n (1-\delta_{ik}) \times \\ \times \int_x^{x+a} \exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)p_{2ik}(s,y)z_k(s,y)ds + \\ + \big[\exp(-\gamma_{0i}(y)) - 1\big]^{-1}\int_x^{x+a}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)q_{0i}(s,y)ds \\ (i=1,\dots,n). \end{aligned} \tag{8.22}$$

If we assume

$$\overline{z}(y) = \big( \max_{0\le x\le a} |z_i(x,y)|\big)_{i=1}^n, \quad \overline{q} = \beta\big( \exp(a\|p_{2ii}\|_{L_\infty})\big)_{i=1}^n,$$

then with regard to (8.16) and (8.21), from (8.22) we find

$$\overline{z}(y) \le A(y)\overline{z}(y) + \overline{q},$$

whence in view of the condition

$$r(A(y)) < \alpha < 1$$

and the essential boundedness of $A$ it follows that

$$z(y) \le \big[E - A(y)\big]^{-1}\overline{q}$$

and

$$\operatorname*{ess\,sup}_{0\le y\le b} \|\overline{z}(y)\| < +\infty.$$

Consequently, condition (8.18) holds. ■

The following assertions follow from Theorem 8.4 and Corollary 7.1′.

′ *Let $\mathcal{P}_j$ and $q$ be continuous, $\varphi$ be continuously differentiable and*

$$\gamma_{0i}(y) \equiv \int_0^a p_{2ii}(s,y)ds \ne 0 \quad \textit{for} \quad 0 \le y \le b \quad (i = 1,\dots,n).$$

*Let, besides, there exist continuous functions $a_{ik} : [0,b] \to \mathbb{R}_+$ $(i \ne k,\ i,k = = 1,\dots,n)$, such that the spectral radius of the matrix $A(y) = (a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y) \equiv 0$ $(i = 1,\dots,n)$, is less than unity for every $y \in [0,b]$ and inequalities* (8.16) *hold in $\mathcal{D}_b$. Then problems* (8.1),(8.2) *and* (8.1),(8.4) *are uniquely solvable and they have one and the same solution which is classical.*

*Let the inequalities*

$$\sigma_i(y)p_{2ii}(x,y) \le l_{ii} \quad (i = 1,\dots,n),$$
$$|p_{2ik}(x,y)| \le l_{ik} \quad (i \ne k;\ i,k = 1,\dots,n),$$

*where $\sigma_i : [0,b] \to \{-1,1\}$ $(i = 1,\dots,n)$ are measurable functions and $l_{ik}$ $(i,k = 1,\dots,n)$ constants such that the real parts of eigenvalues of the matrix $(l_{ik})_{i,k=1}^n$ are negative. Then problems* (8.1),(8.2) *and* (8.1),(8.4) *are uniquely solvable and their solutions coincide.*

′ *Let $\mathcal{P}_j$ $(j = 0,1,2)$ and $q$ be continuous, $\varphi$ be continuously differentiable and the inequalities*

$$\sigma_i p_{2ii}(x,y) \le l_{ii} \quad (i = 1,\dots,n),$$
$$|p_{2ik}(x,y)| \le l_{ik} \quad (i \ne k;\ i,k = 1,\dots,n),$$

*hold in $\mathcal{D}_b$, where $\sigma_i \in \{-1,1\}$ and $l_{ik}$ $(i,k = 1,\dots)$ are constants and the real parts of eigenvalues of matrix $(l_{ik})_{i,k=1}^n$ are negative. Then problems*

*(8.1),(8.2) and (8.1),(8.4) are uniquely solvable and they have one and the same solution which is classical.*

## § 9.

In this section we investigate the problem of almost-periodicity in the first argument of a solution of the problem

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \mathcal{P}_0(x,y)u(x,y) + \mathcal{P}_1(x,y)\frac{\partial u(x,y)}{\partial x} + \\ + \mathcal{P}_2(x,y)\frac{\partial u(x,y)}{\partial y} + q(x,y), \tag{9.1}$$

$$u(x,0) = \varphi(x), \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \left( \left\| \frac{\partial u(x,y)}{\partial x} \right\| + \left\| \frac{\partial u(x,y)}{\partial y} \right\| \right) < +\infty. \tag{9.2}$$

In addition, it is assumed everywhere that

$$\mathcal{P}_j = \left(p_{jik}\right)_{i,k=1}^n \in L_\infty(\mathcal{D}_b;\mathbb{R}^{n\times n}) \ \ (j=0,1,2),$$
$$q = (q_i)_{i=1}^n \in L_\infty(\mathcal{D}_b;\mathbb{R}^n), \ \ \varphi = (\varphi_i)_{i=1}^n \in \widetilde{\mathbb{C}}_\infty(\mathbb{R};\mathbb{R}^n).$$

The concepts of almost-periodicity and $S$-almost-periodicity in the first argument of a matrix function of two variables which are introduced below, are the modification of Bohr's and Stepanov's concepts of almost-periodicity of a function of one variable ([30], Ch.1,§1 and Ch.5,§2).

A continuous matrix function $Z : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ is called almost- periodic in the first argument if for an arbitrary $\varepsilon > 0$ there exists $l > 0$ such that an arbitrary segment $[x_0, x_0+l]$ contains at least one number $\tau$ for which the inequality

$$\sup_{(x,y)\in\mathcal{D}_b} \|Z(x+\tau,y) - Z(x,y)\| < \varepsilon$$

takes place.

A locally summable matrix function $Z : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ is called $S$-almost-periodic in the first argument if for an arbitrary $\varepsilon > 0$ there exists $l > 0$ such that an arbitrary segment $[x_0, x_0+l]$ contains at least one number $\tau$ for which the inequality

$$\sup_{x\in\mathbb{R}} \int_0^b \int_x^{x+1} \|Z(s+\tau,t) - Z(s,t)\| ds dt < \varepsilon$$

takes place.

For any locally summable matrix function $Z : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ we put

$$\|Z\|_S = \sup_{x\in\mathbb{R}} \int_0^b \int_x^{x+1} \|Z(s,t)\| ds dt.$$

A sequence of locally summable matrix functions $Z_k : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ $(k = 1, 2, \dots)$ is called $S$-convergent to $Z$ if

$$\lim_{k\to+\infty} \|Z_k - Z\|_S = 0.$$

A continuous (locally summable) matrix function $Z : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ is called normal ($S$-normal) if for any sequence of real numbers $(\lambda_k)_{k=1}^{+\infty}$ the sequence of matrix functions $(Z_k)_{k=1}^{+\infty}$, where $Z_k(x, y) = Z(x + \lambda_k, y)$, contains uniformly convergent ($S$-convergent) subsequence.

*A continuous (locally summable) matrix function $Z : \mathcal{D}_b \to \mathbb{R}^{m\times n}$ is almost-periodic ($S$-almost-periodic) in the first argument if and only if it is normal ($S$-normal).*

This lemma is an analogue of Bochner's well-known theorem ([30], Theorem 5.4.2) for matrix functions of two variables and can be proved in the same way as Bochner's theorem.

Let $\overline{\mathcal{P}}_i \in L_\infty(\mathcal{D}_b; \mathbb{R}^{n\times n})$ $(i = 0, 1, 2)$ and there exists a sequence of real numbers $(\lambda_k)_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty} \|\mathcal{P}_{ik} - \overline{\mathcal{P}}_i\|_S = 0,$$

where

$$\mathcal{P}_{ik}(x, y) = \mathcal{P}_i(x + \lambda_k, y) \quad (i = 0, 1, 2;\ k = 1, 2, \dots).$$

Then we shall say that the triple of matrix functions $(\overline{\mathcal{P}}_0, \overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2)$ belong to the class $H(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$.

*Let $\mathcal{P}_i$ $(i = 0, 1, 2)$ be $S$-almost-periodic in the first argument and for any $(\overline{\mathcal{P}}_0, \overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2) \in H(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ the homogeneous problem*

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} = \overline{\mathcal{P}}_0(x, y) u(x, y) + \overline{\mathcal{P}}_1(x, y) \frac{\partial u(x, y)}{\partial x} + \overline{\mathcal{P}}_2(x, y) \frac{\partial u(x, y)}{\partial y}, \quad (9.3)$$

$$u(x, 0) = 0, \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \left( \left\| \frac{\partial u(x, y)}{\partial x} \right\| + \left\| \frac{\partial u(x, y)}{\partial y} \right\| \right) < +\infty \quad (9.4)$$

*have only the trivial solution. Then for any almost-periodic $\varphi : \mathbb{R} \to \mathbb{R}^n$ and $S$-almost-periodic in the first argument $q : \mathcal{D}_b \to \mathbb{R}^n$ a solution of problem* (9.1), (9.2)*, if it exists, is almost-periodic in the first argument.*

*Proof.* Let for an almost-periodic $\varphi : \mathbb{R} \to \mathbb{R}^n$ and $S$-almost-periodic in the first argument $q : \mathcal{D}_b \to \mathbb{R}^n$ problem (9.1),(9.2) have a solution $u$. According to Lemma 9.1, to prove the theorem it suffices to show that $u$ is normal.

Assume on the contrary that $u$ is not normal. Then by Lemma 9.1 there exist sequences of real numbers $(\lambda_k)_{k=1}^{+\infty}$, of natural numbers $(k_l)_{l=1}^{+\infty}$ and $(j_l)_{l=1}^{+\infty}$ and a positive number $\delta$ such that the sequences of matrix and vector functions

$$(\mathcal{P}_{ik})_{k=1}^{+\infty} \quad (i = 0, 1, 2), \quad (q_k)_{k=1}^{+\infty} \quad (9.5)$$

and

$$(\varphi_k)_{k=1}^{+\infty}, \tag{9.6}$$

where

$$\mathcal{P}_{ik}(x,y) = \mathcal{P}_i(x+\lambda_k, y) \quad (i=0,1,2),$$
$$q_k(x,y) = q(x+\lambda_k, y) \quad \varphi_k(x) = \varphi(x+\lambda_k)$$

are, respectively, $S$-convergent and uniformly convergent and

$$\sup_{(x,y)\in\mathcal{D}_b} \|u(x+\lambda_{k_l}, y) - u(x+\lambda_{j_l}, y)\| > \delta \ \ (l=1,2,\dots),$$

from which the existence of sequences $x_l \in [0,a]$ and $y_l \in [0,b]$ $(l=1,2,\dots)$ such that

$$\|u(x_l+\lambda_{k_l}, y_l) - u(x_l+\lambda_{j_l}, y_l)\| > \delta \ \ (l=1,2,\dots) \tag{9.7}$$

becomes evident.

Assume

$$u_l(x,y) = u(x+x_l+\lambda_{k_l}, y) - u(x+x_l+\lambda_{j_l}, y),$$
$$\overline{\varphi}_l = \varphi_{k_l}(x+x_l) - \varphi_{j_l}(x+x_l),$$
$$\overline{\mathcal{P}}_{il}(x,y) = \mathcal{P}_i(x+x_l+\lambda_{k_l}, y) \ \ (i=0,1,2)$$

and

$$\begin{aligned}\overline{q}_l(x,y) = {} & [\mathcal{P}_0(x+x_l+\lambda_{k_l}, y) - \mathcal{P}_0(x+x_l+\lambda_{j_l}, y)]u(x+x_l+\lambda_{j_l}, y) + \\ & +\big[\mathcal{P}_1(x+x_l+\lambda_{k_l}, y) - \mathcal{P}_1(x+x_l+\lambda_{j_l}, y)\big]\frac{\partial u(x+x_l+\lambda_{j_l}, y)}{\partial x} + \\ & +\big[\mathcal{P}_2(x+x_l+\lambda_{k_l}, y) - \mathcal{P}_1(x+x_l+\lambda_{j_l}, y)\big]\frac{\partial u(x+x_l+\lambda_{j_l}, y)}{\partial y} + \\ & +q(x+x_l+\lambda_{k_l}, y) - q(x+x_l+\lambda_{j_l}, y).\end{aligned} \tag{9.8}$$

Then, in view of (9.1) and (9.2), for any natural $l$ we have

$$\begin{aligned}\frac{\partial^2 u_l(x,y)}{\partial x \partial y} = {} & \overline{\mathcal{P}}_{0_l}(x,y)u_l(x,y) + \overline{\mathcal{P}}_{1_l}(x,y)\frac{\partial u_l(x,y)}{\partial x} + \\ & +\overline{\mathcal{P}}_{2_l}(x,y)\frac{\partial u_l(x,y)}{\partial y} + \overline{q}_l(x,y),\end{aligned} \tag{9.9}$$

$$u_l(x,0) = \overline{\varphi}_l(x) \tag{9.10}$$

and

$$\sup_{(x,y)\in\mathcal{D}_b} \|u_l(x,y)\| \le \gamma, \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \left(\left\|\frac{\partial u_l(x,y)}{\partial x}\right\| + \left\|\frac{\partial u_l(x,y)}{\partial y}\right\|\right) < \gamma, \tag{9.11}$$

where $\gamma$ is a positive constant independent of $l$. On the other hand, in view of (9.7)

$$\|u_l(0,y_l)\| > \delta \quad (l=1,2,\dots). \tag{9.12}$$

Because of the $S$-convergence of sequences (9.5) and the uniform convergence of sequence (9.6) we have

$$\|\overline{q}_l\|_s \le \gamma \sum_{j=0}^{2} \|\mathcal{P}_{ik_l} - \mathcal{P}_{ij_l}\|_S + \|q_{k_l} - q_{j_l}\|_S \to 0 \ \text{ for } \ l \to +\infty \tag{9.13}$$

and

$$\lim_{l\to+\infty} \overline{\varphi}_l(x) = 0 \quad \text{uniformly in} \quad \mathbb{R}. \tag{9.14}$$

Taking into account Lemma 9.1 and an essential boundedness of $\mathcal{P}_i$ ($i = 0,1,2$), without loss of generality we may assume that the sequences $(\overline{\mathcal{P}}_{il})_{l=1}^{+\infty}$ $(i=0,1,2)$ are $S$-convergent to some matrix functions $\overline{\mathcal{P}}_i \in L_\infty(\mathcal{D}_b;\mathbb{R}^{n\times n})$ $(i=0,1,2)$, i.e.

$$\lim_{l\to+\infty} \|\overline{\mathcal{P}}_{il} - \overline{\mathcal{P}}_i\|_S = 0. \tag{9.15}$$

On the other hand, by virtue of the Arzela-Ascoli lemma and conditions (9.11), again we may assume without loss of generality that the sequence $(u_l)_{l+1}^{+\infty}$ is uniformly convergent in the rectangle $\mathcal{D}_{ab}$ for an arbitrary $a \in \mathbb{R}$.

Put

$$u_0(x,y) = \lim_{l\to+\infty} u_l(x,y). \tag{9.16}$$

From (9.11),(9.12) and (9.14) it is clear that $u_0 : \mathcal{D}_b \to \mathbb{R}^n$ is Lipschitz continuous,

$$u_0(x,0)=0, \quad \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}_b} \left( \left\| \frac{\partial u_0(x,y)}{\partial x} \right\| + \left\| \frac{\partial u_0(x,y)}{\partial y} \right\| \right) \le \gamma \tag{9.17}$$

and

$$\max_{0\le y\le b} \|u_0(0,y)\| > 0. \tag{9.18}$$

Besides, for an arbitrary $a \in \mathbb{R}$ we have

$$\lim_{l\to+\infty} \int_0^x \frac{\partial u_l(s,y)}{\partial s} ds = \int_0^x \frac{\partial u_0(s,y)}{\partial s} ds, \quad \lim_{l\to+\infty} \int_0^y \frac{\partial u_l(x,t)}{\partial t} dt =$$
$$= \int_0^y \frac{\partial u_0(x,t)}{\partial t} dt \quad \text{uniformly in } \mathcal{D}_{ab}. \tag{9.19}$$

It follows from (9.9) and (9.10) that

$$u_l(x,y) = \overline{\varphi}_l(x) - \varphi_l(0) + u_l(0,y) + \int_0^x \int_0^y \Big[\overline{\mathcal{P}}_{0l}(s,t)u_l(s,t) +$$

$$+\overline{\mathcal{P}}_{1l}(s,t)\frac{\partial u_l(s,t)}{\partial s}+\overline{\mathcal{P}}_{2l}(s,t)\frac{\partial u_l(s,t)}{\partial t}+q_l(s,t)\Big]dsdt. \tag{9.20}$$

However, in view of conditions (9.15),(9.16) and (9.19) and Lemma 3.13

$$\lim_{l\to+\infty}\int_0^x\int_0^y\overline{\mathcal{P}}_{0l}(s,t)u_l(s,t)dsdt=\int_0^x\int_0^y\overline{\mathcal{P}}_0(s,t)u_0(s,t)dsdt,$$
$$\lim_{l\to+\infty}\int_0^x\int_0^y\overline{\mathcal{P}}_{1l}(s,t)\frac{\partial u_l(s,t)}{\partial s}dsdt=\int_0^x\int_0^y\overline{\mathcal{P}}_1(s,t)\frac{\partial u_0(s,t)}{\partial s}dsdt,$$
$$\lim_{l\to+\infty}\int_0^x\int_0^y\overline{\mathcal{P}}_{2l}(s,t)\frac{\partial u_l(s,t)}{\partial t}dsdt=\int_0^x\int_0^y\overline{\mathcal{P}}_2(s,t)\frac{\partial u_0(s,t)}{\partial t}dsdt$$

uniformly in $\mathcal{D}_{ab}$.

If, alongside with the latter three equalities, we take into account conditions (9.13) and (9.14), then from (9.20) we obtain

$$u_0(x,y)=u_0(0,y)+$$
$$+\int_0^x\int_0^y\big[\overline{\mathcal{P}}_0(s,t)u_0(s,t)+\overline{\mathcal{P}}_1(s,t)\frac{\partial u_0(s,t)}{\partial s}+\overline{\mathcal{P}}_2(s,t)\frac{\partial u_0(s,t)}{\partial t}\big]dsdt.$$

From this and condition (9.17) it follows that $u_0$ is a solution of problem (9.3),(9.4). But, according to the conditions of the theorem, problem (9.3),(9.4) has only the trivial solution. Consequently, $u_0(x,y)\equiv 0$. But this contadicts inequality (9.18). The obtained contradiction proves the theorem. ■

The theorem proved above is an analogue of J.Favard's well-known theorem [15] for system (9.1).

*Let $\varphi$ be almost-periodic, $\mathcal{P}_i$ $(i=0,1,2)$ and $q$ be $S$-almost-periodic in the first argument and there exist constants $\alpha\in(0,1)$, $\beta>0$ and essentially bounded measurable functions $a_{ik}:[0,b]\to[0,+\infty)$ $(i\neq k;i,k=1,\dots,n)$, $a_i:[0,b]\to(0,+\infty)$ and $\sigma_i:[0,b]\to\{-1,1\}$ $(i=1,\dots,n)$ such that the spectral radius of the matrix $(a_{ik}(y))_{i,k=1}^n$, where $a_{ii}(y)\equiv 0$ $(i=1,\dots,n)$, is less than $\alpha$ almost for all $y\in[0,b]$ and the inequalities*

$$\sigma_i(y)p_{2ii}(x,y)\le -a_i(y)\quad (i=1,\dots,n), \tag{9.21}$$

$$\Big|\int_x^{\chi_i(y)}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)\big(|p_{0ik}(s,y)|+|p_{1ik}(s,y)|+ \\ +|q_i(s,y)|\big)ds\Big|\le\beta\qquad (i,k=1,\dots,n) \tag{9.22}$$

*and*

$$\Big|\int_x^{\chi_i(y)}\exp\Big(\int_s^x p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big|\le a_{ik}(y) \\ (i\neq k;\ i,k=1,\dots,n) \tag{9.23}$$

*take place almost everywhere in $\mathcal{D}_b$, where $\chi_i(y) = +\infty$ for $\sigma_i(y) = -1$ and $\chi_i(y) = -\infty$ for $\sigma_i(y) = 1$, respectively. Then problem (9.1),(9.2) has the unique solution and this solution is almost-periodic in the first argument.*

*Proof.* Let $I_0(p_{2ii})$ and $\chi(p_{2ii})$ $(i = 1, \dots, n)$ be, respectively, the sets and the functions appearing in Theorem 7.1. Because the function $a_i$ $(i = 1, \dots, n)$ is positive, from inequalities (9.21) we have

$$\chi(p_{2ii})(y) = \chi_i(y), \quad \operatorname{mes} I_0(p_{2ii}) = 0.$$

Therefore inequalities (7.4) and (7.5) follow from equalities (9.22) and (9.23). Consequently, all conditions of Theorem 7.1 hold, which guarantees the unique solvability of problem (9.1),(9.2). It remains to show that the solution of the problem is almost-periodic in the first argument. By virtue of Theorems 7.1 and 9.1 it suffices to show that if

$$\overline{\mathcal{P}}_j(x,y) = \big(\overline{p}_{jik}(x,y)\big)_{i,k=1}^{n} \quad (j = 0,1,2)$$

and

$$(\overline{\mathcal{P}}_0, \overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2) \in H(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$$

then the inequalities

$$\sigma_i(y)\overline{p}_{2ii}(x,y) \le -a_i(y) \quad (i = 1, \dots, n), \tag{9.24}$$

$$\Big|\int_x^{\chi_i(y)} \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,y)d\xi\Big)\big(|\overline{p}_{0ik}(s,y)| + |\overline{p}_{1ik}(s,y)|\big)ds\Big| \le \beta \tag{9.25}$$
$$(i, k = 1, \dots, n)$$

and

$$\Big|\int_x^{\chi_i(y)} \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,y)d\xi\Big)|\overline{p}_{2ik}(s,y)|ds\Big| \le a_{ik}(y) \tag{9.26}$$
$$(i \ne k;\ i, k = 1, \dots, n)$$

hold almost everywhere in $\mathcal{D}_b$. According to the definition of the class $H$ there exists a sequence of real numbers $(\lambda_l)_{l=1}^{+\infty}$ such that

$$\varepsilon_l = \max_{\substack{1 \le i,k \le n \\ 0 \le j \le 2}} \Big[\sup_{x \in \mathbb{R}} \int_0^b \int_x^{x+1} |\overline{p}_{jik}(s,t) - p_{jik}(s + \lambda_l, t)|dsdt\Big| \to 0 \tag{9.27}$$
$$\text{for} \quad l \to +\infty.$$

By (9.21)

$$\int_y^t \int_x^s \sigma_i(\tau)p_{2ii}(\xi + \lambda_l, \tau)d\xi d\tau \le -(s - x)\int_y^t a_i(t)dt$$
$$\text{for} \quad 0 \le y < t \le b, \quad x < s \quad (i = 1, \dots, n),$$

whence, in view of (9.27), we have that

$$\int_y^t \int_x^s \sigma_i(\tau)\overline{p}_{2ii}(\xi,\tau)d\xi d\tau \le -(s-x)\int_y^t a_i(\tau)d\tau$$
$$\text{for} \quad 0 \le y < t \le b, \quad x < s \quad (i = 1,\dots,n).$$

It is clear from these estimates that inequalities (9.24) hold almost everywhere in $\mathcal{D}_b$. Inequalities (9.25) and (9.26) can be proved similarly. We shall give here the proof of inequality (9.26) only. For any natural $m$ we put

$$\chi_{im}(y) = \begin{cases} m & \text{for } \sigma_i(y) = -1 \\ -m & \text{for } \sigma_i(y) = 1 \end{cases}.$$

$$\Big| \int\limits_x^{x+\chi_{im}(y)} \exp\Big(\int_s^x p_{2ii}(\xi+\lambda_l,y)d\xi\Big)|p_{2ik}(s+\lambda_l,y)|ds\Big| =$$
$$= \Big| \int\limits_{x+\lambda_l}^{x+\lambda_l+\chi_{im}(y)} \exp\Big(\int\limits_s^{x+\lambda_l} p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \le$$
$$\le \Big| \int\limits_{x+\lambda_l}^{x+\chi_i(y)} \exp\Big(\int\limits_s^{x+\lambda_l} p_{2ii}(\xi,y)d\xi\Big)|p_{2ik}(s,y)|ds\Big| \le a_{ik}(y)$$
$$(i \ne k;\ i,k = 1,\dots,n).$$

Therefore

$$\Big|\int_y^t \int\limits_x^{x+\chi_{im}(y)} \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,\tau)d\xi\Big)|\overline{p}_{2ik}(s,\tau)|dsd\tau\Big| \le$$
$$\le \int_y^t \Big| \int\limits_x^{x+\chi_{im}(y)} \exp\Big(\int_s^x p_{2ii}(\xi+\lambda_l,\tau)d\xi\Big)|p_{2ik}(s+\lambda_l,\tau)| \times$$
$$\times \exp\Big(\int_s^x [\overline{p}_{2ii}(\xi,\tau) - p_{2ii}(\xi+\lambda_l,\tau)]d\xi\Big)ds\Big|d\tau +$$
$$+ \int_y^t \Big| \int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ik}(s,\tau) - p_{2ik}(s+\lambda_l,\tau)| \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,\tau)d\xi\Big)ds\Big|d\tau \le$$
$$\le \int_y^t \exp\Big(\Big| \int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ii}(\xi,\tau) - p_{2ii}(\xi+\lambda_l,\tau)|d\tau\Big|\Big)a_{ik}(\tau)d\tau +$$

$$+\exp(c_0 m)\int_y^t \Big| \int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ik}(s,\tau) - p_{2ik}(s+\lambda_l,\tau)|ds\Big| d\tau \qquad (9.28)$$

$$\text{for} \quad 0 \le y \le t \quad (i \ne k;\ i,k = 1,\dots,n),$$

where

$$c_0 = \max_{1\le i\le n} \|p_{2ii}\|_{L_\infty}.$$

But,

$$\exp\Big(\Big|\int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ii}(\xi,\tau) - p_{2ii}(\xi+\lambda_l,\tau)|ds\Big|\Big) \le$$

$$\le 1 + \Big|\int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ii}(\xi,\tau) - p_{2ii}(\xi+\lambda_m,\tau)|ds\Big| \exp(2c_0 m)$$

and

$$\int_y^t \Big|\int\limits_x^{x+\chi_{im}(y)} |\overline{p}_{2ik}(s,\tau) - p_{2ik}(s+\lambda_l,\tau)|ds\Big| d\tau \le m\varepsilon_l.$$

According to the above arguments from (9.28) we obatin

$$\int_y^t \Big|\int\limits_x^{x+\chi_{im}(y)} \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,y)d\xi\Big)|\overline{p}_{2ik}(s,\tau)|ds\Big| d\tau \le$$

$$\le \int_y^t a_{ik}(\tau)d\tau + m(1+\|a_{ik}\|_{L_\infty})\exp(2c_0 m)\varepsilon_l$$

$$\text{for} \quad 0 \le y \le t \le b \quad (i \ne k;\ i,k = 1,\dots,n).$$

Passing in these inequalities to the limit first for $l \to +\infty$ and then for $m \to +\infty$, on account of (9.27), we find

$$\int_y^t \Big|\int\limits_x^{x+\chi_i(y)} \exp\Big(\int_s^x \overline{p}_{2ii}(\xi,\tau)d\xi\Big)|\overline{p}_{2ik}(s,\tau)|ds\Big| d\tau \le \int_y^t a_{ik}(\tau)d\tau$$

$$\text{for} \quad 0 \le y \le t \le b \quad (i \ne k;\ i,k = 1,\dots,n).$$

from which it is clear that inequalities (9.26) hold almost everywhere in $\mathcal{D}_b$. ■

Similarly to Corollary 7.2 we prove

*Let $\varphi_0$ be almost-periodic and $\mathcal{P}_i$ $(i = 0,1,2)$ and $q$ be $S$-almost- periodic in the first argument. Let, besides, inequalities*

$$\sigma_i(y)p_{2ii}(x,y) \le l_{ii} \quad (i = 1,\dots,n),$$
$$|p_{2ik}(x,y)| \le l_{ik} \quad (i \ne k,\ i,k = 1,\dots,n)$$

*hold almost everywhere in $\mathcal{D}_b$, where $\sigma_i : [0,b] \to \{-1,1\}$ $(i = 1,\dots,n)$ are measurable functions and $l_{ik}$ $(i,k = 1,\dots,n)$ are constants such that the real parts of eigenvalues of the matrix $(l_{ik})_{i,k=1}^n$ are negative. Then problem* (9.1),(9.2) *has the unique solution and this solution is almost-periodic in the first argument.*

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