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CRITICAL CASE OF MULTIPLE PAIRS OF PURE IMAGINARY ROOTS OF A NONAUTONOMOUS ESSENTIALLY NONLINEAR n-th ORDER EQUATION

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We investigate the asymptotic stability in the Lyapunov sense as $t \uparrow \omega$ of the zero solution of a differential equation of the form

$$y^{(n)} + \sum_{k=1}^{n-1} p_k(t) \cdot y^{(n-k)} + p_n(t) \cdot y = F\big(t, y, y', \dots, y^{(n-1)}\big), \tag{1}$$

where $t \in \Delta \equiv [a_0, \omega[$, $-\infty < a_0 < \omega \le +\infty$, $p_s : \Delta \to R$, $R \equiv]-\infty, +\infty[$, $p_s \equiv \pi^s \cdot a_s$, $\pi : \Delta \to R_+$, $R_+ =]0, +\infty[$, $a_s \equiv a_{s0} + o_s(1)$, $a_s^{(1)} = o_{sl}(1)$ as $t \uparrow \omega$, $a_{s0} \in R$, $l \in \{\overline{1,h}\}$, $h \in N$, $N \equiv \{1, 2, \dots\}$, $s = \overline{1,n}$, and the following conditions are fulfilled:

(1) the equation $P_n(\lambda) \equiv \lambda^n + \sum_{s=1}^n a_{s0} \cdot \lambda^{n-s} = 0$ possesses $2 \cdot n_0$, $1 \le n_0 \le \left[\frac{1}{2} \cdot n\right]$ roots λ_0 satisfying $\operatorname{Re} \lambda_0 = 0$; the remaining roots λ possess the property $\operatorname{Re} \lambda < 0$;

(2) $F(t, X) \equiv \sum_{|Q|=2}^m F_Q(t) \cdot X^Q + R_m(t, X)$, $X \equiv (x_1, \dots, x_n)$, $F : \Delta \times S(X, r) \to R$, $S(X, r) \equiv \{X, X^T : \|X\| \le r\}$, $r \in R_+$, $Q = (q_1, \dots, q_n)$, $q_k \in \{0, N\}$, $k = \overline{1,n}$, $\|Q\| = \sum_{k=1}^n q_k$, $X^Q \equiv \prod_{k=1}^n x_k^{q_k}$, $F_Q \in C_\Delta^k$, $h \in N$, $\|Q\| = \overline{2,m}$, $m \in N \setminus \{1\}$, $|R_m| \le L \cdot \left(\sum_{k=1}^n |x_k|\right)^{m+a}$, $L : \Delta \to [0, +\infty[$, $L \in C_\Delta$, $a \in R_+$.

Below the use is made of the following definitions and notation:

Definition 1. The differential equation (1) possesses the property St as $t \uparrow \omega$ if for any arbitrarily small $\varepsilon \in R_+$ there exist $\delta_\varepsilon \in]0, \varepsilon]$ and $T_\varepsilon \in \Delta$ such that any solution $y = y(t)$ of (1) satisfying $|y(T_\varepsilon)| < \delta_\varepsilon \cdot \pi(T_\varepsilon)$, $|y^{(s-1)}(T_\varepsilon)| < \delta_\varepsilon \cdot \pi^s(T_\varepsilon)$, $s = \overline{2,n}$, possesses the property $|y(t)| < \varepsilon_\varepsilon \cdot \pi$, $|y^{(s-1)}(t)| < \varepsilon \cdot \pi^s$ for all $t \in [T_\varepsilon, \omega[$, $s = \overline{2,n}$.

Definition 2. The differential equation (2) possesses the property $AsSt$ as $t \uparrow \omega$ if Definition 1 is fulfilled, and $\pi^{-1} \cdot y(t) = o(1)$ and $\pi^{-s} \cdot y^{(s-1)}(t) = o(1)$ as $t \uparrow \omega$, $s = \overline{2,n}$.

Definition 1$'$. The differential system

$$Y' = f(t, Y), \quad Y \equiv \operatorname{col}(y_1, \dots, y_n), \quad f(t, \overline{0}) \equiv \overline{0}, \quad \overline{0} \equiv \operatorname{col}(0, \dots, 0), \tag{2}$$

possesses the property St as $t \uparrow \omega$ if for any arbitrarily small $\varepsilon \in R$ there exist $\delta_\varepsilon \in]0, \varepsilon]$ and $T_\varepsilon \in \Delta$ such that any solution $Y = Y(t)$ of the differential system (2) with the condition $\|Y(T_\varepsilon)\| < \delta_\varepsilon$ possesses the property $\|Y(t)\| < \varepsilon$ for all $t \in [T_\varepsilon, \omega[$.

For $\omega < +\infty$, the property St of the differential system (2) is defined by a rephrasing of this property for $\omega = +\infty$ [2, p. 168].

Definition 2$'$. The differential system (2) possesses the property $AsSt$ as $t \uparrow \omega$ if Definition 1$'$ is fulfilled, and $\|Y(t)\| = o(1)$ as $t \uparrow \omega$.

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E_k, H_k are respectively the unit and the displacement matrices of the dimension $k \times k$; Y_k is a column vector of the dimension k;

$$Y^{-1} \equiv \mathrm{col}(y_1^{-1}, \ldots, y_n^{-1}), \quad Z = \mathrm{col}(z_1, \ldots, z_n) = \mathrm{col}(Z_{n_1}, \ldots, Z_{n_k}),$$

$$Y \cdot Z \equiv \mathrm{col}(y_1 \cdot z_1, \ldots, y_n \cdot z_n), \quad < Y, Z > \equiv \sum_{k=1}^{n} y_k z_k,$$

$$\|Y\|^2 \equiv \sum_{k=1}^{n} |y_k|^2, \quad \mathrm{grad}\, V(t, Y) \equiv \mathrm{col}\left(\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_n}\right),$$

$$E_k^T \equiv (0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0); \quad L_\Delta \equiv \left\{ f : \Delta \to R, \int^{\omega} |f| \cdot dt < +\infty \right\};$$

$$\Lambda \equiv \max i \left\{ f_s : \Delta \to R; s = \overline{1, n} \right\}, \quad \text{for} \ \ \Lambda : \Delta \to R_+,$$

$$\Lambda^{-1} \cdot f_s = c_s + o_s(1) \ \text{ as } \ t \uparrow \omega, \quad c_s \in R, \quad \sum_{s=1}^{n} |c_s| > 0.$$

The results of this paper are effectively applied to the differential equation (1) whose coefficients are slowly varying functions, i.e., the functions whose derivatives are small as $t \uparrow \omega$ in comparison with the functions themselves. For example,

$$p_k \equiv t^{k \cdot \beta} \cdot \left[a_{k0} + b_k \cdot t^{-a_k} \cdot (\ln t)^{\beta_k} \cdot \sin t^{\gamma_k} \right],$$
$$F_Q \equiv t^{\gamma \cdot \|Q\|} \cdot \left[F_{Q0} + g_Q \cdot t^{-a_Q} \cdot (\ln t)^{\beta_Q} \cdot \sin t^{\gamma_Q} \right],$$
$$k, \beta, a_{k0}, b_{k0}, \beta_k, \gamma, F_{Q0}, g_Q, \beta_Q \in R, \quad a_k, a_Q \in \{0, R_+\},$$
$$\gamma_k, \gamma_Q \in]0, 1], \quad k = \overline{1, n}, \quad \|Q\| = \overline{2, m}.$$

Lemma. *If $\pi' \cdot \pi^{-2} = o(1)$ as $t \uparrow \omega$, then the transformation $y = \pi \cdot y_1$, $y^{(s)} = \pi^{s+1} \cdot y_{s+1}$, $s = \overline{1, n-1}$, reduces the differential equation (1) to that of the kind*

$$Y' = \pi \cdot P \cdot Y + G, \tag{3}$$

where $P = \|p_{sk}\|$, $s, k = \overline{1, n}$, $p_{ss} \equiv -s \cdot \pi' \cdot \pi^{-2}$, $s = \overline{1, n-1}$, $p_{nn} \equiv -a_1 - n \cdot \pi' \cdot \pi^{-2}$, $p_{s,s+1} \equiv 1$, $s = \overline{1, n-1}$, $p_{sk} \equiv 0$,

$$s = \overline{1, n-2}, \quad k = \overline{s+2, n}, \quad p_{sk} \equiv 0, \quad s = \overline{2, n-1}, \quad k = \overline{s-1, n-2},$$

$$p_{nk} \equiv -a_{n-k+1}, \quad k = \overline{1, n-1}, \quad G \equiv \mathrm{col}(\overline{0}, G_n).$$

$$G_n \equiv \sum_{|Q|=2}^{m} f_Q \cdot \pi^{-n + \sum_{s=1}^{n} s \cdot q_s} \cdot Y^Q + R_m^*,$$

$$|R_m^*| \le \left(\sum_{k=1}^{n} \pi^k \right)^{m+a} \cdot \pi^{-n} \cdot L \cdot \left(\sum_{k=1}^{n} |y_k| \right)^{m+a},$$

$$\det[P(\omega) - \lambda \cdot E_n] \equiv P_n(\lambda).$$

The proof of the lemma is obvious.

Assume first that using the generalized "shearing" [3], "frozen" [4] and K.P. Persidsky's methods of transformations, we can construct a nondegenerate substitution

$y_s = h_s(t,Z)$, $h_s(t,\overline{0}) \equiv 0$, $s=\overline{1,n}$, which reduces the differential system (3) to that of the special form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + \Omega_{n_s}) \cdot Z_{n_s} + \Phi_{n_s}, \\ i^2 = -1, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0 \\ Z'_{n-2\cdot n_0} = \pi \cdot P_{n-2\cdot n_0} \cdot Z_{n-2\cdot n_0} + \Phi_{n-2\cdot n_0}, \end{cases} \tag{4}$$

where $\pi_s : \Delta \to R_+$, $\mu_s \in R_+$, $s=\overline{1,k_0}$ are known numbers, $\|\Omega_{n_s}\| = o(1)$ as $t \uparrow \omega$, $s=\overline{1,k_0}$, the roots μ of the equation $\det(P_{n-2\cdot n_0} - \mu \cdot E_{n-2\cdot n_0}) = 0$ possess the property $\operatorname{Re}\mu : \Delta \to]-\infty, -\gamma]$, $y \in R_+$; $s=\overline{1,k_0}$, $\Phi_{n-2\cdot n_0}$ are small in a sense.

Theorem 1. *Let the differential equation* (1) *be such that*

(1) $\pi' \cdot \pi^{-2} = o(1)$ *as* $t \uparrow \omega$, *and the transformation* $y = \pi \cdot h_1(t,Z)$, $y^{(s)} = \pi^{s+1} \times h_{s+1}(t,Z)$, $s = \overline{1,n-1}$, *reduces the differential equation* (1) *to the differential system* (4) *with* $\pi_s \cdot \|\operatorname{Re}\Omega_{n_s}\| \in L_\Delta$, $s=\overline{1,k_0}$*;*

(2) *for all* $Z \in S(Z,r)$, *it holds* $\|\Phi_{n_s}\|$, $\|\Phi_{n-2\cdot n_0}\| \in L_D$, $s=\overline{1,k_0}$, $h_k(t,Z) = o(1)$ *as* $t \uparrow \omega$, $k=\overline{1,n}$.

Then the differential equation (1) *possesses the property* $AsSt$ *as* $t \uparrow \omega$.

Proof. Consider the differential system (4) in terms of the quasi-linear differential system and apply the results of [6]. □

Assume now that by the methods of "shearing" [3] and "frozen" [4] transformations we can construct a nondegenerate change of variables

$$y_k = f_s(t,Z) \equiv \sum_{|Q|=2}^{m} f_{sQ} \cdot Z^Q, \quad s = \overline{1,n},$$

reducing the differential system (3) to that of the special form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + H_{n_s}) \cdot Z_{n_s} + \\ \quad + \sum\limits_{\|Q_{n_s}+L_{n_s}\|=2}^{m} g_{n_s,Q_{n_s},L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{l_{n_s}} + \Theta_{n_s}, \\ i^2 = -1, \quad \sum\limits_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k} - L_{n_k}\|) + \mu_s \cdot \pi_s = 0, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0, \\ Z'_{n-2\cdot n_0} = \pi \cdot P_{n-2\cdot n_0} \cdot Z_{n-2\cdot n_0} + \\ + Z_{n-2\cdot n_0} \cdot \sum\limits_{\|Q_{n_s}\|=1}^{m-1} g_{n-2\cdot n_0} \cdot Z_{n_s}^{Q_{n_s}} \overline{Z}_{n_s}^{Q_{n_s}} + \Theta_{n-2\cdot n_0}, \end{cases} \tag{5}$$

where $\pi_s : \Delta \to R_+$, $\mu_s \in R_+$, $g_{n_s,Q_{n_s},L_{n_s}}$, $g_{n-2\cdot n_0,Q_{n_s}}$ are known values, the equation $\det(P_{n-2\cdot n_0} - \mu \cdot E_{n-2\cdot n_0}) = 0$ possesses only the roots μ with the property $\operatorname{Re}\mu : \Delta \to]-\infty, \gamma]$, Θ_{n_s}, $\Theta_{n-2\cdot n_0}$ are small in a sense.

Select from the differential system (5) that of the form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + H_{n_s}) \cdot Z_{n_s} + \sum\limits_{\|Q_{n_s}+L_{n_s}\|=2}^{m} g_{n_s,Q_{n_s},L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{l_{n_s}}, \\ \sum\limits_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k}\| - \|L_{n_k}\|) + \mu_s \cdot \pi_s = 0, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0. \end{cases} \tag{6}$$

Assume that the differential system (6) can be substituted by an equivalent $2 \cdot n_0$-th order differential equation with respect to one of the components of the vector $\mathrm{col}(Z_{n_s}, \ldots, Z_{n_{k_0}})$. Then, using a method presented in [7], we can obtain asymptotic representation of all proper solutions of the above obtained differential equation. Denote by $\Psi_{n_s} = \Psi_{n_s}(t)$, $s = \overline{1, k_0}$, an asymptotic representation of one of the proper solutions of the differential system (6).

Theorem 2. *Let the differential equation* (1) *be such that*

(1) $\pi' \cdot \pi^{-2} = o(1)$ *as* $t \uparrow \omega$, *and the transformation* $y = \pi \cdot f_1(t, Z)$, $y^{(s)} = \pi^{s+1} \cdot f_{s+1}(t, Z)$, $s = \overline{1, n-1}$, *reduces the differential equation* (1) *to* (5) *with* $\|P'_{n-2\cdot n_0}\| \cdot \pi^{-1} = o(1)$ *as* $t \uparrow \omega$;

(2) *there exists an asymptotic representation of one of the proper solutions of the differential system* (6), $\Psi_{n_s} = \Psi_{n_s}(t)$, *such that* $\|\Psi_{n_s}\| = o(1)$ *and* $\|\Psi'_{n_s} \cdot \Psi_{n_s}^{-1}\| \cdot \pi_s^{-1} = o(1)$ *as* $t \uparrow \omega$, $s = \overline{1, k_0}$;

(3) *there exist positive definite Lyapunov functions* $V = V_{n_s}(Z_{n_s})$ *such that for all* $t \in \Delta$ *and all* $\mathrm{col}(Z_{n_s}, \ldots, Z_{n_s}, \overline{O}) \in S(Z, r)$

$$\mathrm{Re} < \mathrm{grad}\, V_{n_s}(Z_{n_s}),\ \ \pi_s \cdot \Big[(i \cdot \mu_s - \Psi'_{n_s} \cdot \Psi_{n_s}^{-1} \cdot \pi_s^{-1}) \cdot E_{n_s} + H_{n_s}\Big] \cdot Z_{n_s} +$$
$$+ \sum_{\|Q_{n_s} + L_{n_s}\| = 2}^{m} g_{n_s, Q_{n_s}, L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E^T_{n_s}} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{L_{n_s}} > \equiv$$
$$\equiv \Lambda_s \cdot \Big[W_{0s}(Z_{n_s}) + W_{1s}(t, Z_{n_s})\Big],$$
$$\Lambda_s \equiv \max\Big\{ \Big\| g_{n_s, Q_{n_s}, L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E^T_{n_s}} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} \Big\|, \sum_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k}\| - \|L_{n_k}\|) +$$
$$+ \mu_s \cdot \pi_s = 0,\ \|Q_{n_s} + L_{n_s}\| = \overline{2, m} \Big\},$$
$$W_{0s}(Z_{n_s}) < 0,\ \ Z_{n_s} \neq \overline{0},\ \ W_{0s}(\overline{0}) = 0,$$
$$W_{1s}(t, Z_{n_s}) = o(1),\ \ \Lambda_s^{-1} \cdot \pi_s = o(1),$$
$$\Lambda_s^{-1} \cdot g_{n-2\cdot n_0, Q_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E^T_{n_s}} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} = o(1) \ \ \textit{as}\ \ t \uparrow \omega,\ \ s = \overline{1, k_0};$$

(4) *there exists* $\nu : \Delta \to R_+$, $\nu \in C_\Delta^{-1}$, *such that* $\nu = o(1)$, $\nu' \cdot \nu^{-1} \cdot \pi^{-1} = o(1)$, $t \uparrow \omega$ *and for all* $Z \in S(Z, r) \setminus \overline{O}$

$$\Big[\sum_{s=1}^{k_0} \Big\| \Theta_{n_s}(t, \Psi_{n_1} \cdot Z_{n_1}, \ldots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, \nu \cdot Z_{n-2\cdot n_0}) \cdot \Psi_{n_s}^{-1} \Big\| +$$
$$+ \Big\| \Theta_{n-2\cdot n_0}(t, \Psi_{n_1} \cdot Z_{n_1}, \ldots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, \nu \cdot Z_{n-2\cdot n_0}) \cdot v^{-1} \Big\| \Big] \times$$
$$\times \Big[\sum_{s=1}^{k_0} \Lambda_s \cdot W_{0s}(Z_{n_s}) - \pi \cdot \|Z_{n-2\cdot n_0}\|^2\Big]^{-1} = o(1),$$
$$f_s(t, \Psi_{n_1} \cdot Z_{n_1}, \ldots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, v \cdot Z_{n-2\cdot n_0}) = o(1) \ \ \textit{as}\ \ t \uparrow \omega,\ \ s = \overline{1, k_0}.$$

Then the differential equation (1) *possesses the property AsSt as* $t \uparrow \omega$.

Proof. In the differential system (5), we make the substitution $Z_{n_s} = \Psi_{n_s} \cdot Y_{n_s}$, $s = \overline{1, k_0}$, $Z_{n-2\cdot n_0} = v \cdot Y_{n-2\cdot n_0}$ and apply to the differential system with respect to Y_{n_s}, $s = \overline{1, k_0}$, $Y_{n-2\cdot n_0}$ the analogue of the lemma [4] on the stability in a ring-shaped domain involving the origin. □

Remark. If the coefficients of the differential equation (1) are slowly varying functions, then applying several times the method of "frozen' transformations, one can attain for fixed Z that the functions Φ_{n_s}, $s = \overline{1, k_0}$, $\Phi_{n-2\cdot n_0}$ and Θ_{n_s}, $s = \overline{1, k_0}$, $\Theta_{n-2\cdot n_0}$ in the differential systems (4) and (5), respectively, would tend rapidly enough to zero as $t \uparrow \omega$.

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