Mem. Differential Equations Math. Phys. 10(1997), 157–161

I. Kiguradze and B. Puža

ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

(Reported on November 18–25, 1996)

In the present note, basing on the results of our previous work [1] we establish sufficient conditions for the existence and uniqueness of a solution of the boundary value problem

$$\frac{dx(t)}{dt} = g\big(t, x(\tau_1(t)), \dots, x(\tau_m(t))\big), \tag{1}$$

$$h(x) = 0, \tag{2}$$

where $g : [a,b] \times R^{nm} \to R^n$ is a vector function satisfying the local Carathéodory conditions, $\tau_i : [a,b] \to [a,b]$ $(i = 1, \dots, m)$ are measurable functions and $h : C([a,b]; R^n) \to R^n$ is a continuous operator.

Under solution of the system (1) we understand an absolutely continuous vector function $x : [a,b] \to R^n$ which almost everywhere on $[a,b]$ satisfies it, and under solution of the problem (1), (2) we mean a solution of the system (1) which satisfies the condition (2).

The use is made of the following notation:

$I = [a,b]$, $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$;

R^n – the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with $x_i \in R$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$R^{n\times n}$ – the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with $x_{ik} \in R$ $(i,k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$$R_+^n = \Big\{(x_i)_{i=1}^n \in R^n : \ x_i \geq 0 \ (i = 1, \dots, n)\Big\},$$

$$R_+^{n\times n} = \Big\{(x_{ik})_{i,k=1}^n \in R^{n\times n} : \ x_{ik} \geq 0 \ \ (i,k = 1, \dots, n)\Big\};$$

if $x, y \in R^n$ and $X, Y \in R^{n\times n}$, then

$$x \leq y \Longleftrightarrow y - x \in R_+^n \quad \text{and} \quad X \leq Y \Longleftrightarrow Y - X \in R_+^{n\times n};$$

if $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \quad |X| = (|x_{ik}|)_{i,k=1}^n;$$

1991 *Mathematics Subject Classification.* 34K10.

Key words and phrases. System of nonlinear differential equations with deviating arguments, boundary value problem, existence theorem, uniqueness theorem.

$C(I;R^n)$ – the space of continuous vector functions* $x : I \to R^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\| : t \in I\};$$

$$C(I;R^n_+) = \Big\{x \in C(I;R^n) : x(t) \geq 0 \text{ for } a \leq t \leq b\Big\};$$

$L(I;R^n)$ – the space of summable vector functions $x : I \to R^n$ with the norm

$$\|x\|_L = \int_a^b \|x(t)\|\,dt.$$

Definition 1. Let $\mathcal{P} : I \times R^{n_0} \to R^{n\times n}$ be a matrix function satisfying the local Carathéodory conditions. We say that a summable matrix function $\mathcal{P}_0 : I \to R^{n\times n}$ belongs to the set $\mathcal{E}^{n_0}_{\mathcal{P}}$ if there exists a sequence $u_k \in C(I;R^{n_0})$ $(k = 1,2,\dots)$ such that

$$\lim_{k\to\infty} \int_a^t \mathcal{P}(s,u_k(s))\,ds = \int_a^t \mathcal{P}_0(s)\,ds \quad \text{uniformly on} \quad I.$$

Definition 2. Let $l : C(I;R^{n_0}) \times C(I;R^n) \to R^n$ be a continuous operator. We say that a linear operator $l_0 : C(I;R^n) \to R^n$ belongs to the set $\mathcal{E}^{n_0}_l$ if there exists a sequence $u_k \in C(I;R^{n_0})$ $(k = 1,2,\dots)$ such that

$$\lim_{k\to\infty} l(u_k,v) = l_0(v) \quad \text{for} \quad v \in C(I;R^n).$$

Definition 3. An operator $h_0 : C(I;R^n_+) \to R^n_+$ is said to be positively homogeneous if for any $\lambda \in R_+$ and $u \in C(I;R^n_+)$ we have $h_0(\lambda u) = \lambda h_0(u)$. However, if for any $u,v \in C(I;R^n_+)$ satisfying $u(t) \leq v(t)$ for $t \in I$ the inequality $h_0(u) \leq h_0(v)$ is fulfilled, then h_0 is said to be nondecreasing.

Definition 4. Let $Q_k : I \to R^{n\times n}_+$ $(k = 1,\dots,m)$ be summable matrix functions, $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,m)$ be measurable functions and $h_0 : C(I;R^n_+) \to R^n_+$ be a positively homogeneous continuous nondecreasing operator. Then the writing

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{n_1,n_2}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}$$

means that

(i) $\mathcal{P}_k : I \times R^{n_1} \to R^{n\times n}$ $(k = 1,\dots,n)$ are matrix functions satisfying the local Carathéodory conditions and $l : C(I;R^{n_2}) \times C(I;R^n) \to R^n$ is a continuous operator; moreover, $l(u,\cdot) : C(I;R^n) \to R^n$ is linear for arbitrarily fixed $u \in C(I;R^{n_2})$.

(ii) there exist a summable function $\alpha : I \to R_+$ and a positive number α_0 such that the inequalities

$$\|\mathcal{P}_k(t,x)\| \leq \alpha(t) \quad (k = 1,\dots,m) \quad \text{and} \quad \|l(u,v)\| \leq \alpha_0\|v\|_C$$

are fulfilled on $I \times R^{n_1}$ and $C(I;R^{n_2}) \times C(I;R^n)$, respectively;

(iii) for any $\mathcal{P}_{0k} \in \mathcal{E}^{n_1}_{\mathcal{P}}$ $(k = 1,\dots,m)$ and $l_0 \in \mathcal{E}^{n_2}_l$, the problem

$$\left|\frac{dv(t)}{dt} - \sum_{k=1}^m \mathcal{P}_{0k}(t)v(\tau_k(\tau))\right| \leq \sum_{k=1}^m Q_k(t)|v(\tau_k(t)|, \quad |l_0(v)| \leq h_0(|v|)$$

has only the trivial solution.

---

*A vector or matrix function is said to be continuous, absolutely continuous, summable, etc., if all its components have such a property.

Theorem 1. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - \sum_{k=1}^{m} \mathcal{P}_k(t,x_1,\dots,x_m)x_k\Big| \le \sum_{k=1}^{m} Q_k(t)|x_k| + \eta(t) \tag{3}$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - l(x,x)\Big| \le h_0(|x|) + \eta_0 \tag{4}$$

be fulfilled, where $Q_k : I \to R_+^{n\times n}$ $(k=1,\dots,m)$ are summable matrix functions, $\eta : I \to R_+^n$ is a summable vector function, $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator, $\eta_0 \in R_+^n$ and

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{mn,n}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}. \tag{5}$$

Then the problem (1), (2) *has at least one solution.*

Scheme of the proof. For any $x,y \in C(I;R^n)$ and $u \in C(I;R_+^n)$, put $f(x)(t) = g(t,x(\tau_1(t),\dots,x(\tau_m(t)))$, $p(x,y)(t) = \sum_{k=1}^m \mathcal{P}_k(t,x_1(\tau_1(t),\dots,x_m(\tau_m(t)))y(\tau_k(t))$ and $q_0(u)(t) = \sum_{k=1}^m Q_k(t)u(\tau_k(t))$. Then the system (1) and the condition (3) take respectively the form

$$\frac{dx(t)}{dt} = f(x)(t), \tag{1'}$$

$$\Big|f(x)(t) - p(x,x)(t)\Big| \le q_0(|x|)(t) + \eta(t). \tag{3'}$$

Owing to the restrictions imposed on g and τ_k $(k=1,\dots,n)$, the operator $f : C(I;R^n) \to L(I;R^n)$ is continuous. On the other hand, by Definition 4 and also by Definition 1.3 of [1], we can show that the condition (5) implies the condition

$$(p,l) \in O^n_{q_0,h_0}. \tag{5'}$$

By Theorem 1.1 from [1], the conditions (3′), (4) and (5′) ensure the solvability of the problem (1′), (2). ∎

According to Theorem 1, we can easily prove

Theorem 2. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - g(t,y_1,\dots,y_m) - \sum_{k=1}^{m} \mathcal{P}_k(t,x_1,\dots,x_m,y_1,\dots,y_m)(x_k - y_k)\Big| \le$$
$$\le \sum_{k=1}^{m} Q_k(t)|x_k - y_k|,$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - h(y) - l(x,y,x-y)\Big| \le h_0(|x-y|)$$

be fulfilled, where $Q_k : I \to R_+^{n\times n}$ $(k=1,\dots,m)$ are summable matrix functions, $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator and

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{2mn,2n}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}.$$

Then the problem (1), (2) *has a unique solution.*

In the case where the matrix functions $\mathcal{P}_k$ $(k=1,\dots,m)$ depend only on t and $l : C(I;R^n) \to R^n$ is a linear operator, Theorems 1 and 2 will respectively take the following form.

Corollary 1. *Let on $I \times R^{mn}$ the inequality*

$$\Big| g(t, x_1, \dots, x_m) - \sum_{k=1}^{m} \mathcal{P}_k(t) x_k \Big| \le \sum_{k=1}^{m} Q_k(t) |x_k| + \eta(t)$$

and on $C(I; R^n)$ the inequality

$$\Big| h(x) - l(x) \Big| \le h_0(|x|) + \eta_0$$

be fulfilled, where $\mathcal{P}_k : I \to R^{n\times n}$, $Q_k : I \to R_+^{n\times m}$ $(k = 1, \dots, m)$ are summable matrix functions, $\eta : I \to R_+^n$ is a summable vector function, $l : C(I; R^n) \to R^n$ is a linear bounded operator and $h_0 : C(I; R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator. Let, moreover, the problem

$$\Big| \frac{dv(t)}{dt} - \sum_{k=1}^{m} \mathcal{P}_k(t) v(\tau_k(t)) \Big| \le \sum_{k=1}^{m} Q_k(t) |v(\tau_k(t))|, \quad |l(v)| \le h_0(|v|) \tag{6}$$

have only the trivial solution. Then the problem (1), (2) *has at least one solution.*

Corollary 2. *Let on $I \times R^{mn}$ the inequality*

$$\Big| g(t, x_1, \dots, x_m) - g(t, y_1, \dots, y_m) - \sum_{k=1}^{m} \mathcal{P}_k(t)(x_k - y_k) \Big| \le \sum_{k=1}^{m} Q_k(t) |x_k - y_k|$$

and on $C(I; R^n)$ the inequality

$$\Big| h(x) - h(y) - l(x - y) \Big| \le h_0(|x - y|)$$

be fulfilled, where $\mathcal{P}_k : I \to R^{n\times n}$, $Q_k : I \to R_+^{n\times n}$ $(k = 1, \dots, m)$ are summable matrix functions, $l : C(I; R^n) \to R^n$ is a linear bounded operator and $h_0 : C(I; R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator such that the problem (6) *has only the trivial solution. Then the problem* (1), (2) *has a unique solution.*

Consider now the case where the boundary conditions (2) have the form

$$\varphi\big(x(t_1), \dots, x(t_{m_0})\big) = 0, \tag{7}$$

where $\varphi : R^{m_0 n} \to R^n$ is a continuous vector function and $t_k \in I$ $(i = 1, \dots, m_0)$.

For the problem (1), (7), we have from Theorem 2 the following

Corollary 3. *Let:* (i) *for almost all $t \in I$ there exist $\frac{\partial g(t, x_1, \dots, x_m)}{\partial x_k}$ $(k = 1, \dots, m)$ which are continuous with respect to $x_1, \dots, x_m$ in R^{mn} and satisfy*

$$\mathcal{P}_{1k}(t) \le \frac{\partial g(t, x_1, \dots, x_m)}{\partial x_k} \le \mathcal{P}_{2k}(t) \quad (k = 1, \dots, m),$$

where $\mathcal{P}_{1k}$ and $\mathcal{P}_{2k} : I \to R^{n\times n}$ $(k = 1, \dots, m)$ are summable matrix functions;

(ii) *the vector function φ have the first order continuous partial derivatives and*

$$A_{1k} \le \frac{\partial \varphi(x_1, \dots, x_{m_0})}{\partial x_k} \le A_{2k} \quad (k = 1, \dots, m_0)$$

on $R^{m_0 n}$, where A_{1k} and $A_{2k} \in R^{n\times n}$ $(k = 1, \dots, m_0)$; (iii) *for any summable matrix functions $\mathcal{P}_k : I \to R^{n\times n}$ $(k = 1, \dots, m)$ and matrices $A_k \in R^{n\times n}$ $(k = 1, \dots, m_0)$ satisfying*

$$\begin{gathered} \mathcal{P}_{1k}(t) \le \mathcal{P}_k(t) \le \mathcal{P}_{2k}(t) \quad \textit{for} \quad t \in I \quad (k = 1, \dots, m), \\ A_{1k} \le A_k \le A_{2k} \quad (k = 1, \dots, m_0), \end{gathered}$$

the boundary value problem

$$\frac{dv(t)}{dt} = \sum_{k=1}^{m} \mathcal{P}_k(t)v(\tau_k(t)), \quad \sum_{k=1}^{m_0} A_k v(t_k) = 0$$

have only the trivial solution. Then the problem (1), (7) *has a unique solution.*

Remark. This work was supported by Grant 201/96/0410 of the Grant Agency of the Czech Republic (Prague) and by Grant 619/1996 of the Development Fund of Czech Universities.

References

1. I. T. Kiguradze and B. Puža, Conti-Opial type theorems for nonlinear functional differential equations. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 2 (*to appear*).

Authors' addresses:

I. Kiguradze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia

B. Puža
Department of mathematics
Masaryk University
Janáčkovo nám. 2a
66 295 Brno
Czech Republic