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Ye. Galakhov

ON DIFFERENTIAL OPERATORS WITH INTEGRAL CONDITIONS

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Ordinary differential equations with nonlocal conditions are studied in [1 - 4]. Such conditions are usually written in the form of the Stieltjes integral containing an atomic measure at the ends of an interval. Using methods of the theory of nonlocal elliptic problems [5], we can avoid such restrictions. First this method was employed in [6].

In the present paper, we consider the equation

$$\begin{aligned}(Au-\lambda u)(t) &= -a_0(t)u''(t)+a_1(t)u'(t)+\\ &+a_2(t)u(t)-\lambda u(t)=f_0(t) \quad (t\in(0,1))\end{aligned} \tag{1}$$

with integral conditions

$$B_j u=\int\limits_0^1 \varphi_j(t)u(t)dt=f_j \quad (j=1,2). \tag{2}$$

Here a_i are real-valued functions $(i=0,1,2)$; $a_0(t)\ge k>0$ $(0\le t\le 1)$; $a_1,a_2\in C[0,1]$; $f_0\in L_2(0,1)$ is a complex-valued function; $f_j\in\mathbb{C}$ $(j=1,2)$ are constants; $\lambda\in\mathbb{C}$ is a spectral parameter; φ_j are linearly independent real-valued functions.

The functions a_0 and φ_j have to satisfy some conditions. In order to formulate them, we introduce some notation:

$C^{0,\alpha}[a,b]$ is the Hölder space consisting of the functions $u(t)$ such that $u(t)$ are continuous on $[a,b]$ and

$$|u(t_1)-u(t_2)|\le c|t_1-t_2|^\alpha \quad (t_1,t_2\in[a,b]),$$

where $c>0$ and $0<\alpha\le 1$ do not depend on t_1, t_2;

$W^{1,\infty}(a,b)$ is the space of the functions $u(t)$ which are absolutely continuous on $[a,b]$ and $u'\in L_\infty(a,b)$;

$$C^{0,\alpha}_\beta[0,1]=\Big\{u\in L_2(0,1):\ u\in C^{0,\alpha}[0,\beta]\cap C^{0,\alpha}[1-\beta,1]\Big\},$$
$$W^{1,\infty}_\beta(0,1)=\Big\{u\in C[0,1]: u\in W^{1,\infty}(0,\beta)\cap W^{1,\infty}(1-\beta,1)\Big\},$$

where $0<\beta<1/2$;

$W^m(0,1)$ is the Sobolev space of complex-valued functions which are absolutely continuous on $[0,1]$ along with their derivatives up to the order $(m-1)$ inclusively and whose m-th derivative belongs to $L_2(0,1)$. The norm is defined by

$$\|u\|_{W^m(0,1)}=\Big\{\sum_{j=0}^m\int\limits_0^1|u^{(j)}(t)|^2dt\Big\}^{1/2}.$$

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$V(0,1) = L_2(0,1) \times \mathbb{C} \times \mathbb{C}$;

$\Omega_{\varepsilon,q} = \{\lambda \in \mathbb{C} : |\arg\lambda| \geq \varepsilon, |\lambda| \geq q\}$, where ε, $q > 0$.

In the spaces $W^m(0,1)$ and $V(0,1)$, we introduce the following equivalent norms depending on λ:

$$|||u|||_{W^m(0,1)} = \Big\{ \|u\|^2_{W^m(0,1)} + |\lambda|^m \|u\|^2_{L_2(0,1)} \Big\}^{1/2},$$

$$|||f|||_{V(0,1)} = \Big\{ \|f_0\|^2_{L_2(0,1)} + |\lambda|^2 \Big(|f_1|^2 + |f_2|^2 \Big) \Big\}^{1/2},$$

where $f = (f_0, f_1, f_2)$.

We define a bounded operator $L(\lambda) : W^2(0,1) \to V(0,1)$ by the formula $L(\lambda)u = (Au - \lambda u, B_1 u, B_2 u)$. We also introduce an unbounded operator $A_B : L_2(0,1) \to L_2(0,1)$ with the domain $\mathcal{D}(A_B) = \{u \in W^2(0,1) : B_j u = 0, j = 1,2\}$ by the formula $A_B u = Au$ $(u \in \mathcal{D}(A_B))$.

Suppose that $a_0 \in W^{1,\infty}_\beta(0,1)$, $\varphi_j \in C^{0,\alpha}_\beta[0,1]$ $(1/2 < \alpha \leq 1)$, and

$$\Delta_\varphi = \varphi_1(0)\varphi_2(1) - \varphi_1(1)\varphi_2(0) \neq 0.$$

Theorem 1. *Let $\Delta_\varphi \neq 0$. Then for any $\varepsilon > 0$ there exists $q > 1$ such that for all $\lambda \in \Omega_{\varepsilon,q}$ a solution $u \in W^2(0,1)$ of the problem (1), (2) satisfies*

$$|||u|||_{W^2(0,1)} \leq c\lambda^{1/4} |||f|||_{V(0,1)}, \tag{3}$$

where $c > 0$ does not depend on λ and f.

From Theorem 1 it follows:

Theorem 2. *Let $\Delta_\varphi \neq 0$. Then the spectrum $\sigma(A_B)$ consists of isolated eigenvalues. For any $\varepsilon > 0$, there exists $q > 0$ such that*

$$\sigma(A_B) \subset \Big\{ \lambda \in \mathbb{C} : |\arg\lambda| < \varepsilon \Big\} \cup \Big\{ \lambda \in \mathbb{C} : |\lambda| < q \Big\}.$$

$L(\lambda) : W^2(0,1) \to V(0,1)$ is a Fredholm operator and $\operatorname{ind} L(\lambda) = 0$ for every $\lambda \in \mathbb{C}$. For $\lambda \notin \sigma(A_B)$, the operator $L(\lambda)$ has a bounded inverse $L^{-1}(\lambda) : V(0,1) \to W^2(0,1)$.

Under the assumptions $a_0 \in C^2[0,1]$, $a'(0) = a'(1) = 0$, $\phi_j \in C^1[0,1]$, these theorems were proved in [6].

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Author's address:
Moscow State Aviation Institute
(Technical University)
4, Volokolamskoe shosse, Moscow 125871
Russia