Mem. Differential Equations Math. Phys. 10(1997), 115–118

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ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER NONLINEAR EQUATIONS

(Reported on March 11–18, 1996)

Below we will use the following notation:

R is the set of real numbers.

$L([a,b])$ is the set of the functions $p :]a,b[\to R$ which are Lebesgue integrable on $[a,b]$.

$L_{loc}(]a,b[)$ is the set of the functions $p :]a,b[\to R$ which are Lebesgue integrable on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$K_0(]a,b[\times R^2)$ is the set of the functions $q :]a,b[\times R^2 \to R$ for which the mapping $t \longmapsto g(t, x_1(t), x_2(t))$ is measurable for any continuous functions $x_i :]a,b[\to R$ $(i = 1,2)$.

$\sigma : L_{loc}(]a,b[) \to L_{loc}(]a,b[)$ is an operator defined by the equality

$$\sigma(p)(t) = \exp\Big[\int\limits_{\frac{a+b}{2}}^{t} p(s)ds\Big].$$

If $\sigma(p) \in L([a,b])$, $\alpha \in [a,b]$ and $\beta \in]\alpha, b]$, then

$$\sigma_\alpha(p)(t) = \frac{1}{\sigma(p)(t)}\Big|\int\limits_\alpha^t \sigma(p)(s)ds\Big|,$$

$$\sigma_{\alpha\beta}(p)(t) = \frac{1}{\sigma(p)(t)}\int\limits_\alpha^t \sigma(p)(s)ds \cdot \int\limits_t^\beta \sigma(p)(s)ds.$$

$u(s+)$ and $u(s-)$ are the limits of the function u at the point s from the right and from the left, respectively.

If $\mu : [a,b] \to R$ is a function of bounded variation, then by $\mu^*(t)$ we denote the full variation of the function μ on the segment $[a,t]$.

Under solution of the equation

$$u'' = f(t,u,u'), \tag{1}$$

where $f :]a,b[\times R^2 \to R$ satisfies the Carathéodory conditions on every compactum contained in $]a,b[\times R^2$, we understand a function $u :]a,b[\to R$ which is absolutely continuous along with its first derivative on every segment from $]a,b[$, and satisfies (1) a.e.

1991 *Mathematics Subject Classification.* 34K10.

Key words and phrases. Second order equation, nonlocal boundary value problem, existence and uniqueness.

In the present paper, we consider the problem of existence and uniqueness of solution of the equation (1) satisfying the boundary conditions

$$u(a+) = 0, \quad u(b-) = \int_a^b u(s)d\mu(s), \tag{2}$$

where $\mu : [a, b] \to R$ is a function of bounded variation.

Some criteria for unique solvability of the problem in a linear case are contained in [1, 2]. In the nonlinear case, a problem of the type (1), (2) has been considered in [3–5]. However, in those works μ is assumed to be a piecewise constant function ($\mu(t) = 0$ for $a \le t \le t_0$ and $\mu(t) = 1$ for $t_0 < t \le b$).

Theorems of existence and uniqueness of solution of the problem (1), (2) given in the present paper cover the case where μ is not, generaly speaking, piecewise constant, and f is not integrable in the first argument on the segment $[a, b]$, having singularities at the points $t = a$ and $t = b$.

Before passing to the formulation of basic results, let us introduce the following definitions.

Definition 1. We say that a vector-function $(p_1, p_2) :]a, b[\to R^2$ belongs to the class $U_\mu(]a, b[)$ if

$$\sigma(p_2), \ \sigma_{ab}(p_2)p_1 \in L(]a, b[)$$

and the solution u_1 of the singular Cauchy problem

$$u'' = p_1(t)u + p_2(t)u'; \quad u(a+) = 0, \quad \lim_{t \to a+} \frac{u'(t)}{\sigma(p_2)(t)} = 1$$

satisfies the conditions

$$u_1(t) > 0 \quad \text{for} \quad a < t < b, \quad u_1(b-) > \int_a^b u_1(s)d\mu^*(s).$$

Definition 2. We say that a vector-function $(p_1, p_{12}, p_{22}) :]a, b[\to R^3$ belongs to the class $V_\mu(]a, b[)$ if

$$p_{12}(t) \le p_{22} \quad \text{for} \quad a < t < b, \tag{3}$$

$$p_{i2}, \ p_1 \in L_{loc}(]a, b[), \quad i = 1, 2, \tag{4}$$

$$\sigma(p_{i2}) \in L([a, b]), \quad \sigma_{ab}(p_{i2})p_1 \in L([a, b]), \quad i = 1, 2, \tag{5}$$

and $(p_1, p_2) \in U_\mu(]a, b[)$ for any measurable function $p_2 :]a, b[\to R$ satisfying

$$p_{12}(t) \le p_2(t) \le p_{22}(t) \quad \text{for} \quad a < t < b.$$

Theorem 1. *On the set* $]a, b[\times R^2$, *let the inequalities*

$$\Big[f(t, x, y) - p_1(t)x - p_2(t, x, y)y\Big] \operatorname{sgn} x \ge -p(t),$$
$$p_{12}(t) \le p_2(t, x, y) \le p_{22}(t)$$

be fulfilled, where $p_2 \in K_0(]a, b[\times R^2)$ *and* $(p_1, p_2, p_{22}) \in V_\mu(]a, b[)$. *Furthermore, let* $\sigma_{ab}(p_{i2})p \in L([a, b])$ $(i = 1, 2)$, *and for some point* $t_1 \in]a, b[$ *let*

$$\Big|f(t, x, y) - p_1(t)x - p_2(t, x, y)y\Big| \le p(t) \quad \textit{for} \quad t_1 < t < b, \ \ x \in R, \ \ y \in R. \tag{6}$$

Then the problem (1), (2) *has at least one solution.*

Remark 1. Let μ be nondecreasing, the conditions (3)–(5) be fulfilled, and let $(p_1, p_{12}, p_{22}) \notin V_\mu(]a,b[)$. Then there exists a function f satisfying the conditions of Theorem 1 for which the problem (1), (2) has no solution.

Remark 2. The condition (6) can be replaced by the condition

$$\left|f(t,x,y) - p_1(t)x - \widetilde{p}_2(t,x,y)y\right| \le p(t) \quad \text{for} \quad t_1 < t < b, \quad x \in R, \quad y \in R,$$

where $\widetilde{p}_2 \in K_0(]a,b[\times R^2)$ and $p_{12}(t) \le \widetilde{p}_2(t) \le p_{22}(t,x,y)$ for $(t,x,y) \in]t_1,b[\times R^2$.

As an example, let us consider the problem (1), (2) in the case where μ increases, $\mu(b) - \mu(a) < 1$, and

$$f(t,x,y) = p_0(t) + p_1(t)x + p_2(t)y + p_3(t)x^{2n+1}|y|^k, \quad p_i \in L_{loc}(]a,b[) \;\; i = \overline{0,3},$$

where n and k are positive integers. Assume that $\lambda > 0$, $0 \le \delta < 1$, $p_1(t) \ge 0$, $p_3(t) \ge 0$, $|p_2(t)| \le \lambda + \frac{\delta}{t-a} + \frac{\delta}{b-t}$ for $a < t < b$,

$$\int_a^b (s-a)(b-s)|p_i(s)|ds < +\infty, \quad i = 0, 1,$$

and $p_3(t) \equiv 0$ in a neighborhood of the point b. Taking into account Theorem 1.2 in [2], we obtain from Theorem 1 that in this case the problem (1), (2) has at least one solution. As it is seen from the example, the function f may have nonintegrable singularities for $t = a$ and $t = b$.

Corollary 1. *On the set* $]a,b[\times R^2$*, let the inequality*

$$f(t,x,y)\operatorname{sgn} x \ge p_1(t)|x| - p_2(t)|y| - p(t) \tag{7}$$

be fulfilled, where $(p_1, -p_2, p_2) \in V_\mu(]a,b[)$ *and*

$$\sigma_{ab}((-1)^i p_2)p \in L([a,b]), \quad i = 1, 2.$$

Furthermore, let for some point $t_1 \in]a,b[$

$$\left|f(t,x,y) - p_1(t)x - p_2(t)y\right| \le p(t) \quad \textit{for} \quad t_1 < t < b, \quad x \in R, \quad y \in R. \tag{8}$$

Then the problem (1), (2) *has at least one solution.*

Corolary 2. *Let there exist numbers* $\lambda_i \in [0,1[$, $l_i \in [0,+\infty[$, $\gamma_i \in [0,+\infty[$, $i = 1,2$, $c \in]a,b[$ *and the function* $p :]a,b[\to]0,+\infty[$ *such that*

$$\int_0^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} \ge \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1}, \quad \int_{\gamma_1}^{\gamma_2} \frac{ds}{l_1 + l_2 s + s^2} \ge \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2},$$

$$\int_{\gamma_1}^{\gamma_2} \frac{sds}{l_1 + l_2 s + s^2} < -\ln(\mu^*(b) - \mu^*(a)),$$

let the function $t \longmapsto (t-a)(b-t)p(t)$ *be summable on* $[a,b]$*, and on the set* $]a,b[\times R^2$ *let the inequality* (7) *be fulfilled, where*

$$p_1(t) = \begin{cases} -l_1(t-a)^{-2\lambda_1} & \textit{for} \quad a < t \le c \\ -l_1(b-t)^{-2\lambda_2} & \textit{for} \quad c < t < b \end{cases},$$

$$p_2(t) = \begin{cases} l_2(t-a)^{-\lambda_1} + \lambda_1(t-a)^{-1} & \textit{for} \quad a < t \le c \\ l_2(b-t)^{-\lambda_2} + \lambda_2(b-t)^{-1} & \textit{for} \quad c < t < b \end{cases}.$$

Moreover, let (8) *be fulfilled for some point* $t_1 \in]a, b[$. *Then the problem* (1), (2) *has at least one solution.*

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