Memoirs on Differential Equations and Mathematical Physics
Volume 10, 1997, 1–53

Tengiz Buchukuri and $\boxed{\text{Tengiz Gegelia}}$

SOME DYNAMIC PROBLEMS OF THE THEORY OF ELECTROELASTICITY

Abstract. Boundary value problems of electroelasticity for equations of pseudo-oscillation and dynamics are studied by using the methods of potential and of singular integral equations. Existence and uniqueness theorems are proved. Properties of solutions at infinity are estimeted.

1991 Mathematics Subject Classification. 73C30, 73C15, 73C35.

Key words and Phrases. Boundary value problems of dynamics and pseudooscillation for the equation of electroelasticity, fundamental solutions, potentials, singular integral equations.

რეზიუმე. ნაშრომში შესწავლილია ელექტროდრეკადობის სასაზღვრო ამოცანები ფსევდორხევისა და დინამიკის განტოლებებისათვის პოტენციალებისა და სინგულარული ინტეგრალური განტოლებების მეთოდებით. დამტკიცებულია არსებობისა და ერთადერთობის თეორემები. დადგენილია ამონახსნების ყოფაქცევა უსასრულობის მიდამოში.

1. Introduction

The phenomenon of piezoelectricity discovered by brothers Jacques and Pierre Curie (1880) forms the basis of the theory of electroelasticity. The piezoelectric effect consists in the fact that under deformation of some crystals there appear on their surfaces electric charges depending on the deformation magnitude. The reverse effect – generation of stresses in crystals due to the action of an electric field – has also been found out (J. Curie, P. Curie, H. G. Lippmann, 1881).

At present, the phenomenon of piezoelectricity is of great importance. It is used in electromechanical transducers transforming mechanical energy to electric one, and vice versa, in radioelectronics, electroacoustics, instrument-making and measuring equipment.

W. Voigt [1] was the first who constructed a mathematical model of the elastic medium taking linear interaction of electric and mechanical fields into account. In their works, R. Toupin [2,3], R. Mindlin [4,5,6], W. Nowacki [7], S. Kaliski and J. Petikiewicz [8], and A. Ulitko [9] suggested new, more refined models (for details, see [7], [9] and [10]).

An elastic medium with piezoelectric effect is referred to as electroelastic medium, while a mathematical model of this medium taking interaction of electric fields into account is called the theory of electroelasticity.

Despite a great number of works on electroelasticity which have appeared in the last years, not many strict mathematical results are available. These works deal mainly with the problems of statics and oscillations, but little attention is given to the dynamic problems.

The present paper is devoted to this very matter. We investigate dynamic problems for a homogeneous anisotropic electroelastic medium as well as associated problems of pseudo-oscillations. In particular, existence and uniqueness theorems are proved and asymptotic properties of solutions are established. The investigation is performed by employing the Laplace transform, the potential theory and the theory of singular integral equations. We stick mainly to the scheme used for investigation of dynamic problems in the classical theory of elasticity [11]. Nevertheless, there exist intrinsic differences connected with the fact that the fourth equation of the system of equations of dynamics of electroelasticity does not contain time derivatives (in which case the system is sometimes called quasi-static). This circumstance, when investigating the solvability of the problems of pseudo-oscillation and dynamics, gives rise to complications and requires some changes to be put in the proofs of the corresponding theorems, in comparison with the classical theory of elasticity.

In this work, the use will be made of the following notation:

If x is an element of $\mathbb{R}^m$ – the m-dimensional real Euclidean space with $|x| = (\sum\limits_{i=1}^{m} x_i^2)^{1/2}$ – then $B(x,r) = \{y \in \mathbb{R}^m, |x-y| < r\}$. The boundary of a domain $\Omega \subset \mathbb{R}^m$ is denoted by $\partial\Omega$. In particular, $C(x,r) = \partial B(x,r)$.

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\alpha_i \in \mathbb{N} \cup \{0,\}$, $i = 1, \dots, m$, then $|\alpha| = \sum\limits_{i=1}^{m} \alpha_i$, $\alpha! = \alpha_1!\alpha_2!\dots\alpha_m!$, and for $\xi \in \mathbb{R}^m$, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_m^{\alpha_m}$, $\partial_\xi^\alpha = \frac{\partial^{|\alpha|}}{\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\dots\partial_{\xi_m}^{\alpha_m}}$.

For any set Ω, we denote by $C^k(\Omega)$ the space of continuous functions with continuous on Ω derivatives up to the k-th order inclusive. Functions of the class $C^2(\Omega) \cap C^1(\overline{\Omega})$ will be referred to as regular.

If $k \in N$, $0 < \alpha \leq 1$, then $C^{k,\alpha}(\Omega) = \{f \in C^k(\Omega) : |f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\alpha, x_1, x_2 \in \Omega\}$ is the Hölder space in which we introduce the norm

$$\|f\|_{(\Omega,k,\alpha)} = \sum_{|\beta|=0}^{k} \sup_{x\in\Omega} |\partial^\beta f(x)| +$$
$$+ \sum_{|\beta|=k} \sup_{\substack{x_1,x_2\in\Omega \\ |x_1-x_2|<1}} \big(|\partial^\beta f(x_1) - \partial^\beta f(x_2)|\, |x_1 - x_2|^{-\alpha}\big).$$

$C^k(\Omega)$ is identified with $C^{k,0}(\Omega)$. By $C^{k,\alpha}$ we denote the corresponding class of Lyapunov surfaces [17]. And finally, $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ denote respectively the scalar product and the norm in the space $L_2(\Omega)$.

2. Basic Problems of Dynamics

The basic equations of motion and of electric field in the classical electroelasticity (the Voigt model [1]) have the form

$$\frac{\partial \tau_{ij}}{\partial X_j} + \rho x_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2, 3, \tag{2.1}$$
$$\frac{\partial D_i}{\partial x_i} = 0, \tag{2.2}$$

where $(\tau_{ij})_{3\times3}$ is the stress tensor, $D = (D_1, D_2, D_3)$ is the vector of electric displacement (introduction), $X = (X_1, X_2, X_3)$ is the mass force and ρ is the density of the medium. $x = (x_1, x_2, x_3)$ are the coordinates of the point x and t is time. In the equations (2.1) and (2.2) as well as in what follows, we will stick to the conventional agreement that summation is performed with respect to the repeating indices.

The equations (2.1) and (2.2) are supplemented with the determining relations

$$\tau_{ij} = c_{ijkl}s_{kl} - e_{kij}E_k, \quad i, j = 1, 2, 3, \tag{2.3}$$
$$D_i = e_{ikl}s_{kl} + \varepsilon_{ik}E_k, \quad i = 1, 2, 3, \tag{2.4}$$

where C_{ijkl}, e_{kij} and ε are respectively elastic, piezo-electric and dielectric constants. The strain tensor $(s_{kl})_{3\times3}$ and the electric field vector $E =$

(E_1, E_2, E_3) are connected with the displacement vector $u = (u_1, u_2, u_3)$ and the electric potential φ by the relation

$$s_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad E_k = -\frac{\partial \varphi}{\partial x_k}, \quad i,j,k = 1,2,3. \tag{2.5}$$

The coefficients c_{ijkl}, e_{kij}, ε_{ik} satisfy the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad i,j,k,l = 1,2,3, \tag{2.6}$$

as well as the condition of positiveness of the internal energy

$$U = \frac{1}{2} c_{ijkl} s_{ij} s_{kl} + \frac{1}{2}\varepsilon_{ij} E_i E_j > 0 \quad \text{for} \quad s_{ij} s_{ij} + E_i E_i \neq 0.$$

Since $s_{ij} = s_{ji}$, this condition is equivalent to the conditions

$$\begin{gathered} \forall (\xi_{ij}), \ \ (\eta_i), \ \ \xi_{ij} = \xi_{ji}, \ \ \exists c_0 > 0 \\ c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0 \xi_{ij}\xi_{ij}, \quad \varepsilon_{ij}\eta_i\eta_j \geq c_0 \eta_i \eta_i. \end{gathered} \tag{2.7}$$

Taking into account (2.3)–(2.5), we obtain the dynamic equations of electroelasticity with respect to the displacement and the electric potential

$$\begin{gathered} c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + e_{kij}\frac{\partial^2 \varphi}{\partial x_k \partial x_j} + \rho X_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1,2,3, \\ -e_{jkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \varepsilon_{jk}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} = 0, \quad i = 1,2,3. \end{gathered} \tag{2.8}$$

Let us formulate the boundary value problems which will be considered in the sequel.

Denote by Ω^+ a finite domain of $\mathbb{R}^3$, containing the point 0, with the piecewise-smooth boundary $S \equiv \partial\Omega$, $\Omega^- \equiv \mathbb{R}^3 \setminus \overline{\Omega}^+$. Let $n(y) = (n_1(y), n_2(y), n_3(y))$ be the unit normal to S at the point y, external with respect to Ω^+.

Problem (1)$^\pm$. In the cylinder $\Omega^\pm \times]0,+\infty[$, find a solution $U \equiv (u_1, u_2, u_3, \varphi) = (u, \varphi)$ of the system (2.8) belonging to the class $C^2(\Omega^\pm \times]0,+\infty[) \cap C^1(\overline{\Omega}^\pm \cap [0,+\infty[)$, and satisfying the boundary condition

$$U(y,t) = f(y,t), \quad y \in S, \quad t \in [0,+\infty[, \tag{2.9}$$

and the initial conditions

$$u(x,0) = \overset{(0)}{u}(x), \quad \frac{\partial u(x,0)}{\partial t} = \overset{(1)}{u}(x), \quad x \in \overline{\Omega}^\pm, \tag{2.10}$$

where $X = (X_1, X_2, X_3)$ are given vectors.

Problem (2)$^\pm$. In the cylinder $\Omega^\pm \times]0,+\infty[$, find a solution $U = (u_1, u_2, u_3, \varphi)$ of the system (2.8) belonging to the class $C^2(\Omega^\pm \times]0,+\infty[) \cap C^1(\overline{\Omega}^\pm \times [0,+\infty[)$ and satisfying the boundary conditions

$$n_j(y)\tau_{ij}(y,t) = f_i(y,t), \quad i = 1,2,3, \tag{2.11}$$

$$n_i(y) D_i(y,t) = f_4(y,t), \quad y \in S, \quad t \in [0,+\infty[, \tag{2.12}$$

and the initial conditions (2.10).

Remark. For the equation of electroelasticity (2.8), one can also consider some other boundary value problems. In particular, the boundary condition (2.11) can be replaced by one of the boundary conditions for the problems of classical theory of elasticity [11] by adding as "an electric" boundary condition either the condition (2.12) or

$$\varphi(y,t) = f_4(y,t), \quad y \in S, \quad t \in [0,+\infty[.$$

The investigation of these boundary value problems does not differ from that of Problems $(1)^\pm$ and $(2)^\pm$. Some other boundary value problems are quoted in [7] and [9].

3. Uniqueness Theorems for Dynamic Problems

Let $U = (u_1, u_2, u_3, \varphi)$ be a solution of one of the internal homogeneous dynamic problems under consideration. Thus it satisfies the equation on $\Omega^+ \times]0,\infty[$

$$\begin{aligned} c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + e_{kij}\frac{\partial^2 \varphi}{\partial x_k \partial x_j} - \rho\frac{\partial^2 u_i}{\partial t^2} = 0, \quad i = 1,2,3, \\ -e_{jkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \varepsilon_{jk}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} = 0, \quad i = 1,2,3, \end{aligned} \tag{2.8$_0$}$$

the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \tag{2.10$_0$}$$

and one of the two homogeneous boundary conditions on $S \times [0,+\infty[$:

$$u(y,t) = 0, \quad y \in S, \quad t \in [0,+\infty[, \tag{2.9$_0$}$$

or

$$n_j(y)\tau_{ij}(y,t) = 0, \quad i = 1,2,3, \tag{2.11$_0$}$$

$$n_i(y)D_i(y,t) = 0, \quad y \in S, \quad t \in [0,+\infty[. \tag{2.12$_0$}$$

From $(2.8)_0$ it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \Big(c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l} + \varepsilon_{ik}\frac{\partial \varphi}{\partial x_i}\cdot\frac{\partial \varphi}{\partial x_k} + \rho\frac{\partial u_k}{\partial t}\cdot\frac{\partial u_k}{\partial t}\Big)dx + \\ +\frac{d}{dt}\int_\Omega D_j n_j \varphi dS = \int_{\partial\Omega} \tau_{ij}\frac{\partial u_j}{\partial t} n_i dS + \int_{\partial\Omega} D_j \frac{\partial \varphi}{dt} n_j dS, \end{aligned} \tag{3.1}$$

where Ω is an arbitrary domain such that $\overline{\Omega} \subset \Omega^+$. We integrate this equality with respect to t from τ_0 to τ $(0<\tau_0<\tau)$ and then pass to limit as $\Omega \to \Omega^+$. By virtue of the boundary conditions $(2.9)_0$–$(2.12)_0$, we obtain

$$\int\limits_{\Omega^+} \Big(c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}+\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+\rho\frac{\partial u_k}{\partial t}\cdot\frac{\partial u_k}{\partial t}\Big)\Big|_{t=\tau_0}^{t=\tau} dx=0. \quad (3.2)$$

Pass in (3.2) to limit as $\tau_0\to 0$. Owing to the initial conditions $(2.10)_0$,

$$\lim_{\tau_0\to 0}\int\limits_{\Omega^+} \Big(c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}+\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+\rho\frac{\partial u_k}{\partial t}\cdot\frac{\partial u_k}{\partial t}\Big)_{t=\tau_0} dx=$$

$$=\lim_{\tau_0\to 0}\int\limits_{\Omega^+} \Big(\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}\Big)_{t=\tau_0} dx=0,$$

since from the second equation $(2.8)_0$ and the boundary conditions $(2.9)_0$–$(2.12)_0$ it follows

$$\int\limits_{\Omega^+} \Big(\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}\Big)_{t=\tau_0} dx=\int\limits_{\Omega^+} \Big(e_{ikl}\frac{\partial u_k}{\partial x_l}\cdot\frac{\partial\varphi}{\partial x_i}\Big)_{t=\tau_0} dx.$$

Thus from (3.2) we have

$$\int\limits_{\Omega^+} \Big(c_{ijkl}\frac{\partial u_i(x,t)}{\partial x_j}\cdot\frac{\partial u_k(x,t)}{\partial x_l}+$$
$$+\varepsilon_{ik}\frac{\partial\varphi(x,t)}{\partial x_i}\cdot\frac{\partial\varphi(x,t)}{\partial x_k}+\rho\frac{\partial u_k(x,t)}{\partial t}\cdot\frac{\partial u_k(x,t)}{\partial t}\Big)dx=0$$

for $t>0$. This, due to the condition (2.7), yields

$$\frac{\partial u_k(x,t)}{\partial t}=0,\quad \frac{\partial\varphi(x,t)}{\partial x_k}=0,\quad k=1,2,3,$$
$$u(x,t)=u(x,0)=0,\quad \varphi(x,t)=c(t). \quad (3.3)$$

In the case of Problem $(1)^+$, we in addition have

$$\varphi(x,t)=\varphi(y,t)=0,\quad x\in\Omega^+,\quad y\in S.$$

Thus the following theorem is proved.

Theorem 3.1. *Problem* $(1)^+$ *has a unique solution of the class* $C^2(\Omega^+\times]0,+\infty[)\cap C^1(\overline{\Omega}^+\times[0,+\infty[)$. *The solution of Problem* $(2)^+$ *of the class* $C^2(\Omega^+\times]0,+\infty[)\cap C^1(\overline{\Omega}^+\times[0,+\infty[)$ *is defined uniquely to within a summand of the form* $V(x,t)=(0,0,0,c(t))$. *In particular, if the condition*

$$\varphi(0,t)=0,\quad t\in[0,+\infty[, \quad (3.4)$$

is fulfilled, then Problem $(2)^+$ *has a unique solution.*

In the sequel, the condition (3.4) is assumed to be fulfilled in the case of Problem $(2)^+$. From the physical viewpoint, this means that the values of the electric potential φ are calculated with respect to the point $x = 0$.

Consider now the external Problems $(1)^-$ and $(2)^-$. The proof for the uniqueness of their solutions can be obtained in a standard way [12] – by considering the formula (3.1) in the domain $\Omega_R = \Omega^- \cap B(0,R)$ and by passing to limit as $R \to \infty$. Moreover, we require of the solution to satisfy certain conditions of decrease near the infinity which would ensure tending of the surface integrals

$$\int\limits_{\partial B(0,R)} \tau_{ij} \frac{\partial u_j}{\partial t} n_i dS, \qquad \int\limits_{\partial B(0,R)} D_j \frac{\partial \varphi}{\partial t} n_j dS$$

to zero. However, in the classical theory of elasticity, one can get rid of such conditions. In particular, in their work [13] L. Wheeler and E. Sternberg have proved uniqueness theorems for external dynamic problems without imposing upon the solution the conditions of decreasing at infinity. In the proof, the use has been made of the fact that the rate of propagation of mechanical perturbations in the elastic medium is finite. The uniqueness was proved similarly in [14].

Due to the "quasi-static" character of the equation of dynamics in electroelasticity, it is impossible to get completely rid of such conditions. For example, the vector $V = (0,0,0,\varphi)$ is obviously a solution of the homogeneous problem $(2)_0^-$ for any function $\phi(x,t) = \phi(t)$, $\phi \in C^2([0,\infty[)$. Nevertheless, one can prove the following theorem in which the conditions of displacement vector's decreasing at infinity are omitted.

Theorem 3.2. *Let $U = (u_1, u_2, u_3, \varphi)$ be a solution of one of the external homogeneous problems of dynamics of the class $C^2(\Omega^- \times]0,\infty[) \cap C^1(\overline{\Omega}^- \times [0,+\infty[)$ which on every interval $[0,t_0]$ satisfies the conditions*

$$\begin{gathered} |x|\,|\varphi(x,t)| \le c, \quad x \in \Omega^-, \\ \lim_{|x|\to\infty} |x| \Big(\sum_{|\alpha|=1} \Big| \frac{\partial \varphi(x,t)}{\partial x_\alpha} \Big| + \Big| \frac{\partial \varphi(x,t)}{\partial t} \Big| \Big) = 0 \end{gathered} \tag{3.5}$$

uniformly with respect to t. Then $U = 0$.

Proof. Introduce the notation

$$Z(r,a,\tau) \equiv \big\{ (x,t) : x \in B(0,r), t \in [0,\tau] \big\} \cup \Big\{ (x,t) : x \in B(0,r+a) \backslash B(0,r), \\ 0 \le t \le \frac{r+a-|x|}{a} \tau \Big\},$$

$$Z_1(r,a,\tau) \equiv \Big\{ (x,t) : x \in B(0,r) \backslash B(0,r-a),\ 0 \le t \le \frac{|x|+a-r}{a} \tau \Big\} \cup \\ \cup \Big\{ (x,t) : x \in B(0,r+a) \backslash B(0,r), \quad 0 \le t \le \frac{r+a-|x|}{a} \tau \Big\},$$

$$M(r,a,\tau)\equiv Z(r,a,\tau)\cap(\Omega^-\times[0,\tau]),\ \ 0<a<r,\ \ 0<\tau<t_0.$$

Choose r so that $\partial\Omega\subset B(0,r)$. From $(2.8)_0$, it follows that

$$-\frac{1}{2}\int\limits_{M(r,a,\tau)}\frac{\partial}{\partial t}\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}-\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+$$
$$+e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big)dv+$$
$$+\int\limits_{M(r,a,\tau)}\frac{\partial}{\partial x_j}\Big(\tau_j\frac{\partial u_i}{\partial t}+D_j\frac{\partial\varphi}{\partial t}\Big)dv=0. \tag{3.6}$$

Denote by $n=(n_1,n_2,n_3,n_4)$ the external with respect to $M(r,a,\tau)$ unit normal to the manifold $\partial M(r,a,\tau)$. Passing in (3.6) to surface integrals, we have

$$-\frac{1}{2}\int\limits_{\partial M(r,a,\tau)}n_4\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}-\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+2e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big)dS+$$
$$+\int\limits_{\partial M(r,a,\tau)}n_j\Big(\tau_{ij}\frac{\partial u_i}{\partial t}+D_j\frac{\partial\varphi}{\partial t}\Big)dS=0. \tag{3.7}$$

$\partial M(r,a,\tau)$ can be represented as the union $S_\Omega\cup S_0\cup S_\tau\cup S_{r,a}$, where $S_\Omega=\partial\Omega\times[0,\tau]$, $S_0=\{(x,t):x\in B(0,r+a)\cap\Omega^-,t=0\}$, $S_\tau=\{(x,t):x\in B(0,r)\cap\Omega^-,t=\tau\}$, $S_{r,a}=\{(x,t):x\in B(0,r+a)\backslash B(0,r),t=\frac{r+a-x}{a}\tau\}$.

Calculating the external normal on every surface and substituting in (3.7), we obtain

$$-I(S_\tau)+I(S_0)-\frac{ma}{\tau}I(S_{r,a})+\int\limits_{S_\Omega}n_j(y)\Big(\tau_{ij}\frac{\partial u_i}{\partial t}+D_j\frac{\partial\varphi}{\partial t}\Big)dS+$$
$$+m\int\limits_{S_{r,a}}\frac{y_j}{|y|}\Big(\tau_{ij}\frac{\partial u_i}{\partial t}+D_j\frac{\partial\varphi}{\partial t}\Big)dS=0, \tag{3.8}$$

where

$$I(S)=\frac{1}{2}\int\limits_S\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}-\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+$$
$$+2e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big)dS,\quad m=\tau(\tau^2+a^2)^{-\frac{1}{2}}.$$

It is not difficult to prove that

$$I(S_\tau)=\int\limits_{\partial B(0,r)}(n_kD_k\varphi)(y,\tau)d_yS+$$

$$+\frac{1}{2}\int\limits_{\Omega^-\cap B(0,r)}\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}+\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}\Big)(x,\tau)dx,$$

$$I(S_0)=-\frac{1}{2}\int\limits_{\partial B(0,r+a)}\varepsilon_{ik}\Big(n_k\frac{\partial\varphi}{\partial y_i}\varphi\Big)(y,0)d_yS,$$

Therefore from (3.8) we have

$$F(r,a,\tau)-\frac{1}{2}\int\limits_{\Omega^-\cap B(0,r)}\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}+$$
$$+\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}\Big)(x,\tau)dx+\int\limits_{\partial B(0,r)}(n_kD_k\varphi)(y,\tau)d_yS-$$
$$-\frac{1}{2}\int\limits_{\partial B(0,r)}\Big(\varepsilon_{ik}\frac{\partial\varphi}{\partial y_i}\varphi n_k\Big)(y,0)d_yS=0,\tag{3.9}$$

where

$$F(r,a,\tau)=$$
$$=m\int\limits_{S_{r,a}}\Big[\frac{y_j}{|y|}\Big(\tau_{ij}\frac{\partial u_i}{\partial t}+D_j\frac{\partial\varphi}{\partial t}\Big)-\frac{a}{\tau}\Big(\frac{1}{2}\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+\frac{1}{2}c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}-$$
$$-\frac{1}{2}\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big)\Big]d_yS.$$

Estimate the latter integral. Write the formula (3.6) in the domain $Z_1(r,a,\tau)$. Repeating the foregoing reasoning, we obtain

$$F(r,a,\tau)-F(r,-a,\tau)-I_0(r,a)=0,\tag{3.10}$$

where

$$S_{r,-a}=\Big\{(x,t):r-a\le|x|\le r,\ 0\le t\le\frac{|x|+a-r}{a}\tau\Big\},$$
$$I_0(r,a)=\frac{1}{2}\int\limits_{B(0,r+a)\setminus B(0,r-a)}\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}-\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}+$$
$$+2e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big)(x,0)dx=$$
$$=-\frac{1}{2}\int\limits_{\partial B(0,r+a)\cup\partial B(0,r-a)}\varepsilon_{ik}n_i\Big(\frac{\partial\varphi}{\partial x_i}\varphi\Big)(x,0)dS,$$

Therefore, by (3.5),

$$\lim_{r\to\infty}I_0(r,a)=0.\tag{3.11}$$

Introduce the notation

$$|u|_{(r,a,\tau)}=\Big\{\int\limits_{\partial Z_1(r,a,\tau)}\Big[\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\Big]dS\Big\}^{1/2},$$

$$|\varphi|_{(r,a,\tau)}=\Big\{\int\limits_{\partial Z_1(r,a,\tau)}\Big[\Big(\frac{\partial\varphi}{\partial t}\Big)^2+\frac{\partial\varphi}{\partial x_k}\cdot\frac{\partial\varphi}{\partial x_k}\Big]dS\Big\}^{1/2}.$$

The condition (3.5) yields

$$|\varphi|_{(r,a,\tau)}\le c(r,a,\tau),\quad \lim_{r\to\infty}c(r,a,\tau)=0. \tag{3.12}$$

Estimate now $|u|_{(r,a,\tau)}$. From (3.10), owing to (2.7) and $(2.10)_0$, we have

$$|u|^2_{(r,a,\tau)}\le c_1\int\limits_{S_{r,a}\cup S_{r,-a}}\Big[\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}\Big]dS=$$

$$=\int\limits_{S_{r,a}\cup S_{r,-a}}\Big[\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}-2e_{kij}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}\Big]dS-$$

$$-\frac{\tau}{ma}\int\limits_{S_0(r,a)}\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}dS+$$

$$+\frac{2\tau}{a}\int\limits_{S_{r,a}\cup S_{r,-a}}\frac{x_j}{|x|}\Big(c_{ijkl}\frac{\partial u_k}{\partial x_l}\cdot\frac{\partial u_i}{\partial t}+e_{jkl}\frac{\partial u_k}{\partial x_l}\cdot\frac{\partial\varphi}{\partial t}-\varepsilon_{jk}\frac{\partial\varphi}{\partial x_k}\cdot\frac{\partial\varphi}{\partial t}\Big)dS,$$

where

$$S_0(r,a)=\{(x,t):r-a\le|x|\le r+a,\ t=0\}\subset\partial Z_1(r,a,\tau).$$

Then for any $\delta>0$,

$$|u|^2_{(r,a,\tau)}\le c_2\Big(1+\frac{1}{\delta}+\frac{\tau}{ma}+\frac{\tau}{a}\Big)|\varphi|^2_{(r,a,\tau)}+c_2\Big(\delta+\frac{\tau}{a}\Big)|u|^2_{(r,a,\tau)}.$$

Choose δ and a such that $c_2(\delta+\frac{\tau}{a})<\frac12$ for $\tau\le t$. We have the estimate

$$|u|_{(r,a,\tau)}\le c_3(a,\tau)|\varphi|_{(r,a,\tau)}, \tag{3.13}$$

where $c_3(a,\tau)$ is uniformly bounded on $[0,t_0]$ with respect to τ.

From (3.12) and (3.13) it follows that

$$|F(r,a,\tau)|\le c_4(r,a,\tau),\quad \lim_{r\to\infty}c_4(r,a,\tau)=0. \tag{3.14}$$

Let $\rho(r,a,\tau)=r+a-\frac{\tau}{t_0}a$, $0\le\tau\le t_0$. Consider the integral

$$I(r,a,\tau)=\int\limits_{\Omega^-\cap B(0,\rho(r,a,\tau))}\Big(\rho\frac{\partial u_i}{\partial t}\cdot\frac{\partial u_i}{\partial t}+c_{ijkl}\frac{\partial u_i}{\partial x_j}\cdot\frac{\partial u_k}{\partial x_l}+\varepsilon_{ik}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_k}\Big)(x,\tau)dx.$$

Returning to (3.9) where r is replaced by $\rho(r,a,\tau)$, we get

$$I(r,a,\tau) = 2F(\rho(r,a,\tau),a,\tau) + 2 \int\limits_{\partial B(0,\rho(r,a,\tau))} (n_k D_k \varphi)(x,\tau) d_x S -$$
$$- \int\limits_{\partial B(0,\rho(r,a,\tau))} \Big(\varepsilon_{ik} \frac{\partial \varphi}{\partial y_i} \varphi n_k\Big)(x,\tau) d_x S.$$

By virtue of the above-proved estimates

$$|I(r,a,\tau)| \leq 2 \int\limits_{\partial B(0,\rho(r,a,\tau))} |(D_k\varphi)(x,\tau)| d_x S + c_5(r,a,\tau),$$
$$\lim_{r\to\infty} c_5(r,a,\tau) = 0 \tag{3.15}$$

and because of the equality

$$\int\limits_0^{t_0} \int\limits_{\partial B(0,\rho(r,a,\tau))} f(x,\tau) d_x S d\tau = \frac{t_0}{\sqrt{t_0^2+a^2}} \int\limits_{S_{r,a}} f(x,\tau) dS,$$

we have

$$\int\limits_{Z(r,a,t_0)} \Big(\rho \frac{\partial u_i}{\partial t} \cdot \frac{\partial u_i}{\partial t} + c_{ijkl} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_k}{\partial x_l} + \varepsilon_{ik} \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_k}\Big) dx dt \leq$$
$$\leq \int\limits_0^{t_0} I(r,a,\tau) d\tau \leq \frac{t_0}{\sqrt{t_0^2+a^2}} \int\limits_{S_{r,a}} |(D_k\varphi)(x,\tau)| dS + c_6(r,a,t_0) \leq$$
$$\leq c_7 \Big(|u|_{(r,a,t_0)} + |\varphi|_{(r,a,t_0)}\Big) \Big\{ \int\limits_{S_{r,a}} |\varphi(x,\tau)|^2 dS \Big\}^{1/2} + c_6(r,a,t_0).$$

Evidently,

$$\int\limits_{S_{r,a}} |\varphi(x,\tau)|^2 dS \leq c_7(a,t_0), \quad \lim_{r\to\infty} c_6(r,a,t_0) = 0.$$

Therefore, passing to limit as $r \to \infty$, we obtain

$$\int\limits_0^{t_0} \int\limits_{\Omega^-} \Big(\rho \frac{\partial u_i}{\partial t} \cdot \frac{\partial u_i}{\partial t} + c_{ijkl} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_k}{\partial x_l} + \varepsilon_{ik} \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_k}\Big) dx d\tau = 0.$$

The remainder of the proof of Theorem 3.2 is obvious. ■

4. Reduction of the Boundary Value Problems of Dynamics to the Problems of Pseudo-oscillation

To investigate dynamic problems of electroelasticity, we have to reduce them by the Laplace transform to the corresponding problems of pseudo-oscillation. We require of the data of the problem that the following conditions be fulfilled:

$$\begin{gathered} X\in c^7(\overline{\Omega}\times[0,+\infty[), \quad |\partial_t^p\partial_x^\alpha X(x,t)|\leq c(1+|x|)^{-2-q}e^{\sigma_0 t}, \\ x\in\overline{\Omega}, \quad p+|\alpha|\leq 7, \quad q>0, \quad c>0, \quad \sigma_0>0, \end{gathered} \tag{4.1}$$

$$\begin{gathered} \overset{(k)}{u}\in c^7(\overline{\Omega}), \quad |\partial^\alpha \overset{(k)}{u}(x)|\leq c(1+|x|)^{-2-q}, \quad k=0,1, \\ |\alpha|\leq 7, \quad x\in\overline{\Omega}; \end{gathered} \tag{4.2}$$

$$\begin{gathered} \|\partial_t^7 f(\cdot,t)\|_{(S,1,\beta)}\leq ce^{\sigma_0 t}, \quad |\partial_t^p f(y,t)|\leq ce^{\sigma_0 t}, \\ p=0,1,\dots,\sigma, \quad \beta>0. \end{gathered} \tag{4.3}$$

Here $X, f, \overset{(k)}{u}, k=1,2$, are the vectors appearing in (2.8)–(2.10), $\Omega=\Omega^+$ for Problems $(1)^+$ and $(2)^+$ and $\Omega=\Omega^-$ for Problems $(1)^-$ and $(2)^-$. In what follows, we assume $S=\partial\Omega\in C^{2,\gamma}$, $\gamma>\beta$.

Note that for $\Omega=\Omega^+$, by (2.2) we have

$$\int_S n_i(y)D_i(y,t)d_yS=\int_{\Omega^+}\frac{\partial D_i(x,t)}{\partial x_i}dx=0.$$

Therefore, in the case of Problem $(2)^+$, we have to add the condition

$$\int_S f_4(y,t)d_yS=0. \tag{4.4}$$

Determine now compatibility conditions for initial and boundary data. Let $\overset{(r)}{u}=(\overset{(r)}{u}_1,\overset{(r)}{u}_2,\overset{(r)}{u}_3)$ be some functions,

$$\begin{gathered} \overset{(r)}{u}\in C^2(\Omega)\cap C^1(\overline{\Omega}); \quad |\partial^\alpha\overset{(r)}{u}(x)|\leq c(1+|x|)^{-2-q}, \\ \text{for} \quad \Omega=\Omega^-, \quad q>0, \quad |\alpha|\leq 2. \end{gathered} \tag{4.5}$$

Denote by $\varphi^{(r,m)}$, $m=1,2$, the solution of the following boundary value problem:

$$\varepsilon_{ik}\frac{\partial^2}{\partial x_i\partial x_k}\varphi^{(r,m)}(x)=e_{ijk}\frac{\partial^2}{\partial x_i\partial x_k}\overset{(r)}{u}_j(x), \quad x\in\Omega; \tag{4.6}$$

$$\varphi^{(r,m)}(y)=\frac{\partial^r f_4(y,0)}{\partial t^r}, \quad y\in S, \quad \text{for} \quad m=1; \tag{4.7}$$

$$n_i(y)\varepsilon_{ik}\frac{\partial\varphi^{(r,m)}}{\partial y_k}(y) = -\frac{\partial^r f_4(y,0)}{\partial t^r} + e_{ijk}n_i(y)\frac{\partial \overset{(r)}{u}_j(y)}{\partial y_k},$$

$$y \in S, \quad \text{for} \quad m = 2, \tag{4.8}$$

$$\varphi^{(r,2)}(0) = 0, \quad \Omega = \Omega^+, \tag{4.9}$$

$$\lim_{|x|\to\infty} \varphi^{(r,m)}(x) = 0, \quad m = 1, 2, \quad \Omega = \Omega^-. \tag{4.10}$$

For $m = 1$, the function $\varphi^{(r,q)}$ is a solution of the homogeneous Dirichlet problem for the elliptic equation (4.6) (problem (4.6), (4.7) for $\Omega = \Omega^+$ and problem (4.6), (4.7), (4.10) for $\Omega = \Omega^-$). By virtue of (4.5), there exists a unique solution of this problem, and therefore the functions $\varphi^{(r,q)}$ are defined correctly.

If $m = 2$, then $\varphi^{(r,m)}$ is the solution of the Neumann problem for the equation (4.6). For $\Omega = \Omega^-$ it also satisfies (4.10). Obviously, there exists a unique suchlike solution belonging to $C^2(\Omega^-) \cap C^1(\overline{\Omega}^-)$.

Consider finally the case $m = 2$, $\Omega = \Omega^+$. Then $\varphi^{(r,q)}$ satisfies (4.6), (4.8) and (4.9). In order to this problem to be solvable, it is necessary and sufficient that the condition

$$\int_{\Omega+} e_{ijk}\frac{\partial^2 \overset{(r)}{u}_j(x)}{\partial x_i \partial x_k}dx = \int_S \Big(e_{ijk}n_i(y)\frac{\partial \overset{(r)}{u}_j(y)}{\partial y_k} - \frac{\partial^r}{\partial t^r}f_4(y,0)\Big)d_yS$$

or

$$\int_S \frac{\partial^r f_4(y,0)}{\partial t^r}d_yS = 0, \tag{4.11}$$

be fulfilled. The latter condition is satisfied due to (4.4). Then the problem (4.6), (4.8) has a solution defined to within the constant summand. If we fix its value at the point 0 by means of (4.9), then $\varphi^{(r,m)}$ is defined uniquely.

Remark. All the assertions regarding the problem (4.6)–(4.10) can be proved trivially by reducing the equation (4.6) to the Poisson equation.

Define now by induction the functions $\overset{(r)}{u} = (\overset{(r)}{u}_1, \overset{(r)}{u}_2, \overset{(r)}{u}_3)$ and $\varphi^{(r,m)}$ for Problems $(m)^\pm$, $m = 1, 2$. The functions $\overset{(r)}{u}$, $r = 0, 1$, are defined from the initial conditions (2.10), and the remaining $\overset{(r)}{u}$ from the recurrent relation

$$\overset{(r+2)}{u_i} = \frac{1}{\rho}\Big(c_{ijkl}\frac{\partial^2 \overset{(r)}{u}_k}{\partial x_j \partial x_l} + e_{kij}\frac{\partial^2\varphi^{(r,q)}}{\partial x_k \partial x_j}\Big) + \frac{\partial^r X_i(\cdot,0)}{\partial t^r}. \tag{4.12}$$

It is not difficult to note that if $U = (u,\varphi)$, $U(x,\cdot) \in C^7([0,+\infty])$, $x \in \Omega$, $\frac{\partial_t^r U(\cdot,t)}{\partial t^r} \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $t \in [0,+\infty[$, $r = 0, 1, \dots, 5$, and U is a solution of Problem $(m)^\pm$, then $\overset{(r)}{U} = (\overset{(r)}{u}, \varphi^{(r,q)})$ is the value of $\frac{\partial^r U}{\partial t^r}$ for $t = 0$.

We require that the following compatibility conditions of boundary and initial data for Problem $(q)^{\pm}$ be fulfilled:

$$\begin{gathered}
\overset{(r)}{u}_i(y) = \frac{\partial^r f_i(y,0)}{\partial t^r}, \quad \varphi^{(r,m)}(y) = \frac{\partial^r f_4(y,0)}{\partial t^r}, \\
y \in S, \quad r = 0,1,\ldots,5, \quad \text{for} \quad m = 1, \\
\tau_{ij}^{(r)}(y) n_j(y) = \frac{\partial^r f_i(y,0)}{\partial t^r}, \quad D_k^{(r)}(y) n_k(y) = \frac{\partial^r f_4(y,0)}{\partial t^r}, \\
y \in S, \quad r = 0,1,\ldots,5, \quad \text{for} \quad m = 2.
\end{gathered} \tag{4.13}$$

Here

$$\begin{aligned}
\tau_{ij}^{(r)} &= c_{ijkl} \frac{\partial \overset{(r)}{u}_k(y)}{\partial y_l} + e_{kij} \frac{\partial \varphi^{(r,q)}}{\partial y_k}, \\
D_k^{(r)}(y) &= e_{kij} \frac{\partial \overset{(r)}{u}_i}{\partial y_j} - \varepsilon_{kj} \frac{\partial \varphi^{(r,q)}(y)}{\partial y_j}.
\end{aligned}$$

If Problem $(m^{\pm})$ has a sufficiently smooth solution, then it obviously satisfies the conditions (4.13).

Denote by V the function

$$V(x,t) = \omega(t) \sum_{r=0}^{5} \frac{t^r}{r!} U(x), \tag{4.14}$$

where $w \in C^{\infty}(\mathbb{R})$, $\operatorname{supp} w \subset B(0,2)$, $w(t) = 1$ for $t \in B(0,1)$. If $\overset{(0)}{U} = (\overset{(0)}{u}, \varphi^{(0)})$, where $\overset{(0)}{U} = U - V$ and U is the solution of Problem $(m)^{\pm}$, then

$$\begin{gathered}
c_{ijkl} \frac{\partial^2 \overset{(0)}{u}_k}{\partial x_j \partial x_l} + e_{kij} \frac{\partial^2 \varphi^{(0)}}{\partial x_k \partial x_j} - \rho \frac{\partial^2 \overset{(0)}{u}_i}{\partial t^2} = X_i^{(0)}, \quad i = 1,2,3; \\
-e_{jkl} \frac{\partial^2 \overset{(0)}{u}_k}{\partial x_j \partial x_l} + \varepsilon_{ik} \frac{\partial^2 \varphi^{(0)}}{\partial x_i \partial x_k} = 0,
\end{gathered} \tag{4.15}$$

where

$$X_i^{(0)} = \rho \frac{\partial^2 V_i}{\partial t^2} - c_{ijkl} \frac{\partial^2 V_k}{\partial x_j \partial x_l} - e_{kij} \frac{\partial^2 V_4}{\partial x_k \partial x_j} - X_i, \quad i = 1,2,3.$$

Moreover,

$$\overset{(0)}{u}_i(x,0) = 0, \quad \frac{\partial \overset{(0)}{u}_i}{\partial t}(x,0) = 0, \quad \varphi^{(0)}(x,0) = 0, \quad x \in \Omega^{+}. \tag{4.16}$$

Let us prove the last equality in (4.16). It can be easily verified that $\varphi^{(0,m)}$ is a solution of the problem

$$\varepsilon_{ij}\frac{\partial^2\varphi^{(0,m)}}{\partial x_i\partial x_j}=0,$$
$$\varphi^{(0,m)}(y)=f_4(y,0),\quad y\in S,\quad \text{for}\quad m=1,$$
$$n_i(y)\varepsilon_{ij}\frac{\partial\varphi^{(0,m)}(y)}{\partial y_j}=f_4(y,0),\quad y\in S,\quad \text{for}\quad m=2,$$
$$\lim_{|x|\to\infty}\varphi^{(0,m)}(x)=0,\quad \text{for}\quad \Omega=\Omega^-,$$
$$\varphi^{(0,m)}(0)=0,\quad \Omega=\Omega^+,\quad \text{for}\quad m=2.$$

The function $\varphi(\cdot,0)$ is a solution of the same problem. Since the problem has a unique solution,

$$\varphi^{(0)}(x,0)=\varphi(x,0)-\varphi^{(0,m)}(x)=0.$$

Boundary conditions satisfied by $\overset{(0)}{U}$ are of the form

$$\overset{(0)}{u}(y,t)=f^{(0)}(y,t),\quad y\in S,\quad t\in[0,+\infty[,\quad \text{for}\quad m=1, \tag{4.17}$$
$$f^{(0)}(y,t)=f(y,t)-V(y,t),$$

$$c_{ijkl}n_j(y)\frac{\partial\overset{(0)}{u}_k(y,t)}{\partial y_l}+e_{kij}n_j(y)\frac{\partial\varphi^{(0)}(y,t)}{\partial y_k}=f_i^{(0)}(y,t),\quad i=1,2,3,$$
$$-e_{ikl}n_i(y)\frac{\partial\overset{(0)}{u}_k(y,t)}{\partial y_l}+\varepsilon_{ik}n_i(y)\frac{\partial\varphi^{(0)}(y,t)}{\partial y_k}=f_4^{(0)}(y,t), \tag{4.18}$$
$$y\in S,\quad m=2,$$
$$f_i^{(0)}(y,t)=f_i(y,t)-c_{ijkl}\frac{\partial V_k(y,t)}{\partial y_l}n_j(y)-e_{ikj}\frac{\partial V_4(y,t)}{\partial y_k}n_j(y),\quad i=1,2,3,$$
$$f_4^{(0)}(y,t)=f_4(y,t)+e_{ikj}n_i(y)\frac{\partial V_k(y,t)}{\partial y_j}-\varepsilon_{ik}n_i(y)\frac{\partial V_4(y,t)}{\partial y_k}.$$

Initial and boundary data of the problem (4.15)–(4.18) satisfy the conditions

$$\frac{\partial^r X_i^{(0)}(x,0)}{\partial t^r}=0,\quad r=0,1,2,3,\quad i=1,2,3,\quad x\in\Omega,$$
$$\frac{\partial^r f^{(0)}(y,0)}{\partial t^r}=0,\quad r=0,1,\dots,5,\quad y\in S,$$

along with the same smoothness and decrease at infinity conditions as X_i and f do (see (4.1)–(4.3)).

Let $\mathbb{C}$ be the complex plane, $\sigma_0 > 0$ and $\mathbb{C}_{\sigma_0} = \{\tau \in \mathbb{C} : \operatorname{Re}\tau > \sigma_0\}$. Apply formally to the solution $\overset{(0)}{U} = (\overset{(0)}{u}, \varphi^{(0)})$ of the problem (4.15)–(4.18) the Laplace transform on the set $\mathbb{C}_{\sigma_0}$:

$$\widetilde{u}(x,\tau) = \int_0^\infty e^{-\tau t} \overset{(0)}{u}(x,t)dt.$$

Then $\widetilde{U}$ will be the solution of the problem

$$\begin{gathered}
c_{ijkl}\frac{\partial^2 \widetilde{u}_k(x,\tau)}{\partial x_j \partial x_l} + e_{kij}\frac{\partial^2 \widetilde{\varphi}(x,\tau)}{\partial x_k \partial x_j} - \rho\tau^2 \widetilde{u}_i(x,\tau) = \widetilde{X}_i(x,\tau), \quad i = 1,2,3, \\
-e_{jkl}\frac{\partial^2 \widetilde{u}_k(x,\tau)}{\partial x_j \partial x_l} + \varepsilon_{ik}\frac{\partial^2 \widetilde{\varphi}(x,\tau)}{\partial x_i \partial x_k} = 0, \quad x \in \Omega, \quad \tau \in \mathbb{C}_{\sigma_0}, \qquad (4.19)
\end{gathered}$$

$$\widetilde{u}(y,\tau) = \widetilde{f}(y,\tau), \quad y \in S, \quad \tau \in \mathbb{C}_{\sigma_0}, \quad \text{for} \quad m = 1, \qquad (4.20)$$

$$\begin{gathered}
c_{ijkl}\frac{\partial \widetilde{u}_k(y,\tau)}{\partial y_l} n_j(y) + e_{kij}\frac{\partial \widetilde{\varphi}(y,\tau)}{\partial y_k} n_j(y) = \widetilde{f}_i(y,\tau), \quad i = 1,2,3, \\
-e_{ikl}\frac{\partial \widetilde{u}_k(y,\tau)}{\partial y_l} n_i(y) + \varepsilon_{ik}\frac{\partial \widetilde{\varphi}(y,\tau)}{\partial y_k} n_i(y) = \widetilde{f}_4(y,\tau), \quad y \in S, \qquad (4.21) \\
\widetilde{\varphi}(0,\tau) = 0, \quad \tau \in \mathbb{C}_{\sigma_0}, \quad \text{for} \quad m = 2.
\end{gathered}$$

Here $\widetilde{X}_i$, $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4)$ is the Laplace transform of the functions $X_i^{(0)}$ and $f^{(0)}$, respectively.

Note that from (4.19) and (4.20) it follows that for Problem $(2)^+$ to be solvable it is necessary that the condition

$$\int_S \widetilde{f}_4(y,\tau)d_y S = 0 \qquad (4.22)$$

be fulfilled.

In the sequel, we assume this condition to be fulfilled. We can easily verify that $\widetilde{X}_i$, $i = 1,2,3$, $\widetilde{f}$ satisfy the conditions

$$\begin{gathered}
\widetilde{X}_i(\cdot,\tau) \in C^1(\overline{\Omega}), \quad \widetilde{f}(\cdot,\tau) \in C^{1,\beta}(S), \\
|\widetilde{X}_i(x,\tau)| \le c|\tau|^{-5}, \quad \Big|\frac{\partial \widetilde{X}_i(x,\tau)}{\partial x_k}\Big| \le c|\tau|^{-5}, \quad i,k = 1,2,3, \qquad (4.23) \\
x \in \overline{\Omega}, \quad \tau \in \mathbb{C}_{\sigma_0},
\end{gathered}$$

$$|\partial^\alpha \widetilde{X}_i(x,\tau)| \le c(1+|x|)^{-2-q}, \quad |\alpha| \le 2, \quad i = 1,2,3, \quad x \in \Omega^-, \qquad (4.24)$$

$$\|\widetilde{f}(\cdot,\tau)\|_{(S,1,\gamma)} \le c|\tau|^{-7}, \quad \tau \in \mathbb{C}_{\sigma_0}, \qquad (4.25)$$

$\widetilde{X}_i(x,\cdot)$, $f_i(y,\cdot)$ are the functions analytic in $\mathbb{C}_{\sigma_0}$.

The system (4.19) is said to be the equation of pseudo-oscillation if $\operatorname{Re}\tau > 0$. If $\operatorname{Re}\tau = 0$, $\operatorname{Im}\tau \ne 0$, then (4.19) is reduced to the system of equations of harmonic oscillations of electroelasticity. If $\tau = 0$, then (4.19) is the equation of the static state of electroelastic medium.

Write the system (4.19) in terms of a matrix. To this end, we introduce the following matrix differential operator

$$A(\partial_x, \tau^2) = \|A_{ik}(\partial_x, \tau^2)\|_{4\times 4}, \tag{4.26}$$
$$A_{ik}(\partial_x, \tau^2) = c_{ijkl}\frac{\partial^2}{\partial x_j \partial x_l} - \delta_{ik}\rho\tau^2, \quad i,k = 1,2,3,$$
$$A_{i4}(\partial_x, \tau^2) = e_{kij}\frac{\partial^2}{\partial x_k \partial x_j}, \quad i = 1,2,3,$$
$$A_{4k}(\partial_x, \tau^2) = -e_{ikl}\frac{\partial^2}{\partial x_i \partial x_l}, \quad k = 1,2,3,$$
$$A_{44}(\partial_x, \tau^2) = \varepsilon_{ik}\frac{\partial^2}{\partial x_i \partial x_k}.$$

Then (4.19) takes the form

$$A(\partial_x, \tau^2)U = X, \tag{4.27}$$

where $X = (X_1, X_2, X_3, 0)$.

Our further aim is to investigate the problems (4.20), (4.21) for the system of pseudo-oscillation (4.19).

5. Fundamental Solution of the Equation of Pseudo-oscillation

Consider the operator $A(\xi, \tau^2)$ obtained from $-A(\partial_x, \tau^2)$ by the Fourier transform:

$$A(\xi, \tau^2) = \|A_{ik}(\xi, \tau^2)\|_{4\times 4}, \quad \xi = (\xi_1, \xi_2, \xi_3), \tag{5.1}$$
$$A_{ik}(\xi, \tau^2) = c_{ijkl}\xi_i\xi_l + \delta_{ik}\rho\tau^2, \quad i,k = 1,2,3,$$
$$A_{i4}(\xi, \tau^2) = e_{kij}\xi_k\xi_j, \quad i = 1,2,3,$$
$$A_{4k}(\xi, \tau^2) = -e_{ikl}\xi_i\xi_l, \quad k = 1,2,3,$$
$$A_{44}(\xi, \tau^2) = \varepsilon_{ik}\xi_i\xi_k.$$

The function $\Delta(\xi, \sigma) \equiv \det A(\xi, \sigma)$ is a third degree polynomial with respect to σ. Denote by $\sigma(\xi)$ a solution of the equation

$$\Delta(\xi, \sigma(\xi)) = 0. \tag{5.2}$$

Then the following assertion is valid.

Lemma 5.1. *Every solution $\sigma(\xi)$ of the equation* (5.2) *is a real number, and there exists a positive constant c_1 such that*

$$\forall \xi \in \mathbb{R}^3 : \sigma(\xi) < -c_1|\xi|^2. \tag{5.3}$$

Proof. The homogeneous system

$$A_{ik}(\xi,\sigma(\xi))\zeta_k = 0, \quad i = 1,2,3,4, \tag{5.4}$$

has a non-trivial solution $\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{C}^4$. Moreover, $A_{4k}(\xi,\sigma) = A_{4k}(\xi) \in \mathbb{R}$. Therefore

$$A_{4k}(\xi,\sigma(\xi))\overline{\zeta}_k = 0. \tag{5.5}$$

Multiplying (5.4) and (5.5) by $\overline{\zeta}_i$ and ζ_4, respectively, and summing up, we obtain

$$c_{ijkl}\xi_j\xi_l\overline{\zeta}_i\zeta_k + \varepsilon_{ik}\xi_i\xi_k\zeta_4\overline{\zeta}_4 + \rho\sigma(\xi)\zeta_i\overline{\zeta}_i = 0. \tag{5.6}$$

By (2.7), we have

$$\varepsilon_{ik}\xi_i\xi_k\zeta_4\overline{\zeta}_4 \geq c_0|\xi|^2|\zeta_4|^2. \tag{5.7}$$

Let $\xi_i \in \mathbb{R}$, $i = 1,2,3$. Then (s_{ij}), $s_{ij} = \frac{1}{2}(\xi_i\zeta_j + \xi_j\zeta_i)$ is a real symmetric tensor, and

$$c_{ijkl}s_{ij}s_{kl} \geq c_0 s_{ij}s_{ij}.$$

Hence

$$c_{ijkl}\xi_j\xi_l\overline{\zeta}_i\zeta_k \geq \frac{c_0}{2}(|\xi|^2|\zeta|^2 + 2(\xi_j\zeta_j)^2) \geq \frac{c_0}{2}|\xi|^2|\zeta|^2. \tag{5.8}$$

The estimate (5.8) is also valid if $\zeta_i \in \mathbb{C}$, $i = 1,2,3$, because if $\zeta_j = \zeta_j^{(1)} + i\zeta_j^{(2)}$, $\zeta_j^{(k)} \in \mathbb{R}$, $k = 1,2$, $j = 1,2,3$, then owing to (2.6),

$$c_{ijkl}\xi_j\xi_l\overline{\zeta}_i\zeta_k = c_{ijkl}\xi_j\xi_l\zeta_1^{(1)}\zeta_k^{(1)} + c_{ijkl}\xi_j\xi_l\zeta_i^{(2)}\zeta_k^{(2)} \geq \frac{c_0}{2}|\xi|^2|\zeta|^2.$$

Taking (5.7) and (5.8) into account, from (5.6) we have

$$\rho\sigma(\xi)|\zeta|^2 \leq -\frac{c_0}{2}|\xi|^2|\zeta|^2 - c_0|\xi|^2|\zeta_4|^2 \leq -c_1|\xi|^2|\zeta|^2, \quad |\zeta|^2 \neq 0,$$

whence it follows (5.3). ■

From the above proven lemma it follows that the operator $A(\partial_x,\tau^2)$ is elliptic. Therefore, there exists its fundamental solution $\phi(x,\tau^2)$ which has the form

$$\Phi(x,\tau^2) = -F^{-1}(A^{-1}(\cdot,\tau^2))(x), \tag{5.9}$$

where F^{-1}, an inverse Fourier transform, is the continuous extension of the operator

$$F^{-1}: S(\mathbb{R}^3) \to S(\mathbb{R}^3), \quad F^{-1}(f)(x) = (2\pi)^{-3}\int\limits_{\mathbb{R}^3} e^{-ix\xi}f(\xi)d\xi$$

from the space $S(\mathbb{R}^3)$ of the rapidly decreasing functions into the space $S'(\mathbb{R}^3)$ of the moderate growth distributions [16], A^{-1} is a matrix inverse to A.

To state the properties of ϕ, we have to study the matrix A^{-1}. Let us prove the following assertions.

Lemma 5.2. *For some δ, on the set $\xi \in \mathbb{R}^3 \backslash B(0,\delta)$ we can represent the components of the matrix $A^{-1}(\xi,\sigma)$ as follows:*

$$A_{ik}^{-1}(\xi,\sigma) = A_{ik}^{-1}(\xi,0) + F_{ik}(\xi,\sigma) + G_{ik}(\xi,\sigma), \quad i,k = 1,2,3,4, \tag{5.10}$$

where $F_{ik}(\cdot,\sigma)$, $G_{ik}(\cdot,\sigma) \in C^\infty(\mathbb{R}^3 \backslash B(0,\delta))$, F_{ik} is a homogeneous with respect to ξ function of the order -4 and G_{ik} admits the estimate

$$|G_{ik}(\xi,\sigma)| \le c|\xi|^{-6}. \tag{5.11}$$

Lemma 5.3. *For some δ, the following representation is valid on the set $B(0,\delta)$:*

$$A_{ik}^{-1}(\xi,\sigma) = \sum_{p=0}^{3} |\xi|^{2(p-1)} f_{ik}^{(p)}(\xi,\sigma), \tag{5.12}$$

where $f_{ik}^{(p)}$ are bounded functions, and

$$\begin{gathered} f_{ik}^{(0)}(\xi,\sigma) = 0, \quad i+k \neq 8, \\ f_{ik}^{(1)}(\xi,\sigma) = 0, \quad i,k = 1,2,3; \quad i \neq k. \end{gathered} \tag{5.13}$$

Proof of Lemma 5.2. Denote by $\Delta_{ik}(\xi,\sigma)$ the algebraic complement of the element $A_{ik}(\xi,\sigma)$. Then

$$A_{ik}^{-1}(\xi,\sigma) = \frac{\Delta_{ik}(\xi,\sigma)}{\Delta(\xi,\sigma)}, \tag{5.14}$$

where

$$\begin{gathered} \Delta(\xi,\sigma) = \sum_{j=0}^{3} a_j(\xi)\sigma^{(3-j)}, \\ \Delta_{ik}(\xi,\sigma) = \sum_{j=0}^{3} b_{ik}^{(j)}(\xi)\sigma^{3-j}. \end{gathered} \tag{5.15}$$

Obviously, a_j and b_{ik}^j are homogeneous functions:

$$a_j(t\xi) = t^{2(j+1)}a_j(\xi), \quad b_{ik}^{(j)}(t\xi) = t^{2j}b_{ik}(\xi), \quad t \in \mathbb{R}. \tag{5.16}$$

In particular,

$$
\begin{gathered}
a_0(\xi) = \rho^3 \varepsilon_{ik} \xi_i \xi_k, \quad a_3(\xi) = \Delta(\xi, 0), \\
b_{ik}^{(0)}(\xi) = 0 \quad \text{for} \quad i + k \neq 8, \\
b_{ik}^{(1)}(\xi) = 0 \quad \text{for} \quad 1 \leq i, \quad k \leq 3, \quad i \neq k, \\
b_{ik}^{(3)}(\xi) = \Delta_{ik}(\xi, 0).
\end{gathered}
\tag{5.17}
$$

Prove the inequality

$$
|\Delta(\xi, \tau^2)| \geq c |\operatorname{Re} \tau|^3 |\xi|^5. \tag{5.18}
$$

Denote the solutions of the equation $\Delta(\xi, \sigma) = 0$ by $\sigma_k(\xi)$, $k = 1, 2, 3$. Then

$$
\Delta(\xi, \sigma) = a_0(\xi)(\sigma - \sigma_1(\xi))(\sigma - \sigma_2(\xi))(\sigma - \sigma_3(\xi)). \tag{5.19}
$$

Note the following properties of σ_k, $k = 1, 2, 3$:
(1) σ_k are real negative functions, $\xi \in \mathbb{R}^3$;
(2) $\sigma_k(t\xi) = t^2 \sigma_k(\xi)$;
(3) there exist positive numbers c_1, c_2 such that

$$
-c_1 |\xi|^2 \leq \sigma_k(\xi) \leq -c_2 |\xi|^2. \tag{5.20}
$$

(The last assertion follows from Lemma 5.1.)
Let $\tau = \tau_1 + i\tau_2$. Then

$$
|\tau^2 - \sigma_k(\xi)|^2 = (\tau_1^2 - \tau_2^2 - \sigma_k(\xi))^2 + 4\tau_1^2 \tau_2^2 \geq -4\tau_1^2 \sigma_k(\xi).
$$

From (5.19), we have

$$
|\Delta(\xi, \tau^2)| \geq 8 |a_0(\xi)| |\tau_1|^3 \sqrt{-\sigma_1(\xi)\sigma_2(\xi)\sigma_3(\xi)}.
$$

But

$$
\begin{gathered}
a_0(\xi) = \rho^3 \varepsilon_{ik} \frac{\xi_i}{|\xi|} \cdot \frac{\xi_k}{|\xi|} \cdot |\xi|^2 \geq c|\xi|^2, \\
-\sigma_1(\xi)\sigma_2(\xi)\sigma_3(\xi) \geq c|\xi|^6,
\end{gathered}
$$

and hence the inequality (5.18) is valid.
Rewrite (5.14) as

$$
A_{ik}^{-1}(r, \theta, \sigma) = \frac{1}{r^2} M_{ik}\Big(\frac{1}{r^2}, \theta, \sigma\Big), \tag{5.21}
$$

where

$$
\begin{gathered}
r = |\xi|, \quad \theta = \xi/|\xi|, \\
A_{ik}^{-1}(r, \theta, \sigma) = A_{ik}^{-1}(\xi, \sigma) = \frac{\Delta_{ik}(\xi, \sigma)}{\Delta(\xi, \sigma)},
\end{gathered}
$$

$$M_{ik}(t,\theta,\sigma) = \frac{\sum\limits_{j=0}^{3} b_{ik}^{(j)}(\theta) t^{3-j}\sigma^{3-j}}{\sum\limits_{j=0}^{3} a_j(\theta) t^{3-j}\sigma^{3-j}}.$$

Denote the denominator $M_{ik}(t,\theta,\sigma)$ by $P(t)$. Then

$$P(t) = t^4\Delta(\xi,\sigma), \quad t = |\xi|^{-2}.$$

The estimate (5.18) yields

$$|P(t)| \geq c|\operatorname{Re}\tau|^3 t^{3/2} > 0, \quad t > 0.$$

Moreover,

$$P(0) = a_3(\theta) = -a_0(\theta)\sigma_1(\theta)\sigma_2(\theta)\sigma_3(\theta) \geq c > 0.$$

Consequently, M_{ik} is an analytic function with respect to t in the interval $]-\varepsilon,\infty[$ for some ε. This results in the estimate

$$\Big|\frac{\partial^\alpha M_{ik}(t,\theta,\sigma)}{\partial t^\alpha}\Big| \leq c_\alpha, \quad \alpha \geq 0, \quad |\theta| = 1, \quad t \in]-\varepsilon,+\infty[,$$
$$\operatorname{Re}\tau > 0, \quad i,k = 1,2,3,4.$$

Using the Taylor formula, expand M_{ik} in the neighborhood of $t = 0$:

$$M_{ik}(t,\theta,\sigma) = M_{ik}(0,\theta,\sigma) + t\frac{\partial M_{ik}(0,\theta,\sigma)}{\partial t} + \frac{t^2}{2}\frac{\partial^2 M(\gamma t,\theta,\sigma)}{\partial t^2},$$
$$0 < \gamma < 1.$$

Substituting in this expansion $t = |\xi|^{-2}$, we obtain all the assertions of Lemma 5.2. ∎

Proof of Lemma 5.3. We rewrite (5.14) as

$$A_{ik}^{-1}(r,\theta,\sigma) = r^{-2}N_{ik}(r^2,\theta,\sigma),$$

where

$$N_{ik}(t,\theta,\sigma) = \frac{\sum\limits_{j=0}^{3} b_{ik}^{(j)}(\theta) t^{j}\sigma^{3-j}}{\sum\limits_{j=0}^{3} a_j(\theta) t^{j}\sigma^{3-j}}.$$

Since $|\sigma| = |\tau|^2 > \sigma_0^2$ and $a_0(\theta) \geq c > 0$, the denominator $N_{ik}(t,\theta,\sigma)$ does not vanish on the set $t \in [-\delta,\delta]$ for sufficiently small δ. Hence on this set the representation

$$A_{ik}^{-1}(\xi,\sigma) = \frac{N_{ik}(0,\theta,\sigma)}{|\xi|^2} + \frac{\partial N_{ik}(0,\theta,\sigma)}{\partial t} +$$
$$+\frac{|\xi|^2}{2}\frac{\partial^2 N_{ik}(0,\theta,\sigma)}{\partial t^2} + \frac{|\xi|^4}{6}\frac{\partial^3 N_{ik}(\gamma|\xi|^2,\theta,\sigma)}{\partial t^3}, \quad 0 < \gamma < 1, \tag{5.22}$$

is valid. Moreover, $\forall t \in [-\delta, \delta]$

$$\Big|\frac{\partial^2 N_{ik}(t,\theta,\sigma)}{\partial t^\alpha}\Big| \leq c_\alpha, \quad \alpha \geq 0,$$

$$N_{ik}(0,\theta,\sigma) = \frac{b_{ik}^{(0)}(\theta)}{a_0(\theta)} = 0, \quad i+k \neq 8,$$

$$\frac{\partial N_{ik}(0,\theta,\sigma)}{\partial t} = \frac{a_0(\theta)b_{ik}^{(1)}(\theta) - a_1(\theta)b_{ik}^{(0)}(\theta)}{\sigma a_0^2(\theta)} = 0, \quad i,k=1,2,3, \quad i \neq k.$$

This directly results in Lemma 5.3. ■

Consider the fundamental solution Φ of the equation (4.27) given by (5.9). Since the operator $A(\partial_x, \tau)$ is elliptic, we have

$$\Phi(\cdot,\sigma) \in C^\infty(\mathbb{R}^3 \backslash \{0\}), \quad \sigma = \tau^2. \tag{5.23}$$

Properties of ϕ at the point $x=0$ are described by the following

Theorem 5.1. *The fundamental solution Φ of the pseudo-oscillation equation in the neighborhood of $x=0$ is represented as*

$$\Phi(x,\sigma) = \Phi^{(s)}(x) + \Phi^{(r)}(x,\sigma), \tag{5.24}$$

where $\phi^{(s)}(x) = \phi(x,0)$ is the fundamental solution of the equation of statics of electroelasticity

$$\Phi^{(s)}(x) = -F^{-1}(A^{-1}(\cdot,0))(x), \tag{5.25}$$

and $\phi^{(r)}(\cdot,\sigma)$ satisfies at $x=0$ the conditions

$$\begin{gathered} |\Phi^{(r)}(x,\sigma)| \leq c, \\ |\partial_x^\alpha \Phi^{(r)}(x,\sigma)| \leq c|\log|x||, \quad |\alpha| = 1, \\ |\partial_x^\alpha \Phi^{(r)}(x,\sigma)| \leq c|x|^{-1}, \quad |\alpha| = 2. \end{gathered} \tag{5.26}$$

Proof. Let δ be a positive number such that if $|\xi| \geq \delta$, then the representation (5.10) is valid and if ω is a finite function of the class $C^\infty(\mathbb{R}^3)$ satisfying $\omega(\xi)=1$, $|\xi| \leq \delta$, then (5.9) and (5.10) imply

$$\begin{gathered} -\partial_x^\alpha \Phi_{ik}(x,\sigma) = -\partial_x^\alpha \Phi_{ik}^{(s)}(x) + \\ +F^{-1}\Big(\omega(\xi)(i\xi)^\alpha \big(A_{ik}^{-1}(\xi,\sigma) - A_{ik}^{-1}(\xi,0)\big)\Big)(x) + \\ +F^{-1}\Big((1-\omega(\xi))(i\xi)^\alpha F_{ik}(\xi,\sigma)\Big)(x) + \\ +F^{-1}\Big((1-\omega(\xi))(i\xi)^\alpha G_{ik}(\xi,\sigma)\Big)(x). \end{gathered} \tag{5.27}$$

The function $\xi \to \omega(\xi)(i\xi)^\alpha(A_{ik}^{-1}(\xi,\sigma) - A_{ik}^{-1}(\xi,0))$ vanishes near $|\xi| = \infty$, and therefore its inverse Fourier transform belongs to $C^\infty(\mathbb{R}^3)$.

By virtue of (5.11), the distribution $(1-\omega(\xi))(i\xi)^\alpha G_{ik}(\xi,\sigma)$ is integrable in $\mathbb{R}^3$ for $|\alpha|\le 2$, and hence the function $\partial_x^\alpha F((1-\omega(\xi))(i\xi)^\alpha G_{ik}(\xi,\sigma))$ is continuous in $\mathbb{R}^3$ for $|\alpha|\le 2$.

Consider the distribution

$$\xi \to T(\xi) \equiv (i\xi)^\alpha F_{ik}(\xi,\sigma)$$

which is homogeneous of order $|\alpha|-4$ in $\mathbb{R}^3\backslash\{0\}$.

It is known that if $|\alpha|>1$, then there exists the inverse Fourier transform T which is the homogeneous distribution of the order $1-|\alpha|$.

If $|\alpha|\le 1$, then there exists a distribution $\dot{T}$ in $\mathbb{R}^3$ which coincides with T on $\mathbb{R}^3\backslash\{0\}$ and satisfies the condition (see [18], Theorem 3.2.4)

$$\forall\varphi\in C_0^\infty(\mathbb{R}^3),\quad \forall t>0:$$
$$\dot{T}(\varphi)=t^{|\alpha|-4}\dot{T}(\varphi_t)+\frac{\partial^\alpha\varphi(0)}{\alpha!}\log(t)\sum_{|\beta|=1-|\alpha|}\int\limits_{\partial B(0,1)}\xi^\beta T(\xi)d_\xi S,$$

where $\varphi_t(x)=t^3\varphi(tx)$.

Thus, if $|\alpha|\ge 1$, then the inverse Fourier transform of the distribution

$$\xi\to(1-\omega(\xi))(i\xi)^\alpha F_{ik}(\xi,\sigma)=T(\xi)-\omega(\xi)(i\xi)^\alpha F_{ik}(\xi,\sigma)$$

is the inverse of the order $1-|\alpha|$ plus a function of the class $C^\infty(\mathbb{R}^3)$, since the second summand has a compact support.

However, if $|\alpha|\le 1$, then

$$F^{-1}\Big((1-\omega)T\Big)=F^{-1}\Big((1-\omega)\dot{T}\Big)=F^{-1}(\dot{T})-F^{-1}(\omega\dot{T}),$$

$F^{-1}(\omega\dot{T})\in C^\infty(\mathbb{R}^3)$ as the inverse transform of the distribution with a compact support, and $F^{-1}(\dot{T})$ is expressed by the formula (see [19], (7.1.19))

$$F^{-1}(\dot{T})(x)=T_0(x)+\log|x|\int\limits_{\partial B(0,1)}(-ix\xi)^{1-|\alpha|}T(\xi)d_\xi S,$$

where T_0 is a homogeneous distribution of the order $1-|\alpha|$ in $\mathbb{R}^3$. ■

Remark. By (5.25), $\Phi_{ik}^{(s)}$ $i,k=1,2,3,4$, are the inverse Fourier transforms of the functions $-A_{ik}^{-1}(\xi,0)$ of the order -2, and therefore they possess the following properties:

$$\begin{gathered}\Phi_{ik}^{(s)}\in C^\infty(\mathbb{R}^3\backslash\{0\}),\\ (\partial_x^\alpha\Phi_{ik}^{(s)})(tx)=|t|^{-1}t^{-|\alpha|}(\partial_x^\alpha\Phi_{ik})(x),\quad t\in\mathbb{R},\quad |\alpha|\ge 0.\end{gathered}\tag{5.28}$$

Let us investigate properties of the fundamental solution Φ near infinity.

Theorem 5.2. *In the neighborhood of infinity, the following estimates are valid:*

$$\partial_x^\alpha \Phi_{ik}(x,\sigma) = O(|x|^{-3-|\alpha|}), \tag{5.29}$$

if either $i = k$ and $1 \le i,\ k \le 3$, or $1 \le i \le 3$ and $k = 4$, or $i = 4$ and $1 \le k \le 3$;

$$\partial_x^\alpha \Phi_{ik}(x,\sigma) = O\Big(|x|^{-5-|\alpha|}\Big), \tag{5.30}$$

if $i \ne k$ and $1 \le i,\ k \le 3$;

$$\partial_x^\alpha \Phi_{44}(x,\sigma) = O\Big(|x|^{-1-|\alpha|}\Big). \tag{5.31}$$

Proof. By Lemma 5.3 and the representation (5.22), we have

$$\Phi_{ik}(x,\sigma) = -\sum_{p=0}^{3} F^{-1}\Big(|\xi|^{2(p-1)} f_{ik}^{(p)}(\xi,\sigma)\Big)(x),$$

where the functions $|\xi|^{2(p-1)} f_{ik}^{(p)}(\xi,\sigma)$, $p = 0,1,2$, are homogeneous of the order $2(p-1)$. Hence their Fourier transforms are homogeneous functions of the order $-3-2(p-1)$.

As for the fourth summand, we have

$$\begin{aligned} & x^\gamma \partial^\beta F^{-1}\Big(|\xi|^4 f_{ik}^{(3)}(\xi,\sigma)\Big)(x) = \\ &= (-1)^{|\gamma|} i^{|\beta|+|\gamma|} F^{-1}\Big(\partial^\gamma\big(\omega(\xi)|\xi|^{|\beta|+4} f_{ik}^{(3)}(\xi,\sigma)\big)\Big)(x) + \\ &+ (-1)^{|\gamma|} i^{|\beta|+|\alpha|} F^{-1}\Big(\partial^\gamma\big((1-\omega(\xi))|\xi|^{|\beta|+4} f_{ik}^{(3)}(\xi,\sigma)\big)\Big)(x), \end{aligned}$$

where ω is a function defined in the proof of Theorem 5.1.

If $|\gamma| \le |\beta| + 6$, then the function

$$\xi \to \partial^\gamma\Big(\omega(\xi)|\xi|^{|\beta|+4} f_{ik}^{(3)}(\xi,\sigma)\Big)$$

is absolutely integrable in $\mathbb{R}^3$. Hence its inverse Fourier transform vanishes at infinity. In just the same way, if $|\gamma| \ge |\beta| + 6$, then the inverse transform of the function

$$\xi \to \partial^\gamma\Big(\big(1-\omega(\xi)\big)|\xi|^{|\beta|+4} f_{ik}^{(3)}(\xi,\sigma)\Big)$$

vanishes at infinity. Consequently,

$$\partial^\beta F^{-1}\Big(|\xi|^4 f_{ik}^{(3)}(\xi,\sigma)\Big)(x) = O\Big(|\xi|^{-|\beta|-6}\Big),$$

and the proof is complete. ∎

6. A Formula for Representation of Solutions of the Equation of Pseudo-oscillation

Let Ω be a finite domain in $\mathbb{R}^3$ with the piecewise-smooth boundary $\partial\Omega$, $U, V \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $U = (U_1, U_2, U_3, U_4)$, $V = (V_1, V_2, V_3, V_4)$. Then the following analogue of Somigliana's formula is valid:

$$\int_\Omega U(x) A(\partial_x, \tau^2) V(x) dx = \int_{\partial\Omega} U(y) \mathcal{R}(\partial_y, n) V(y) d_y S - \\ - \int_\Omega E(U, V)(x) dx, \tag{6.1}$$

where $\mathcal{R}(\partial_y, n) = \|\mathcal{R}_{ik}(\partial_y, n)\|_{4\times 4}$ is the operator of electroelastic stress with the components

$$\begin{aligned}
\mathcal{R}_{ik}(\partial_y, n) &= c_{ijkl} n_j \frac{\partial}{\partial y_l}, \quad i, k = 1, 2, 3, \\
\mathcal{R}_{i4}(\partial_y, n) &= e_{lij} n_j \frac{\partial}{\partial y_l}, \quad i = 1, 2, 3, \\
\mathcal{R}_{4k}(\partial_y, n) &= -e_{jkl} n_j \frac{\partial}{\partial y_l}, \quad k = 1, 2, 3, \\
\mathcal{R}_{44}(\partial_y, n) &= \varepsilon_{jl} n_j \frac{\partial}{\partial y_l},
\end{aligned} \tag{6.2}$$

and E is a bilinear form:

$$E(U, V) = c_{ijkl} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial v_k}{\partial x_l} + \rho\tau^2 u_i v_i + e_{kij} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial v_4}{\partial x_k} - \\ - e_{jkl} \frac{\partial u_4}{\partial x_j} \cdot \frac{\partial v_k}{\partial x_l} + \varepsilon_{ik} \frac{\partial u_4}{\partial x_i} \cdot \frac{\partial v_4}{\partial x_k}. \tag{6.3}$$

$n = n(y)$ as conventionally denotes the external unit normal to $\partial\Omega$ at the point y.

Denote by $\widetilde{A}(\partial_x, \tau^2)$ and $\widetilde{\mathcal{R}}(\partial_y, n)$ the matrices obtained from $A(\partial_x, \tau^2)$ and $\mathcal{R}(\partial_y, n)$ respectively by replacing the coefficients e_{kij} by $-e_{kij}$ (in particular, $\widetilde{A}(\partial_x, \tau^2)$ coincides with the transposed matrix $A^\top(\partial_x, \tau^2)$). Then

$$\int_\Omega \Big[U(y) \widetilde{A}(\partial_y, \tau^2) V(y) - V(y) A(\partial_y, \tau^2) U(y) \Big] dy = \\ = \int_{\partial\Omega} \Big[U(y) \widetilde{\mathcal{R}}(\partial_y, n) V(y) - V(y) \mathcal{R}(\partial_y, n) U(y) \Big] d_y S. \tag{6.4}$$

Let now U be a solution of the equation

$$A(\partial_x, \tau^2) U = F \tag*{$(6.5)_F$}$$

in the domain Ω, and V be one of the columns of the matrix of fundamental solutions $\widetilde{\Phi}$ of the operator $\widetilde{A}(\partial_x, \tau^2) = A^\top(\partial_x, \tau^2)$:

$$V_i(y) = \widetilde{\Phi}_{ik}(x-y, \tau^2) = \Phi_{ki}(x-y, \tau^2),$$

where $\Phi(x, \tau^2)$ is the fundamental matrix of the operator $A(\partial_x, \tau^2)$. Then, taking into account (5.24)–(5.26), by standard reasoning [11] we obtain from (6.4) a formula for representation of a solution of the equation $(6.5)_F$ in a finite domain:

$$\begin{aligned} \delta(x)U_k(x) = \int\limits_{\partial\Omega} \Big[& U_j(y)\widetilde{\mathcal{R}}_{ji}(\partial_y, n)\widetilde{\Phi}_{ik}(y-x, \tau^2) - \\ & -\Phi_{kj}(y-x, \tau^2)\mathcal{R}_{ji}(\partial_y, n)U_i(y)\Big] d_yS + \\ + \int\limits_{\Omega} & \Phi_{kj}(y-x, \tau^2)F_j(y)dy, \quad k = 1,2,3,4, \quad x \in \Omega, \end{aligned} \tag{6.6}$$

where

$$\delta(x) = \begin{cases} 1, & x \in \Omega, \\ 1/2, & x \in \partial\Omega, \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{6.7}$$

From (6.6) for $F = 0$ we obtain a representation of a solution of the homogeneous equation $(6.5)_0$ in a finite domain:

$$\begin{aligned} \delta(x)U_k(x) = \int\limits_{\partial\Omega} \Big[& U_j(y)\widetilde{\mathcal{R}}_{ji}(\partial_y, n)\widetilde{\Phi}_{ik}(y-x, \tau^2) - \\ & -\Phi_{kj}(y-x, \tau^2)\mathcal{R}_{ji}(\partial_y, n)U_i(y)\Big] d_yS. \end{aligned} \tag*{$(6.6)_0$}$$

Let us prove that the representation $(6.6)_0$ is also valid in the external domain $\Omega = \Omega^-$ if $U = (U_1, U_2, U_3, U_4)$ for some m satisfies the following decrease at infinity conditions

$$\begin{gathered} U_k(x) = O(|x|^m), \quad k = 1,2,3, \\ U_4(x) = o(1). \end{gathered} \tag{6.8}$$

Proof. Consider the case where $x \in \Omega^-$, and hence $\delta(x) = 1$. By the Taylor formula, we expand Φ as follows:

$$\Phi_{kj}(y-x, \tau^2) = \sum_{|\alpha| \le p} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} \partial_y^\alpha \Phi_{kj}(y, \tau^2) + \Psi_{kj}(x, y, p), \tag{6.9}$$

where

$$\Psi_{kj}(x, y, p) = \sum_{|\alpha| = p+1} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} \, \partial_y^\alpha \Phi_{kj}(y - \theta x, \tau^2), \quad 0 \le \theta \le 1. \tag{6.10}$$

We write $(6.6)_0$ in the domain $\Omega^- \cap B(0,R)$, where R is chosen in such a way that $x \in \Omega^- \cap B(0,R)$. Expressing Φ_{kj} and $\widetilde{\Phi}_{ik} = \Phi_{ki}$ by formula (6.9), we get

$$U_k(x) = \overset{(0)}{U}_k(x) + \sum_{|\alpha|\le p} \frac{(-1)^{|\alpha|} c_k^{(\alpha)}(R)}{\alpha!}\, x^\alpha + I_k(x,R,p), \tag{6.11}$$

where

$$\begin{aligned}
\overset{(0)}{U}_k(x) &= \int\limits_{\partial\Omega} \Big[U_j(y)\mathcal{R}_{ij}(\partial_y,n)\Phi_{ki}(y-x,\tau^2) - \\
&\quad -\Phi_{kj}(y-x,\tau^2)\mathcal{R}_{ji}(\partial_y,n)U_i(y)\Big] d_yS,\\
c_k^{(\alpha)}(R) &= \int\limits_{\partial B(0,R)} \Big[U_j(y)\mathcal{R}_{ij}(\partial_y,n)\partial_y^\alpha\Phi_{ki}(y,\tau^2) - \\
&\quad -\partial_y^\alpha\Phi_{kj}(y,\tau^2)\mathcal{R}_{ji}(\partial_y,n)U_i(y)\Big] d_yS,\\
I_k(x,R,p) &= \int\limits_{\partial B(0,R)} \Big[U_j(y)\mathcal{R}_{ij}(\partial_y,n)\Psi_{ki}(x,y,p) - \\
&\quad -\Psi_{kj}(x,y,p)\mathcal{R}_{ji}(\partial_y,n)U_i(y)\Big] d_yS.
\end{aligned} \tag{6.12}$$

If we write the formula $(6.6)_0$ in the domain $B(0,R_2)\backslash B(0,R_1)$, where $|x| < R_1 < R_2$, and then apply to it the operator ∂_y^α, then for $x=0$ we obtain $C_k^{(\alpha)}(R_1) = C_k^{(\alpha)}(R_2)$, i.e., $C_k^{(\alpha)}(R) = C_k^{(\alpha)}$ does not depend on R.

Let the function $w \in C^\infty(\mathbb{R}^3)$ possess the following properties: $\operatorname{supp} w \subset B(0,3)\backslash B(0,1/3)$, $w(x) = 1$. If $1/2 < |x| < 2$, then for the function $w(x,R) = w(x/R)$ the estimate

$$\partial_x^\alpha \omega(x,R) = d^{(\alpha)} R^{-|\alpha|} \tag{6.13}$$

is valid, while for the matrix $\Psi^R(x,y,p) = w(y,R)\Psi(x,y,p)$ we have

$$\Psi^{(R)}(x,y,p) = \begin{cases} 0, \quad |y| < R/3, \quad \text{or} \quad |y| > 3R,\\ \Psi(x,y,p), \quad R/2 < |y| < 2R. \end{cases}$$

Therefore (6.4) implies

$$I_k(x,R,p) = \int\limits_{B(0,R)\backslash B(0,R/4)} U_j(y)A_{ij}(\partial_y,\tau^2)\Psi_{ki}^{(R)}(x,y,p)dy.$$

Taking into consideration (6.8), (6.10) and (6.13) as well as Theorem 5.2, we obtain the following estimate for I_k:

$$|I_k(x,R,p)| \le cR^3 \sup_{\frac{R}{4}\le y\le R} \Big(|U_i(y)||A_{ij}(\partial_y,\tau^2)\Psi_{ki}^{(R)}(x,y,p)|\Big) = O\Big(R^{m+1-p}\Big).$$

Choose $p > m+1$. Then $\lim\limits_{R\to\infty} I_k(x,R,p) = 0$, and passing in (6.11) to limit, we obtain

$$U(x) = \overset{(0)}{U}(x) + V^{(m)}(x), \tag{6.14}$$

where $\overset{(0)}{U}$ is defined from (6.12) and

$$V_k^{(m)}(x) = \sum_{|\alpha|\le m+1} \frac{(-1)^{|\alpha|} c_k^{(\alpha)}}{\alpha!} x^\alpha, \quad k = 1,2,3,4.$$

Note that in (6.14) both U and $\overset{(0)}{U}$ are solutions of the equation $(6.5)_0$. Therefore $V^{(m)}$ is a polynomial solution of the equation $(6.5)_0$, where

$$\lim_{|x|\to\infty} V_4^{(m)}(x) = 0$$

because of (6.8) and Theorem 5.2. Hence $V_4^{(m)} = 0$, and

$$c_{ijkl} \frac{\partial^2 V_k^{(m)}}{\partial x_j \partial x_l} - \rho\tau^2 V_i^{(m)} = 0, \quad i = 1,2,3. \tag{6.15}$$

It is easily seen that the system (6.15) does not possess a nontrivial polynomial solution. Therefore $V^{(m)} = 0$, and (6.14) implies the validity of $(6.6)_0$ for $x \in \Omega^-$. As for the remaining cases, they trivially follow from the already proven assertion.

Thus the following theorem is valid.

Theorem 6.1. *If U is a regular solution of the equation* $(6.5)_0$ *in a finite domain $\Omega = \Omega^+$ with a piecewise-smooth boundary, then the representation* $(6.6)_0$ *is valid for it. The same representation is also valid in the external domain $\Omega = \Omega^-$ if U at infinity satisfies the conditions* (6.8).

Corollary 6.1. *If U satisfies the conditions of Theorem 6.1 in the exterior domain Ω^-, then in a neighborhood of infinity the following estimates are valid:*

$$\begin{gathered} u_i(x) = O\Big(|x|^{-3}\Big), \quad u_4(x) = O\Big(|x|^{-1}\Big), \\ \tau_{ij}(x) = O\Big(|x|^{-2}\Big), \quad D_i(x) = O\Big(|x|^{-2}\Big), \quad i,j = 1,2,3. \end{gathered} \tag{6.16}$$

The proof follows directly from (6.6) and Theorem 5.2.

As a corollary of the obtained results, let us prove the following uniqueness theorem for the problems of pseudo-oscillation.

Theorem 6.2. *Interior problems of pseudo-oscillation have the unique regular solution. An exterior problem has the unique regular solution satisfying the conditions* (6.8) *in a neighborhood of infinity.*

Proof. Let $U = (U_1, U_2, U_3, \varphi)$ be a solution of the homogeneous equation of pseudo-oscillation in the interior domain Ω^+, and let $\overline{U}$ be the complex-conjugate vector. Then, as is easily verified,

$$\int_{\Omega^+} \Big[c_{ijkl} \frac{\partial \overline{u}_i(x)}{\partial x_j} \frac{\partial u_k(x)}{\partial x_l} + \rho\tau^2 \overline{u}_i u_i + \varepsilon_{ik} \frac{\partial \overline{\varphi(x)}}{\partial x_k} \frac{\partial \varphi(x)}{\partial x_i}\Big] dx =$$
$$= \int_{\partial\Omega} [n_j(y)\tau_{ij}(y)\overline{u}_i(y) - n_i(y)\overline{D}_i(y)\varphi(y)] d_y S.$$

If U is a solution of the homogeneous boundary value problem and hence it satisfies the homogeneous boundary conditions (4.20)–(4.21), then $\overline{U}$ satisfies the same conditions, and hence

$$\int_{\Omega^+} \Big[c_{ijkl} \frac{\partial \overline{u}_i(x)}{\partial x_j} \cdot \frac{\partial u_k(x)}{\partial x_l} + \rho\tau^2 \overline{u}_i u_i +$$
$$+ \varepsilon_{ik} \frac{\partial \overline{\varphi}(x)}{\partial x_k} \cdot \frac{\partial \varphi(x)}{\partial x_i}\Big] dx = 0. \tag{6.17}$$

Due to (2.6)–(2.7), for some $c_0 > 0$ we have

$$c_{ijkl} \frac{\partial \overline{u}_i(x)}{\partial x_j} \cdot \frac{\partial u_k(x)}{\partial x_l} \geq c_0 s_{ij}(x)\overline{s}_{ij}(x),$$
$$\varepsilon_{ik} \frac{\partial \overline{\varphi}(x)}{\partial x_k} \cdot \frac{\partial \varphi(x)}{\partial x_i} \geq c_0 \frac{\partial \overline{\varphi}(x)}{\partial x_i} \cdot \frac{\partial \varphi(x)}{\partial x_i}, \quad x \in \Omega^+,$$

where $s_{ij}(x) = \frac{1}{2}\Big(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\Big)$ (see the proof of Lemma 5.1). Therefore (6.17) for $u_i\overline{u}_i \neq 0$ implies

$$\operatorname{Im} \tau^2 = 0, \quad \operatorname{Re} \tau^2 < 0$$

whence it follows that $\operatorname{Re} \tau = 0$ which contradicts the condition $\operatorname{Re} \tau > 0$ for the system of pseudo-oscillation. Now from (6.17) we have $\varphi = \text{const}$, and taking into account homogeneous conditions (4.20)–(4.21), we obtain $\varphi = 0$. ■

The theorem for the exterior problems is proved analogously if we take into account that,due to Corollary 6.2, the equality (6.17) is fulfilled in the exterior domain as well.

7. Integral Equations for the Problems of Pseudo-oscillation

To solve the problems of pseudo-oscillation, we introduce the following potentials: the simple layer potential

$$V(x, \tau, f) = \int_S \Phi(y - x, \tau^2) f(y) d_y S, \tag{7.1}$$

the double layer potential

$$W(x,\tau,f)=\int\limits_S\Big[\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-x,\tau^2)\Big]^T f(y)d_yS, \tag{7.2}$$

and the Newton potential

$$N(x,\tau,F)=\int\limits_\Omega \Phi(y-x,\tau^2)F(y)dy, \tag{7.3}$$

where f and F are vectors defined on S and Ω, respectively.

The above-mentioned potentials possess the same properties as those of the classical theory of elasticity, namely, for them the following theorems are valid.

Theorem 7.1. *Let Ω^+ be a bounded domain with a boundary $S\in C^{1,\gamma}$, $\Omega^-=\mathbb{R}^3\backslash\overline{\Omega}^+$, $f\in C^{(0,\beta)}(S)$, $0<\beta<\gamma\le 1$. Then*

1) $V(\cdot,\tau,f)\in C^{1,\beta}(\overline{\Omega}^+)\cap C^\infty(\Omega^+), V(\cdot,\tau,f)\in C^{1,\beta}(\overline{\Omega}^-)\cap C^\infty(\Omega^-)$.

2) $A(\partial_x,\tau^2)V(x,\tau,f)=0,\quad x\in\mathbb{R}^3\backslash S.$

3) $\Big[\mathcal{R}(\partial_z,n(z))V(z,\tau,f)\Big]^\pm=\mp\dfrac{1}{2}f(z)+$

$$+\int\limits_S\mathcal{R}(\partial_z,n(z))\Phi(y-z,\tau^2)f(y)d_yS,\quad z\in S. \tag{7.4}$$

4) *if $S\in C^{k+1,\gamma}$, $f\in C^{k,\beta}(S)$, $0<\beta<\gamma\le 1$, $k\ge 0$, and $V(\cdot,\tau,f)\in C^{k+1,\beta}(\overline{\Omega}^\pm)$ then*

$$\|V(\cdot,\tau,f)\|_{(\overline{\Omega}^\pm,k+1,\beta)}\le C\|f\|_{(S,k,\beta)}. \tag{7.5}$$

Here by $[M(z)]^+$ $([M(z)]^-)$ we denote the boundary values

$$\lim_{\Omega^+\ni x\to z}M(x)\quad\big(\lim_{\Omega^-\ni x\to z}M(x)\big).$$

Theorem 7.2. *If Ω^+, S, f satisfy the conditions of Theorem 7.1, then*

1) $W(\cdot,\tau,f)\in C^{0,\beta}(\overline{\Omega}^+)\cap C^\infty(\Omega^+)$, $W(\cdot,\tau,f)\in C^{0,\beta}(\overline{\Omega}^-)\cap C^\infty(\Omega^-)$.

2) $A(\partial_x,\tau^2)W(x,\tau,f)=0,\quad x\in R^3\backslash S.$

3) $\Big[W(z,\tau,f)\Big]^\pm=\pm\dfrac{1}{2}f(z)+$

$$+\int\limits_S\Big[\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-2,\tau^2)\Big]^T f(y)d_yS,\quad z\in S. \tag{7.6}$$

4) *if $S\in C^{k+1,\gamma}$, $f\in C^{k,\beta}(S)$, $0<\beta<\gamma\le 1$, $k\ge 0$, then $W(\cdot,\tau,f)\in C^{k,\beta}(\overline{\Omega}^\pm)$ and*

$$\|W(\cdot,\tau,f)\|_{(\overline{\Omega}^\pm,k,\beta)}\le C\|f\|_{(S,k,\beta)}. \tag{7.7}$$

Theorem 7.3. *If Ω^+ and S satisfy the conditions of Theorem 7.1 and $F \in C^{(0,\beta)}(\Omega^+)$, $0 < \beta < \gamma \leq 1$, then $N(\cdot, \tau, F) \in C^2(\Omega^+) \cap C^{(2,\beta)}(\overline{\Omega}_0^+)$, $\forall \overline{\Omega}_0^+ \subset \Omega^+$ and*

$$A(\partial_x, \tau^2) N(x, \tau, F) = F(x), \quad x \in \Omega^+,$$

Moreover, if $S \in C^{k,\gamma}$ and $F \in C^k(\overline{\Omega}^+)$, then $N(\cdot, \tau, F) \in C^{k+1,\gamma'}(\overline{\Omega}^+)$, $0 < \gamma' < \gamma$, $k \in N$, and

$$\|N(\cdot, \tau, F)\|_{\Omega^+, k+1, \gamma} \leq C \|F\|_{(\Omega, k, \gamma)}. \tag{7.8}$$

The proof of these theorems is based on the representation of the fundamental solution $\Phi(x, \tau^2)$ near the point $x = 0$ (formulas (5.24)–(5.26)), and in fact does not differ from the proof of analogous theorems of the classical theory of elasticity adduced in [11], [19] and [20].

The main difference between the above-considered potentials and those of the classical elasticity lays in their behavior at infinity which is caused by the fact that the matrix of fundamental solutions of the equation of pseudo-oscillation of electroelasticity unlike that of the classical elasticity is not rapidly decreasing at infinity; the degree of its decrease is determined by Theorem 5.2. Owing to this fact, the following theorems on the behavior at infinity of the potentials of electroelasticity are valid.

Theorem 7.4. *The simple layer potential $V(\cdot, \tau, f)$ and the double layer potential $W(\cdot, \tau, f)$ in a neighborhood of infinity admit the estimates*

$$\begin{aligned} &|\partial_x^\alpha V_i(x, \tau, f)| = O\big(|x|^{-3-|\alpha|}\big), \quad i = 1, 2, 3, \\ &|\partial_x^\alpha V_4(x, \tau, f)| = O\big(|x|^{-1-|al|}\big), \\ &|\partial_x^\alpha W(x, \tau, f)| = O\big(|x|^{-2-|\alpha|}\big). \end{aligned} \tag{7.9}$$

The proof follows immediately from the estimates (5.29)–(5.31). Note that these estimates can be improved if we assume that $f_4 = 0$. In this case,

$$\begin{aligned} &|\partial_x^\alpha V_4(x, \tau, f)| = O\Big(|x|^{-3-|\alpha|}\Big), \\ &|\partial_x^\alpha W(x, \tau, f)| = O\Big(|x|^{-4-|\alpha|}\Big). \end{aligned} \tag*{(7.9)$'$}$$

Let us cite now a theorem describing the behavior of the Newton potential at infinity for a density of special kind. Its proof is also based on the estimates (5.29)–(5.31).

Theorem 7.5. *If the conditions of Theorem 7.3 are fulfilled in the domain $\Omega = \Omega^-$ and $F = (F_1, F_2, F_3, F_4)$, where*

$$|F_i(x)| < C\Big(1 + |x|\Big)^{-2-\beta}, \quad x \in \Omega^-, \quad \beta > 0, \tag{7.10}$$

then there exists the Newton potential

$$N(x,\tau,F)=\int\limits_{\Omega^-}\Phi(y-x,\tau^2)F(y)dy,\quad x\in\mathbb{R}^3,$$

which satisfies the estimates

$$|\partial_x^\alpha N(x,\tau,F)|\le c(1+|x|)^{-|\alpha|-\beta'},\quad |\alpha|=0,1,\quad \beta'<\beta,$$

and all the assertions of Theorem 7.3 *are valid. Moreover, if* $F_4=0$, *then*

$$|\partial_x^\alpha N(x,\tau,F)|\le C\Big(1+|x|\Big)^{-2-|\alpha|-\beta'}.$$

From the above theorems it follows, in particular, that if the conditions (7.10) are fulfilled, then the representation (6.6) is valid in the domain Ω^-.

Properties of the potentials under consideration allow us, as in the classical theory of elasticity, to reduce the boundary value problems of electroelasticity to the corresponding singular integral equations.

First of all, note that due to Theorems 7.3 and 7.5, there exists the solution of the equation (4.27) in Ω^+ and Ω^- which possesses the necessary smoothness and the necessary rate of decrease at infinity. Therefore the boundary value problems of pseudo-oscillation for the equation (4.27) can be reduced to the corresponding boundary value problems for the homogeneous equation (6.5).

Consider first Problem $(1)_\tau^-$ for the equation $(6.5)_0$, i.e., the problem $(6.5)_0$, (4.20) in Ω^+. Its solution will be sought in terms of the double layer potential

$$U(x,\tau,\psi)=W(x,\tau,\psi),\quad \psi\in C^{0,\beta}(S).$$

Then by Theorem 7.2, for the density ψ we get the integral equation

$$\frac{1}{2}\psi(z,\tau)+ \\ +\int\limits_S\Big[\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-z,\tau^2)\Big]^T\psi(y,\tau)d_yS=f(z,\tau). \qquad (7.11)_f$$

Similarly, if a solution of Problem $(2)_\tau^-$ (the problem $(6.5)_0$, (4.21)) is sought in terms of the simple layer potential

$$U(x,\tau,\psi)=V(x,\tau,\psi),\quad \psi\in C^{0,\beta}(S),$$

then owing to the properties of the simple layer potential, for the density ψ we obtain the integral equation

$$\frac{1}{2}\psi(z,\tau)+\int\limits_S\mathcal{R}(\partial_z,n(z))\Phi(y-z,\tau^2)\psi(y,\tau)d_yS=f(z,\tau), \qquad (7.12)_f$$

and conversely, if, for example, ψ is a solution of the class $C^{(0,\beta)}(S)$ of the equation $(7.11)_f$, then the vector $W(\cdot,\tau,\psi)$ will be a solution of Problem

$(1)^+_\tau$. Similarly we establish the equivalence of the equation $(7.12)_f$ and of Problem $(2)^-_\tau$.

Introduce the notation

$$\mathcal{K}\psi(z) = \int\limits_S \mathcal{R}(\partial_z, n(z))\Phi(y-z,\tau^2)\psi(y,\tau)d_yS, \tag{7.13}$$

$$\mathcal{K}^*\psi(z) = \int\limits_S \Big[\widetilde{\mathcal{R}}(\partial_y, n(y))\widetilde{\Phi}(y-z,\tau^2)\Big]^T \psi(y,\tau)d_yS. \tag{7.14}$$

Then the equations (7.11) and (7.12) can be respectively written as follows:

$$\frac{1}{2}I\psi + \mathcal{K}^*\psi = f, \qquad (7.11)'_f$$

$$\frac{1}{2}I\psi + \mathcal{K}\psi = f. \qquad (7.12)'_f$$

Here I is an identical operator.

Consider now $\mathcal{K}$ and $\mathcal{K}^*$ as integral operators in the spaces $L_p(S)$ and $L_{p'}(S)$, respectively, where $p^{-1} + {p'}^{-1} = 1$. Then the following theorem is valid.

Theorem 7.6. *Singular integral operators generated by the left-hand sides of the equations* $(7.11)'$ *and* $(7.12)'$ *are mutually conjugate operators of normal type.*

Theorem 7.7. *If* $S \in C^{k+1,\alpha}$ *and* $f \in C^{k,\beta}(S)$, $k \geq 0$, $0 < \beta < \alpha \leq 1$, *then for the equations* $(7.11)'_f$ *and* $(7.12)'_f$ *the Fredholm's theorems are valid in the space* $C^{k,\beta}(S)$.

We do not give the proofs of these theorems which only slightly differ from those of the classical theory of elasticity given in [11], [19] and [20].

Now we pass to the investigation of the equations $(7.11)_f$ and $(7.12)_f$.

Theorem 7.8. *The equations* $(7.11)_0$ *and* $(7.12)_0$ *have only trivial solutions.*

Proof. Let $\psi \in C^{0,\beta}(S)$ be a solution of the equation $(7.11)_0$, $S \in C^{2,\gamma}$, $0 < \beta < \gamma \leq 1$. Then $\psi \in C^{1,\beta}(\partial\Omega)$ (see [22]) and if

$$U(x,\tau) = W(x,\tau,\psi),$$

then $U \in C^{1,\beta}(\overline{\Omega}^+) \cap C^{1,\beta}(\overline{\Omega}^-)$. Moreover,

$$\big[U(z,\tau)\big]^+ = \frac{1}{2}\psi(z,\tau) + \int\limits_{\partial\Omega} \Big[\widetilde{\mathcal{R}}(\partial_y, n(y))\widetilde{\Phi}(y-z,\tau^2)\Big]^T \psi(y,\tau)d_yS = 0.$$

Thus $U(\cdot,\tau)$ is a solution of the homogeneous Problem $(1)_\tau$, and hence $U(x,\tau) = 0$, $x \in \Omega^+$.

By (7.6),

$$\psi(z,\tau) = [W(z,\tau,\psi)]^+ - [W(z,\tau,\psi)]^- = -[W(z,\tau,\psi)]^-. \quad (7.15)$$

Therefore $(6.6)_0$ yields

$$V(x,\tau,g) = 0, \quad x \in \Omega^-,$$

where

$$g(y) \equiv [\mathcal{R}(\partial_y, n(y))U(y,\tau)]^-.$$

Since $g \in C^{0,\beta}(S)$, the function $V(\cdot,\tau,g)$ is continuous in $\mathbb{R}^3$. Then $V(\cdot,\tau,g)$ is a solution of the homogeneous Problem $(1)^+_\tau$, $V(x,\tau,g) = 0$, $x \in \Omega^+$, and

$$g(y) = [\mathcal{R}(\partial_z, n(z))V(z,\tau,g)]^- - [\mathcal{R}(\partial_z, n(z))V(z,\tau,g)]^+ = 0.$$

Thus we have obtained that U is a solution of the homogeneous Problem $(2)^-_\tau$. Consequently $U(x,\tau) = 0$, $x \in \Omega^-$, and by (7.15), $\psi(z,\tau) = 0$, $z \in S$. ■

Consider now the solution $\psi \in C^{0,\beta}(S)$ of the homogeneous equation $(7.12)_0$. Then $\psi \in C^{1,\beta}(S)$ and for the potential $V(x,\tau,\psi)$, we have

$$[\mathcal{R}(\partial_z, n(z))V(z,\tau,\psi)]^- = 0.$$

Hence $V(\cdot,\tau,\psi)$ is a solution of the homogeneous Problem $(2)^-_\tau$, and thus $V(x,\tau,\psi) = 0$, $x \in \Omega^-$. Then by the fact that a simple layer potential is continuous, $V(\cdot,\tau,\psi)$ will also be a solution of the homogeneous Problem $(1)^+_\tau$. Therefore $V(x,\tau,\psi) = 0$, $x \in \mathbb{R}^3$, whence $\psi(z,\tau) = 0$, $z \in S$.

From the above proven theorem, we immediately have

Theorem 7.9. *Equations* $(7.11)_f$ *and* $(7.12)_f$ *are uniquely solvable for any* $f \in C^{0,\beta}(S)$, *and if* $S \in C^{k+2,\gamma}$, $f \in C^{k,\beta}(S)$, $k \geq 0$, $0 < \beta < \gamma \leq 1$, *then the solutions of these equations belong to* $C^{k,\beta}(S)$.

Theorem 7.10. *If* $S \in C^{2,\gamma}$, $f \in C^{1,\beta}(S)$, *then Problem* $(1)^+_\tau$ *has a unique regular solution.*

Theorem 7.11. *If* $S \in C^{2,\gamma}$, $f \in C^{0,\beta}(S)$, *then Problem* $(2)^-_\tau$ *has a unique regular solution.*

Now we pass to Problems $(1)^-_\tau$ and $(2)^+_\tau$. A solution of Problem $(1)^-_\tau$ is sought in the form

$$U(x,\tau) = W(x,\tau,\psi) + \\ +\theta(x,\tau)\int\limits_S \Big[\mathcal{R}(\partial_y, n(y))\Phi(y,\tau^2)\Big]^T \psi(y,\tau)d_yS, \quad (7.16)$$

where $\psi \in C^{0,\beta}(S)$,

$$\theta = \|\theta_{ij}\|_{4\times 4}, \quad \theta_{ij}(x,\tau) = \delta_{4j}\Phi_{i4}(x,\tau^2), \quad i,j = 1,\dots,4.$$

Then U will be a solution of the equation $(6.5)_0$ in Ω^-, and due to the boundary condition (4.20),

$$-\frac{1}{2}\psi(z,\tau)+\int\limits_S\Big[\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-z,\tau^2)\Big]^T\psi(y,\tau)d_yS+$$
$$+\theta(z,\tau)\int\limits_S\Big[\mathcal{R}(\partial_y,n(y))\Phi(y,\tau^2)\Big]^T\psi(y,\tau)d_yS=f(z,\tau). \qquad (7.17)_f$$

Conversely, if ψ is a solution of the equation $(7.17)_f$ of the class $C^{(0,\beta)}(S)$, then U defined by (7.16) will obviously be a regular solution of Problem $(1)^-_\tau$.

Consider Problem $(2)^+_\tau$. We seek its solution in terms of a simple layer potential

$$U(x,\tau)=V(x,\tau,\psi), \quad \psi\in C^{0,\beta}(S).$$

Then, owing to the boundary condition (4.21), ψ will be a solution of the following system of integral equations:

$$-\frac{1}{2}\psi(z,\tau)+\int\limits_S\mathcal{R}(\partial_z,n(z))\Phi(y-z,\tau^2)\psi(y,\tau)d_yS=f(z,\tau), \qquad (7.18)_f$$

$$\int\limits_S\Phi_{4j}(y,\tau^2)\psi_j(y,\tau)d_yS=0. \qquad (7.19)$$

By (5.9), we have $\Phi_{4j}(y,\tau^2)=-\Phi_{j4}(y,\tau^2)$. Therefore ψ is a solution of the equation

$$-\frac{1}{2}\psi(z,\tau)+\int\limits_S\mathcal{R}(\partial_z,n(z))\Phi(y-z,\tau^2)\psi(y,\tau)d_yS+$$
$$+\mathcal{R}(\partial_z,n(z))\Phi(z,\tau^2)\int\limits_S\theta^T(y,\tau)\psi(y,\tau)d_yS=f(z,\tau). \qquad (7.20)_f$$

Let us now prove that if $\psi\in C^{0,\beta}(S)$ is a solution of the equation $(7.20)_f$ and the condition (4.22) is fulfilled, i.e.,

$$\int\limits_S f_4(y,\tau)d_yS=0, \qquad (7.21)$$

then a simple layer potential $V(\cdot,\tau,\psi)$ will be a regular solution of Problem $(2)^+_\tau$.

Indeed, from $(6.6)_0$ we easily have

$$\begin{aligned}\int\limits_S \mathcal{R}_{ij}(\partial_y, n(y))\Phi_{jk}(y,\tau^2)d_yS &= \delta_{ik},\\ \int\limits_S \mathcal{R}_{ij}(\partial_y, n(y))\Phi_{jk}(y-z,\tau^2)d_yS &= \frac{1}{2}\delta_{ik}, \quad z\in S.\end{aligned} \tag{7.22}$$

We integrate $(7.20)_f$ on S. Taking into account (7.21) and (7.22), we get

$$\int\limits_S \theta^T(y,\tau)\psi(y,\tau)d_yS = \int\limits_S \Phi_{4j}(y,\tau^2)\psi_j(y,\tau)d_yS = 0.$$

Therefore ψ is a solution of the system $(7.18)_0$, (7.19). Then $U(\cdot,\tau,\psi)$ will satisfy the boundary condition (4.21), and due to (7.19),

$$U_4(0,\tau) = 0.$$

Consider the equations $(7.17)_f$ and $(7.20)_f$. They are mutually conjugate and their left-hand sides differ from those of the equations

$$\begin{aligned}-\frac{1}{2}I\psi + \mathcal{K}^*\psi &= f,\\ -\frac{1}{2}I\psi + \mathcal{K}\psi &= f\end{aligned}$$

only by completely continuous summands. Therefore for these equations one can prove the validity of Theorems 7.6 and 7.7.

Theorem 7.12. *The equations* $(7.17)_0$ *and* $(7.20)_0$ *have only trivial solutions.*

Proof. Let $\psi \in C^{0,\beta}(S)$ be a solution of the equation $(7.17)_0$. Then $\psi \in C^{1,\beta}(S)$, and U defined by (7.16) will be a solution of the homogeneous Problem $(1)^-_\tau$.

Hence $U(x,\tau) = 0$, $x \in \Omega^-$, and

$$W_i(x,\tau,\psi) = a\Phi_{i4}(x,\tau), \quad i = 1,2,3,4, \tag{7.23}$$

where

$$a = -\int\limits_S \Big[\mathcal{R}(\partial_y, n(y))\Phi(y,\tau^2)\Big]_{j4}\psi_j(y,\tau)d_yS.$$

Passing in (7.23) to limit as $|x| \to \infty$ for $i = 4$, and taking into account Theorem 5.2, we obtain $a = 0$. Thus

$$\begin{aligned}W(x,\tau,\psi) &= 0, \quad x\in\Omega^-,\\ \psi(z,\tau) &= [W(z,\tau,\psi)]^+.\end{aligned}$$

Now from $(6.6)_0$ we obtain $V(x,\tau,g) = 0$, $x \in \Omega^+$, where

$$g(y) = [\mathcal{R}(\partial_y, n(y))W(y,\tau,\psi)]^+.$$

Due to the continuity of a simple layer potential, $V(\cdot,\tau,g)$ is a solution of the homogeneous Problem $(1)^-_\tau$, and $V(x,\tau,g)=0$, $x\in\Omega^-$. Then $g(y)=0$, $y\in S$.

Thus $W(\cdot,\tau,\psi)$ is a solution of the equation $(6.5)_0$ in Ω^+ satisfying the homogeneous boundary condition of Problem $(2)^+_\tau$. As it follows from the proof of Theorem 6.2, it holds $W_i(x,\tau,\psi)=C\delta_{i4}$, $i=1,2,3,4$, $x\in\Omega^+$. Then because of (7.24), we have $\psi_i(z,\tau)=C\delta_{i4}$, $i=1,2,3,4$, $z\in S$. Substituting this value into the expression for a and taking into consideration (7.22), we obtain $c=0$.

If $\psi\in C^{0,\beta}(S)$ is a solution of the equation $(7.20)_0$, then $\psi\in C^{1,\beta}(S)$, and due to the fact that the equations $(7.18)_0$, (7.19) and $(7.20)_0$ are equivalent, it follows that

$$U(x,\tau)=V(x,\tau,\psi)$$

is a solution of the homogeneous Problem $(2)^+_\tau$. Therefore

$$V(x,\tau,\psi)=0,\quad x\in\Omega^+.$$

From this, by using the fact that the potential is continuous, we come to the conclusion that $V(\cdot,\tau,\psi)$ is a solution of the homogeneous Problem $(1)^-_\tau$. Hence $V(x,\tau,\psi)=0$, $x\in\mathbb{R}^3$, and $\psi=0$. ■

From the above proven theorem there follow

Theorem 7.13. *The equations* $(7.7)_f$ *and* $(7.20)_f$ *are uniquely solvable for any* $f\in C^{0,\beta}(S)$, *and if* $S\in C^{k+2,\gamma}$ *and* $f\in C^{k,\beta}(S)$, $k\geq 0$, $0<\beta<\gamma\leq 1$, *then the solutions belong to* $C^{k,\beta}(S)$.

Theorem 7.14. *If* $S\in C^{2,\gamma}$ *and* $f\in C^{1,\beta}(S)$, *then Problem* $(1)^-_\tau$ *has a unique regular solution.*

Theorem 7.15. *If* $S\in C^{1,\gamma}$ *and* $f\in C^{0,\beta}(S)$, *then Problem* $(2)^+_\tau$ *has a unique regular solution if and only if the condition* (7.21) *is fulfilled.*

8. Green Tensors for the Problems of Pseudo-oscillation

Green tensor of Problem $(1)^\pm_\tau$ is said to be the matrix

$$G(x,y,(1)^\pm_\tau)=\Phi(x-y,\tau^2)-g(x,y,(1)^\pm_\tau),$$

where g is the solution of the problem

$$\begin{aligned} &A(\partial_x,\tau^2)g(x,y,(1)^\pm_\tau)=0,\quad x\in\Omega^\pm,\\ &\Big[g(z,y,(1)^\pm_\tau)\Big]^\pm=\Phi(z-y,\tau^2),\quad z\in S,\quad y\in\Omega^\pm, \end{aligned}\tag{8.1}$$

satisfying in the case of Problem $(1)^-_\tau$ the supplementary condition

$$\lim_{x\to\infty} g(x,y,(1)^-_\tau)=0,\quad y\in\Omega^-,\tag{8.2}$$

at infinity.

Green tensor of Problem $(2)^{\pm}_{\tau}$ is said to be the matrix

$$G(x,y,(2)^{\pm}_{\tau}) = \Phi(x-y,\tau^2) - g(x,y,(2)^{\pm}_{\tau}),$$

where $g(\cdot,y,(2)^{-}_{\tau})$ is the solution of the problem

$$\begin{gathered} A(\partial_x,\tau^2)g(x,y,(2)^{-}_{\tau}) = 0, \quad x \in \Omega^- \\ \Big[\mathcal{R}(\partial_z,n(z))g(z,y,(2)^{-}_{\tau})\Big]^- = \Big[\mathcal{R}(\partial_z,n(z))\Phi(z-y,\tau^2)\Big]^-, \\ z \in S, \quad y \in \Omega^-, \\ \lim_{x\to\infty} g(x,y,(2)^{-}_{\tau}) = 0, \end{gathered} \tag{8.3}$$

while $g(\cdot,y,(2)^{+}_{\tau})$ is the solution of the problem

$$A(\partial_x,\tau^2)g(x,y,(2)^{+}_{\tau}) = 0, \quad x,y \in \Omega^+, \tag{8.4}$$

$$\Big[\mathcal{R}(\partial_z,n(z))g(z,y,(2)^{+}_{\tau})\Big]^+ = \Big[\mathcal{R}(\partial_z,n(z))\Phi(z-y,\tau^2)\Big]^+ - D, \tag{8.5}$$

$$z \in S, \quad y \in \Omega^+,$$

$$\int\limits_S g_{4k}(y,x,(2)^{+}_{\tau})d_yS = \int\limits_S \Phi_{4k}(y-x,\tau^2)d_yS, \tag{8.6}$$

$$x \in \Omega^+, \quad k = 1,2,3,4.$$

Here $D = \|D_{ij}\|_{4\times 4}$ and $D_{ij} = (\mathrm{mes}(S))^{-1}\delta_{i4}\delta_{j4}$.

Theorem 8.1. *If $S \in C^{2,\gamma}$, then there exist uniquely defined Green tensors of all problems under consideration.*

Proof. This fact for $G(x,y,(1)^{\pm}_{\tau})$ and $G(x,y,(2)^{-}_{\tau})$ follows directly from the previous results. Let us consider in more detail the case of the tensor $G(x,y,(2)^{+}_{\tau})$.

Since because of (7.22) the right-hand side of (8.5) satisfies the condition (7.21), there exist regular solutions of the problems

$$\begin{gathered} A(\partial_x,\tau^2)g^{(k)}(x,y) = 0, \quad x,y \in \Omega^+, \\ \Big[\mathcal{R}(\partial_z,n(z))g^{(k)}(z,y)\Big]^+ = \Big[\mathcal{R}(\partial_z,n(z))\Phi^{(k)}(z-y,\tau^2)\Big]^+ - D^{(k)}, \\ z \in S, \quad y \in \Omega^+, \quad k = 1,2,3,4, \\ \Phi^{(k)} = (\Phi_{1k},\Phi_{2k},\Phi_{3k},\Phi_{4k}), \quad D^{(k)} = (D_{1k},D_{2k},D_{3k},D_{4k}). \end{gathered}$$

They are defined to within a constant summand of the form $h^{(k)} = (0,0,0,C_k)$, where C_k are arbitrary numbers. We choose them in such a way that to fulfil the condition (8.6):

$$\int\limits_S g_4^{(k)}(y,x)d_yS = \int\limits_S \Phi_{4k}(y-x,\tau^2)d_yS.$$

Then $g^{(k)}$ is defined uniquely and $G(x,y,(2)^+_\tau) = \Phi(x-y,\tau^2) - \|g_i^{(j)}(y,x)\|_{4\times 4}$ is the uniquely defined Green tensor of Problem $(2)^+_\tau$. ■

Denote by $(\widetilde{K})^\pm_\tau$, $K = 1,2$, the problem which is obtained by substituting in Problem $(K)^\pm_\tau$ the operators A and $\mathcal{R}$ respectively by $\widetilde{A}$ and $\widetilde{\mathcal{R}}$ (recall that this substitution is equivalent to that of the coefficients e_{ijk} by $-e_{ijk}$). It is obvious that all the obtained until now results remain valid for Problems $(\widetilde{K})^\pm_\tau$.

Green tensors possess the following property of symmetry:

$$G(x,y,(\widetilde{K})^\pm_\tau) = G^T(y,x,(K)^\pm_\tau). \tag{8.7}$$

The proof follows from (6.4), $(6.6)_0$ and in the case of Problem $(2)^+_\tau$ also from condition (8.6).

Let us quote the formulas for representation of solutions of Problems $(K)^\pm_\tau$ in terms of the corresponding Green tensors.

Denote by $(K)^+_{\tau,f,F}$, $K = 1,2$, the boundary value problem for the nonhomogeneous equation $(6.5)_F$ with the nonhomogeneous boundary conditions (4.20), (4.21) and the right-hand side f.

Theorem 8.2. *The solution U of Problem $(1)^+_{\tau,f,F}$ is given in the form*

$$U(x,\tau) = \int\limits_S \left[\widetilde{\mathcal{R}}(\partial_y, n(y)) G^T(x,y,(1)^+_\tau)\right]^T f(y,\tau) d_y S + \\ + \int\limits_{\Omega+} G(x,y,(1)^+_\tau) F(y,\tau) dy. \tag{8.8}$$

If the conditions (7.10) are fulfilled, then the solution of Problem $(1)^-_{\tau,f,F}$ can be presented in the form

$$U(x,\tau) = -\int\limits_S \left[\widetilde{\mathcal{R}}(\partial_y, n(y)) G^T(x,y,(1)^-_\tau)\right]^T f(y,\tau) d_y S + \\ + \int\limits_{\Omega^-} G(x,y,(1)^-_\tau) F(y,\tau) dy. \tag{8.9}$$

The solution of Problem $(2)^+_{\tau,f,F}$ is given in the form $U = (U_1, U_2, U_3, U_4)$,

$$U_i(x,\tau) = -\int\limits_S \left[G_{ij}(x,y,(2)^+_\tau) - \delta_{i4} G_{4j}(0,y,(2)^+_\tau)\right] f_i(y,\tau) d_y S + \\ + \int\limits_{\Omega_+} \left[G_{ij}(x,y,(2)^+_\tau) - \delta_{i4} G_{4j}(0,y,(2)^+_\tau)\right] F_j(y,\tau) dy. \tag{8.10}$$

If the conditions (7.10) *are fulfilled, then the solution of Problem* $(2)^-_{\tau,f,F}$ *can be given in the form*

$$U(x,\tau) = \int\limits_S G(x,y,(2)^-_\tau) f(y,\tau) d_y S +$$
$$+ \int\limits_{\Omega^-} G(x,y,(2)^-_\tau) F(y,\tau) dy. \tag{8.11}$$

Proof. We will consider Problem $(2)^+_{\tau,f,F}$ (the remaining cases are considered analogously).

By virtue of (6.4), for $i=1,2,3,4$ we have

$$\int\limits_{\Omega^+} g_{ij}(x,y,(2)^+_\tau) F_j(y,\tau) dy = \int\limits_S \Big[g_{ij}(x,y,(2)^+_\tau) f_j(y,\tau) -$$
$$- U_j(y,\tau) \mathcal{R}_{kj}(\partial_y, n(y)) \Phi_{ik}(y-x,\tau^2) + (\mathrm{mes}(S))^{-1} \delta_{i4} U_4(y,\tau) \Big] d_y S.$$

Therefore (6.6) implies

$$U_i(x,\tau) = -\int\limits_S G_{ij}(x,y,(2)^+_\tau) f_j(y,\tau) d_y S +$$
$$+ \frac{\delta_{i4}}{\mathrm{mes}(S)} \int\limits_S U_4(y,\tau) d_y S + \int\limits_{\Omega^+} G_{ij}(x,y,(2)^+_\tau) F_j(y,\tau) dy.$$

Taking into account that in this equality $U_4(0)=0$, we obtain (8.10). ■

In the sequel we will need some estimates of Green tensor. They are collected in the following

Theorem 8.3. *Let* $S \in C^{1,\gamma}$, $0<\gamma\le 1$. *Then for* $|\alpha| = 0,1$

$$\begin{gathered}
|\partial_x^\alpha G_{ij}(x,y,(K)^+_\tau)| \le c|x-y|^{-1-|\alpha|}, \quad x,y \in \Omega^+, \\
|\partial_x^\alpha G_{ij}(x,y,(K)^-_\tau)| \le c|x-y|^{-1-|\alpha|}, \quad x,y \in \Omega^-, \\
K=1,2, \quad i,j=1,2,3,4.
\end{gathered} \tag{8.12}$$

Moreover, near the infinity the following estimates are valid:

$$\begin{gathered}
|\partial_x^\alpha g_{ij}(x,y,(K)^-_\tau)| \le c(1+|x|)^{-3-|\alpha|}(1+|y|)^{-3}, \\
|\partial_x^\alpha g_{i4}(x,y,(K)^-_\tau)| \le c(1+|x|)^{-3-|\alpha|}(1+|y|)^{-1}, \\
|\partial_x^\alpha g_{4j}(x,y,(K)^-_\tau)| \le c(1+|x|)^{-1-|\alpha|}(1+|y|)^{-3}, \\
|\partial_x^\alpha g_{44}(x,y,(K)^-_\tau)| \le c(1+|x|)^{-1-|\alpha|}(1+|y|)^{-1}, \\
K=1,2, \quad i,j=1,2,3.
\end{gathered} \tag{8.13}$$

Proof. Applying Theorem 5.1, we obtain the estimates (8.12) exactly in the same way as those of the classical theory of elasticity [11], [15]. We dwell on the proof of (8.13). As an example, consider the tensor

$$G(x,y,(1)_\tau^-) = \Phi(x-y,\tau^2) - g(x,y,(1)_\tau^-),$$

where g satisfies the conditions (8.1) and (8.2). Therefore its columns $g^{(i)} = (g_{1i}, g_{2i}, g_{3i}, g_{4i})$ are represented as

$$g_j^i(x,y,(1)_\tau^-) = \int\limits_S \mathcal{R}_{kl}(\partial_z, n(z))\Phi_{lj}(z-x,\tau^2)\psi_k^{(i)}(z,y,\tau)d_zS +$$

$$+\Phi_{j4}(x)\int\limits_S \mathcal{R}_{kl}(\partial_z, n(z))\Phi_{l4}(z)\psi_k^{(i)}(z,y,\tau)d_zS, \quad i,j=1,2,3,4, \tag{8.14}$$

where $\psi^{(i)} = (\psi_1^{(i)}, \psi_2^{(i)}, \psi_3^{(i)}, \psi_4^i)$ is a solution of the singular integral equation

$$-\frac{1}{2}\psi^{(i)}(z,y,\tau) + \int\limits_S \Big[\widetilde{\mathcal{R}}(\partial_\eta, n(\eta))\widetilde{\Phi}(\eta - z,\tau^2)\Big]^T \psi^{(i)}(\eta,y,\tau)d_\eta S +$$

$$+\theta(z,\tau)\int\limits_S \Big[\mathcal{R}(\partial_\eta, n(\eta))\Phi(\eta,\tau^2)\Big]^T \psi^{(i)}(\eta,y,\tau)d_\eta S =$$

$$= \Phi^{(i)}(z-y,\tau^2) \tag{8.15}$$

with

$$\Phi^{(i)} = (\Phi_{1i}, \Phi_{2i}, \Phi_{3i}, \Phi_{4i}).$$

As is proved, the operator defined by the left-hand side of (8.15) is invertible. Therefore [11], [21]

$$\|\psi^{(i)}(\cdot,y,\tau)\|_{(S,m,\beta)} \le c\|\Phi^{(i)}(\cdot - y,\tau^2)\|_{(S,m,\beta)}. \tag{8.16}$$

Taking into consideration Theorem 5.2, we obtain the required estimates for $\psi^{(i)}$, and due to (8.14), for G. ■

9. Estimates of Solutions of Boundary Value Problems of Pseudo-oscillation. Proof of the Existence of Solutions of Dynamic Problems

This section is devoted to the investigation of properties of solutions of boundary value problems of pseudo-oscillation. Estimates necessary for the proof of the existence of solutions of the corresponding dynamic problems will also be obtained therein.

Consider first the interior problems, namely Problem $(2)_\tau^+$:

$$A(\partial_x,\tau^2)U(x,\tau) = X(x,\tau), \quad x \in \Omega^+, \tag{9.1}$$

$$\Big[\mathcal{R}(\partial_y, n(y))U(y,\tau)\Big]^+ = f(y,\tau), \quad y \in S, \tag{9.2}$$

$$U_4(0,\tau) = 0, \tag{9.3}$$

where $X = (X_1, X_2, X_3, 0)$, $f = (f_1, f_2, f_3, f_4)$ satisfy the conditions (4.23)–(4.25). Moreover, the condition

$$\int_S f_4(y,\tau) d_y S = 0 \tag{9.4}$$

which is necessary and sufficient for the solvability of Problem $(2)^+_\tau$ is supposed to be fulfilled. In this case, Problem $(2)^+_\tau$ has a unique regular solution. Represent it as follows: $U = U^{(1)} + U^{(2)}$, where $U^{(1)}$ is the solution of the boundary value problem

$$A(\partial_x, \sigma_1^2) U^{(1)}(x,\tau) = 0, \quad x \in \Omega^+, \tag{9.5}$$

$$\Big[\mathcal{R}(\partial_y, n(y)) U^{(1)}(y,\tau)\Big]^+ = f(y,\tau), \quad y \in S, \tag{9.6}$$

$$U_4^{(1)}(0,\tau) = 0, \tag{9.7}$$

with $0 < \sigma_1 < \sigma_0$. By virtue of (9.4), this problem has a unique regular solution represented in terms of (8.10):

$$U_i^{(1)}(x,\tau) =$$
$$= -\int_S \Big[G_{ij}(x,y,(2)^+_{\sigma_1}) - \delta_{i4} G_{4j}(0,y,(2)^+_{\sigma_1})\Big] f_j(y,\tau) d_y S. \tag{9.8}$$

$U^{(2)}$ is obviously the solution of the problem

$$A(\partial_x, \sigma_1^2) U^{(2)}(x,\tau) = \rho(\tau^2 - \sigma_1^2) E U^{(2)}(x,\tau) +$$
$$+ \rho(\tau^2 - \sigma_1^2) E U^{(1)}(x,\tau) + X(x,\tau), \quad x \in \Omega^+, \tag{9.9}$$

$$\Big[\mathcal{R}(\partial_y, n(y)) U^{(2)}(x,\tau)\Big]^+ = 0, \quad y \in S, \tag{9.10}$$

$$U_4^{(2)}(0,\tau) = 0. \tag{9.11}$$

Here $E = \|E_{ij}\|_{4\times 4}$, $E_{ij} = (1 - \delta_{i4})\delta_{ij}$.

Taking into consideration (8.10), the problem (9.9)–(9.11) is reduced to the following system of integral equations with a weak singular kernel:

$$U_i^{(2)}(x,\tau) - \rho(\tau^2 - \sigma_1^2) \sum_{j=1}^{3} \int_{\Omega^+} G_{ij}(x,y,(2)^+_{\sigma_1}) U_j^{(2)}(y,\tau) dy =$$

$$= \sum_{j=1}^{3} \int_{\Omega^+} G_{ij}(x,y,(2)^+_{\sigma_1}) \Big[\rho(\tau^2 - \sigma_1^2) U_j^{(1)}(y,\tau) + X_j(y,\tau)\Big] dy, \tag{9.12}$$

$$i = 1, 2, 3,$$

$$U_4^{(2)}(x,\tau)=\sum_{j=1}^{3}\int_{\Omega+}\Big[G_{4j}(x,y,(2)_{\sigma_1}^{+})-G_{4j}(0,y,(2)_{\sigma_1}^{+})\Big]\times$$
$$\times\Big[\rho(\tau^2-\sigma_1^2)U_j^{(2)}(y,\tau)+\rho(\tau^2-\sigma_1^2)U_j^{(1)}(y,\tau)+X_j(y,\tau)\Big]dy, \quad (9.13)$$

where $U_i^{(2)}$, $i=1,2,3$ and $U_4^{(2)}$ are defined from (9.12) and (9.13), respectively.

By (9.8), the function $U^{(1)}(x,\cdot)$ is analytic in the $\mathbb{C}_{\sigma_0}$.

Consider $U^{(2)}$. Let us prove that the homogeneous system

$$U_i^{(2)}(x,\tau)-\rho(\tau^2-\sigma_1^2)\sum_{j=1}^{3}\int_{\Omega+}G_{ij}(x,y,(2)_{\sigma_1}^{+})U_j^{(2)}(y,\tau)dy=0, \quad (9.12)_0$$
$$i=1,2,3,$$

has only the trivial solution. Let $V=(V_1,V_2,V_3,V_4)$, where V_i, $i=1,2,3$, is a solution of $(9.12)_0$, and

$$V_4(x,\tau)=\rho(\tau^2-\sigma_1^2)\sum_{j=1}^{3}\int_{\Omega+}\Big[G_{4j}(x,y,(2)_{\sigma_1}^{+})-$$
$$-G_{4j}(0,y,(2)_{\sigma_1}^{+})\Big]U_j(y,\tau)dy. \quad (9.14)$$

Then, as is easily verified,

$$A(\partial_x,\tau)V(x,\tau)=0, \quad x\in\Omega^+,$$

and by (8.10),

$$V_i(x,\tau)=-\int_S G_{ij}(x,y,(2)_{\sigma_1}^{+})h_j(y,\tau)d_yS+$$
$$+\rho(\tau^2-\sigma_1^2)\sum_{j=1}^{3}\int_{\Omega}G_{ij}(x,y,(2)_{\sigma_1}^{+})V_j(y,\tau)dy,$$
$$V_4(x,\tau)=-\int_S\Big[G_{4j}(x,y,(2)_{\sigma_1}^{+})-G_{4j}(0,y,(2)_{\sigma_1}^{+})\Big]h_j(y,\tau)d_yS+$$
$$+\rho(\tau^2-\sigma_1^2)\sum_{j=1}^{3}\int_{\Omega+}\Big[G_{4j}(x,y,(2)_{\sigma_1}^{+})-G_{4j}(0,y,(2)_{\sigma_1}^{+})\Big]V_j(y,\tau)dy,$$

where $h(y,\tau)=[\mathcal{R}(\partial_y,n(y))V(y,\tau)]^+$.

Comparing these equalities with $(9.12)_0$ and (9.14), we arrive at

$$\begin{gathered}\int\limits_S G_{ij}(x,y,(2)^+_{\sigma_1})h_j(y,\tau)d_yS=0,\\ \int\limits_S \Big[G_{4j}(x,y,(2)^+_{\sigma_1})-G_{4j}(0,y,(2)^+_{\sigma_1})\Big]h_j(y,\tau)d_yS=0.\end{gathered} \tag{9.15}$$

From (9.15) it follows that the problem

$$\begin{gathered}A(\partial_x,\sigma_1)U(x)=0,\quad x\in\Omega^+;\quad [\mathcal{R}(\partial_y,n(y))U(y)]^+=h(y,\tau),\quad y\in S,\\ U_4(0)=0\end{gathered}$$

has only the trivial solution, and hence $h=0$. Then V is a solution of the homogeneous Problem $(2)^+_\tau$. Therefore $V=0$.

By Fredholm's theorem, the inhomogeneous equation (9.12) is uniquely solvable and its solution is given in terms of Fredholm's formula. The resolvent and the right-hand side of this expression are analytic functions of τ in the half-plane $\mathbb{C}_{\sigma_0}$. Therefore $U^{(2)}(x,\cdot)$, and hence $U(x,\cdot)$ will also be analytic in $\mathbb{C}_{\sigma_0}$.

Let us now pass to asymptotic with respect to τ estimates of the solution. For $U^{(1)}$ the representation

$$U(x,\tau)=V(x,\sigma_1,\psi),$$

is valid, where ψ is the solution of the equation $(7.20)_f$

$$\begin{gathered}-\frac{1}{2}\psi(z,\tau)+\int\limits_S \mathcal{R}(\partial_z,n(z))\Phi(y-z,\sigma_1^2)\psi(y,\tau)d_yS+\\ +\mathcal{R}(\partial_z,n(z))\Phi(z,\sigma_1^2)\int\limits_S \theta^T(y,\sigma_1)\psi(y,\tau)d_yS=f(z,\tau)\end{gathered}$$

which is uniquely solvable. Therefore by virtue of (4.25) and (7.5),

$$\begin{gathered}\|\psi(\cdot,\tau)\|_{(S,0,\beta)}\le c\|f(\cdot,\tau)\|_{(S,0,\beta)}\le c|\tau|^{-7},\\ \|U^{(1)}(\cdot,\tau)\|_{(\Omega^+,k,\beta)}\le c\|\psi(\cdot,\tau)\|_{(S,0,\beta)}\le c|\tau|^{-7},\quad k=0,1,\\ \|U^{(1)}(\cdot,\tau)\|_{(\Omega_0,2,\beta)}\le c\|\psi(\cdot,\tau)\|_{(S,0,\beta)}\le c|\tau|^{-7},\quad \overline{\Omega}_0\subset\Omega^+.\end{gathered} \tag{9.16}$$

Let us pass now to the estimate of $U^{(2)}$. Denote

$$H(x,\tau)=\rho(\tau^2-\sigma_1^2)EU^{(1)}(x,\tau)+X(x,\tau), \tag{9.17}$$

Then because of (4.23) and (9.16), $\forall x\in\overline{\Omega}^+$, $|\alpha|=0,1$,

$$H_4(x,\tau)=0,\quad |\partial^\alpha H_i(x,\tau)|\le c|\tau|^{-5},\quad i=1,2,3. \tag{9.18}$$

The function $U^{(2)}$ satisfies in Ω^+ the equation

$$\begin{gathered}
c_{ijkl}\frac{\partial^2 U_k^{(2)}(x,\tau)}{\partial x_j \partial x_l} - \\
-\rho\tau^2\delta_{ij}U_j^{(2)}(x,\tau) + e_{kij}\frac{\partial^2 U_4^{(2)}(x,\tau)}{\partial x_k \partial x_j} = H_i(x,\tau), \\
-e_{ikj}\frac{\partial^2 U_k^{(2)}(x,\tau)}{\partial x_i \partial x_j} + \varepsilon_{kj}\frac{\partial^2 U_4^{(2)}(x,\tau)}{\partial x_k \partial x_j} = 0.
\end{gathered} \tag{9.19}$$

From (9.19), taking into account the boundary condition (8.10), it is not difficult to obtain

$$\int\limits_{\Omega^+}\Big[c_{ijkl}\frac{\partial \overline{U}_i^{(2)}}{\partial x_j}\cdot\frac{\partial \overset{(2)}{U}_k}{\partial x_l} + \rho\tau^2\overline{U}_i^{(2)}\overset{(2)}{U}_i + \varepsilon_{kj}\frac{\partial \overline{U}_4^{(2)}}{\partial x_k}\cdot\frac{\partial U_4^{(2)}}{\partial x_j}\Big]dx =$$

$$= -\int\limits_{\Omega^+} H_i\overline{U}_i^{(2)}dx. \tag{9.20}$$

Considering the difference between (9.20) and its complex conjugate equality, we can see that

$$\rho(\tau^2-\overline{\tau}^2)\sum_{i=1}^{3}\|U_i^{(2)}(\cdot,\tau)\|_{\Omega^+}^2 = -2\,\mathrm{Im}\,\big(H_i(\cdot,\tau),U_i^{(2)}(\cdot,\tau)\big)_{\Omega^+}$$

If $\tau=\sigma+iw$ and $2w>\sigma>\sigma_0$, then $|\sigma_0||\tau|<\sqrt{2}|\tau^2-\overline{\tau}^2|$. Therefore

$$\sum_{i=1}^{3}\|U_i^{(2)}(\cdot,\tau)\|_{\Omega^+}^2 \le c|\tau|^{-1}\sum_{i=1}^{3}\big(H_i(\cdot,\tau),U_i^{(2)}(\cdot,\tau)\big)_{\Omega^+}.$$

Taking into account the estimates for H, we obtain

$$\|U_i^{(2)}(\cdot,\tau)\|_{\Omega^+} \le c|\tau|^{-6}, \quad i=1,2,3. \tag{9.21}$$

Similarly we can prove (9.21) for $\sigma>2w$. In this case, due to (2.6) and (2.7), from (9.20) we have

$$\rho(\sigma^2-\omega^2)(U_i(\cdot,\tau),U_i(\cdot,\tau))_{\Omega^+} \le -\,\mathrm{Re}\,\Big(H_i(\cdot,\tau),U_i^{(2)}(\cdot,\tau)\Big)_{\Omega^+}.$$

whence by inequality $2^{-1}|\tau|^2\le\sigma^2-w^2$, we again obtain (9.21).

From (9.12) and (9.13), owing to (4.25), (9.16) and Theorem 8.3, we have

$$|U_i^{(2)}(x,\tau)| \le c|\tau^2-\sigma_1^2|\sum_{j=1}^{3}\|U_j^{(2)}(\cdot,\tau)\|_{\Omega^+} + c|\tau^2-\sigma_1^2|\|U^{(1)}(\cdot,\tau)\|_{\Omega^+} +$$

$$+c\|X(\cdot,\tau)\|_{\Omega^+} \le c|\tau|^{-4}, \quad x\in\Omega^+, \quad i=1,2,3,4. \tag{9.22}$$

The equalities (9.12) and (9.13) enable us to estimate the first derivatives of $U^{(2)}$. We have

$$\Big|\frac{\partial U_i^{(2)}(x,\tau)}{\partial x_p}\Big| \le c|\tau^2-\sigma_1^2|\sum_{j=1}^{3}\int\limits_{\Omega+}\Big|\frac{\partial G_{ij}(x,y,(2)_{\sigma_1}^{+})}{\partial x_p}\Big||U_j^{(2)}(y,\tau)|dy +$$

$$+c\sum_{j=1}^{3}\int\limits_{\Omega+}\Big|\frac{\partial G_{ij}(x,y,(2)_{\sigma_1}^{+})}{\partial x_p}\Big||H_j(y,\tau)|dy, \quad j=1,2,3,4.$$

Let Ω be some domain in $\mathbb{R}^3$, $F\in C(\Omega)$ and satisfies at infinity the condition $F(x)=O(|x|^{-2-q})$ for some $q>0$. Then for sufficiently small ε,

$$\int\limits_{\Omega}|x-y|^{-2}|F(x)|dx \le \Big[\int\limits_{\Omega}|x-y|^{-3+\varepsilon}(1+|y|)^{-2\varepsilon}dy\Big]^{\frac{2}{3-\varepsilon}} \times$$

$$\times\Big[\int\limits_{\Omega}(1+|y|)^{\frac{4\varepsilon}{1-\varepsilon}}|F(y)|^{\frac{3-\varepsilon}{1-\varepsilon}}dy\Big]^{\frac{1-\varepsilon}{3-\varepsilon}} \le$$

$$\le c\sup_{y\in\Omega}\Big((1+|y|)^{\frac{4\varepsilon}{3-\varepsilon}}|F(y)|^{\frac{1+\varepsilon}{1-\varepsilon}}\Big)\|F\|_{\Omega}^{\frac{2(1-\varepsilon)}{3-\varepsilon}}. \qquad (9.23)$$

Using this estimate for $\Omega=\Omega^+$, we obtain

$$\Big|\frac{\partial U_i^{(2)}(x,\tau)}{\partial x_p}\Big| \le$$

$$\le c|\tau|^2\cdot|\tau|^{-4\frac{1+\varepsilon}{1-\varepsilon}}|\tau|^{-12\frac{1-\varepsilon}{1+\varepsilon}}+c|\tau|^{-5}\le c|\tau|^{-5},\ i=1,2,3,4. \qquad (9.24)$$

Finally, let us estimate the second derivatives of $U^{(2)}$ in Ω_0, $\overline{\Omega}_0\subset\Omega^+$. If $x\in\Omega_0$, then

$$\frac{\partial U_i^{(2)}(x,\tau)}{\partial x_p} = \rho(\tau^2-\sigma_1^2)\sum_{j=1}^{3}\int\limits_{\Omega+}\frac{\partial G_{ij}(x,y,(2)_{\sigma_1}^{+})}{\partial x_p}U_j^{(2)}(y,\tau)dy +$$

$$+\sum_{j=1}^{3}\int\limits_{\Omega+}\frac{\partial G_{ij}(x,y,(2)_{\sigma_1}^{+})}{\partial x_p}H_j(y,\tau)dy =$$

$$= -\sum_{j=1}^{3}\int\limits_{S}n_p(y)\Phi_{ij}(y-x,\sigma_1^2)\Big[\rho(\tau^2-\sigma_1^2)U_j^{(2)}(y,\tau)+H_j(y,\tau)\Big]d_yS -$$

$$-\sum_{j=1}^{3}\int\limits_{\Omega+}\Phi_{ij}(y-x,\sigma_1^2)\Big[\rho(\tau^2-\sigma_1^2)\frac{\partial U_j^{(2)}(y,\tau)}{\partial y_p}+\frac{\partial H_j(y,\tau)}{\partial y_p}\Big]dy +$$

$$+\sum_{j=1}^{3}\int\limits_{\Omega+}\frac{\partial g_{ij}(x,y,(2)_{\sigma_1}^{+})}{\partial x_p}\Big[\rho(\tau^2-\sigma_1^2)U_j^{(2)}(y,\tau)+H_j(y,\tau)\Big]dy,$$

$$\frac{\partial^2 U_i^{(2)}(x,\tau)}{\partial x_p \partial x_q} = -\sum_{j=1}^{3} \int\limits_S n_p(y) \frac{\partial \Phi_{ij}(y-x,\sigma_1^2)}{\partial x_q} \times$$
$$\times \Big[\rho(\tau^2-\sigma_1^2) U_j^{(2)}(y,\tau) + H_j(y,\tau)\Big] d_y S -$$
$$-\sum_{j=1}^{3} \int\limits_{\Omega+} \frac{\partial \Phi_{ij}(y-x,\sigma_1^2)}{\partial x_q} \Big[\rho(\tau^2-\sigma_1^2) \frac{\partial U_j^{(2)}(y,\tau)}{\partial y_p} + \frac{\partial H_j(y,\tau)}{\partial y_p}\Big] dy +$$
$$+\sum_{j=1}^{3} \int\limits_{\Omega+} \frac{\partial^2 g_{ij}(x,y,(2)^+_{\sigma_1})}{\partial x_p \partial x_q} \Big[\rho(\tau^2-\sigma_1^2) U_j^{(2)}(y,\tau) + H_j(y,\tau)\Big] dy. \quad (9.25)$$

Taking into account the earlier obtained estimates for Φ, $U^{(2)}$ and H, from (9.25) we have

$$\Big| \frac{\partial^2 U_i^{(2)}(x,\tau)}{\partial x_p \partial x_q} \Big| \le c|\tau|^{-2}, \quad i=1,2,3,4, \quad x \in \Omega_0.$$

Thus the following theorem is proved.

Theorem 9.1. *If U is a solution of Problem $(2)^+_\tau$, then $U(x,\cdot)$ is an analytic function in the half-plane $\mathbb{C}_{\sigma_0}$ for which the estimates*

$$\begin{aligned} &|\partial_x^\alpha U(x,\tau)| \le c|\tau|^{-4}, \quad x \in \overline{\Omega}^+, \quad |\alpha| = 0,1, \\ &|\partial_x^\alpha U(x,\tau)| \le c|\tau|^{-2}, \quad x \in \overline{\Omega}_0 \subset \Omega^+, \quad |\alpha| = 2, \end{aligned} \quad (9.26)$$

are valid.

Exactly in the same way, we can prove that the assertions of Theorem 9.1 are also valid for a solution of Problem $(1)^+_\tau$.

Consider now the exterior boundary value problems, for example, Problem $(1)^-_\tau$:

$$\begin{gathered} A(\partial_x,\tau^2)U(x,\tau) = X(x,\tau), \quad x \in \Omega^-, \\ [U(y,\tau)]^- = f(y,\tau), \quad y \in S, \\ \lim_{|x|\to\infty} U(x,\tau) = 0, \end{gathered} \quad (9.27)$$

where X and f satisfy (4.23)–(4.25). Then, by Theorem 8.2, we have

$$U_i(x,\tau) = -\int\limits_S \mathcal{R}_{kj}(\partial_y, n(y)) G_{ik}(x-y,\tau^2) f_j(y,\tau) d_y S +$$
$$+\sum_{j=1}^{3} \int\limits_{\Omega-} \Phi_{ij}(x-y,\tau^2) X_j(y-\tau) dy - \sum_{j=1}^{3} \int\limits_{\Omega-} g_{ij}(x,y,(1)^-_\tau) X_j(y,\tau) dy,$$
$$i = 1,2,3,4.$$

Taking in this representation into account the estimates obtained in Theorems 5.2, 7.5 and 8.3, we obtain for U the following estimates at infinity:

$$\begin{gathered}\partial^\alpha U_i(x,\tau) = O(|x|^{-2-q'}), \quad i=1,2,3, \quad 0<q'<q,\\ \partial^\alpha U_4(x,\tau) = O(|x|^{-1-|\alpha|}).\end{gathered} \tag{9.28}$$

Here $|\alpha| = 0,1$.

Represent U as the sum $U = U^{(1)} + U^{(2)}$, where $U^{(1)}$ is the solution of the problem

$$\begin{gathered}A(\partial_x,\sigma_1^2)U^{(1)}(x,\tau) = 0, \quad x\in\Omega^-,\\ \Big[U^{(1)}(y,\tau)\Big]^- = f(y,\tau), \quad y\in S,\\ \lim_{|x|\to\infty} U^{(1)}(x,\tau) = 0,\end{gathered} \tag{9.29}$$

and $U^{(2)}$ satisfies the conditions

$$\begin{gathered}A(\partial_x,\sigma_1^2)U^{(2)}(x,\tau) = \rho(\tau^2-\sigma_1^2)EU^{(2)}(x,\tau) + H(x,\tau), \quad x\in\Omega^-,\\ [U^{(2)}(y,\tau)]^- = 0, \quad y\in S,\\ \lim_{|x|\to\infty} U^{(2)}(x,\tau) = 0,\end{gathered} \tag{9.30}$$

where H is defined from (9.17).

As it is already proved, the solution of the problem (9.29) can be represented in the form of (7.16):

$$\begin{gathered}U^{(1)}(x,\tau) = \int\limits_S [\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-x,\sigma_1^2)]^T\psi(y,\tau)d_yS + \\ +\theta(x,\sigma_1)\int\limits_S [\mathcal{R}(\partial_y,n(y))\Phi(y,\sigma_1^2)]^T\psi(y,\tau)d_yS, \quad x\in\Omega^-,\end{gathered} \tag{9.31}$$

where ψ is the solution of the uniquely solvable equation

$$\begin{gathered}-\frac{1}{2}\psi(z,\tau) + \int\limits_S [\widetilde{\mathcal{R}}(\partial_y,n(y))\widetilde{\Phi}(y-z,\sigma_1^2)]^T\psi(y,\tau)d_yS + \\ +\theta(z,\tau)\int\limits_S [\mathcal{R}(\partial_y,n(y))\Phi(y,\sigma_1^2)]^T\psi(y,\tau)d_yS = f(z,\tau), \quad z\in S.\end{gathered}$$

Therefore estimates (9.16) are valid in this case as well:

$$\begin{gathered}\|U^{(1)}(\cdot,\tau)\|_{(\Omega^-,k,\beta)} \le c|\tau|^{-7}, \quad k=0,1,\\ \|U^{(1)}(\cdot,\tau)\|_{(\Omega_0,2,\beta)} \le c|\tau|^{-7}, \quad \overline{\Omega}_0\subset\Omega^+.\end{gathered} \tag{9.32}$$

Moreover, (9.31) implies that the following estimates are valid at infinity:

$$\begin{gathered}|\partial^\alpha U_i^{(1)}(x,\tau)| \le c(1+|x|)^{-3-|\alpha|}, \quad i=1,2,3,\\ |\partial^\alpha U_4^{(1)}(x,\tau)| \le c(1+|x|)^{-1-|\alpha|}.\end{gathered} \tag{9.33}$$

Consider now $U^{(2)}$. It satisfies the conditions (9.30), and due to (9.28) and (9.33), admits the following estimates at infinity:

$$\begin{gathered}\partial^\alpha U_i^{(2)}(x,\tau) = O(|x|^{-2-q'}), \quad i=1,2,3, \quad 0<q'<q,\\ \partial^\alpha U_4^{(2)}(x,\tau) = O(|x|^{-1-|\alpha|})\end{gathered} \tag{9.34}$$

for $|\alpha| = 0,1$.

By virtue of (9.34) and (9.23), all the arguments adduced for the solution $U^{(2)}$ of Problem $(2)^+_\tau$ remain also valid in our case. As a result, we obtain the following

Theorem 9.2. *The solution U of Problem $(1)^-_\tau$ is an analytic in the half-plane $\mathbb{C}_{\sigma_0}$ function with respect to the parameter τ for which the estimates*

$$\begin{aligned}&|\partial_x^\alpha U(x,\tau)| \le c|\tau|^{-4}, \quad x\in\overline{\Omega}^-, \quad |\alpha|=0,1,\\ &|\partial_x^\alpha U(x,\tau)| \le c|\tau|^{-2}, \quad x\in\overline{\Omega}_0\subset\Omega^+, \quad |\alpha|=2,\end{aligned} \tag{9.35}$$

as well as the estimates (9.28) *are valid.*

It is not difficult to see that this theorem is true for the solution of Problem $(2)^-_\tau$ as well.

Let us pass now to the proof of the existence of solutions for dynamic problems.

Theorem 9.3. *If $S\in C^{2,\gamma}$, $0<\gamma\le 1$, then all the above-considered dynamic problems are uniquely solvable.*

Proof. If U is a solution of, for example, Problem $(1)^+$, then, as it has already been proved,

$$\widetilde{U}(x,\tau) = \int_0^\infty e^{-\tau t}\Big(U(x,t)-V(x,t)\Big)dt,$$

where V defined by (4.14), is a regular solution of Problem $(1)^+_\tau$ with $\operatorname{Re}\tau > \sigma_0$, and for it the assertions of Theorem 9.1 are valid.

Let us consider the inverse Laplace transform

$$\overset{(0)}{U}(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\tau t}\widetilde{U}(x,\tau)dx, \quad \sigma>\sigma_0. \tag{9.36}$$

From Theorem 9.1, it follows (see [11]) that $\overset{(0)}{U}$ belongs to the class $C^2(\Omega^+ \times \times]0,\infty[) \cap C^1(\overline{\Omega}^+ \times [0,+\infty[)$ and satisfies the conditions (4.15)–(4.17). Then $U = \overset{(0)}{U} + V$ will be the desired solution of Problem $(1)^+$. ■

Theorems for the remaining problems are proved analogously.

References

1. W. Voigt, Lehrbuch der Kristall-Physik. *Teubner, Leipzig,* 1910.

2. R. A. Toupin, A dynamical theory of elastic dielectrics. *Internat. J. Engrg. Sci.* **1**(1963), No. 1, 101–126.

3. R. A. Toupin, The elastic dielectrics. *J. Rat. Mech. Analysis* **5**(1956), 849–915.

4. R. D. Mindlin, Polarization gradient in elastic dielectrics. *Internat. J. Solids Structures* **4**(1968), 637–663.

5. R. D. Mindlin, Continuum and lattice theories of influence of electromechanical coupling on capacitance of thin dielectric films. *Internat. J. Solids Structures* **5**(1969), 1197–1213.

6. R. D. Mindlin, Elasticity, piezoelasticity and crystal lattice dynamics. *J. Elasticity* **2**(1972), No. 4, 217–280.

7. W. Nowacki, Efekty electromagnetyczne w statych cialach odksztalcalnych. *Panstwowe Wydawnictwo Naukowe, Warszawa*, 1983.

8. S. Kaliski and J. Petikiewicz, Equations of motion coupled with the field of temperature in a magnetic field involving mechanical and electromagnetic relaxations for anisotropic bodies. *Proc. Vibr. Probl.* **1**(1960), No. 4, 17–28.

9. A. F. Ulitko, To the theory of oscillations of piezokeramic bodies. Heat stresses in the construction elements. (Russian) *Respublikanskiĭ Mezhvedomstv. Sbornik* (1975), 110–114.

10. R. Holland and E. P. EerNisse, Design of resonant piezoelectric devices. *M.I.T. Press Cambridge–London,* 1969.

11. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. *North-Holland Publ. Company, Amsterdam*, 1979.

12. R. J. Knops, and L. E. Payne, Uniqueness theorems in linear elasticity. *Springer tracts in natural philosophy* **19**, *Springer-Verlag, Berlin-Heidelberg-New York,* 1971.

13. L. Wheeler, and E. Sternberg, Some theorems in classical elastodynamics. *Arch. Rat. Mech. Anal.* **31**(1968), No. 1, 51–90.

14. D. Natroshvili, On the theorem of uniqueness of a solution of the Cauchy problem for the equation of dynamics of homogeneous anisotropic elastic medium. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **87**(1977), No. 2, 321–324.

15. D. Natroshvili, Green tensor's estimates of the theory of elasticity. (Russian) *Differentsial'nye Uravneniya* **14**(1978), No. 7, 1272–1284.

16. W. Rudin, Functional Analysis. *McGraw-Hill Book Company, New York,* 1973.

17. N. M. Gyunter, Potential theory and its application to the basic problems of mathematical physics. (Russian) *Gostekhizdat, Moscow*, 1953.

18. L. Hörmander, The Analysis of Linear Partial Differential Operators, I. *Springer-Verlag, Berlin-Heidelberg-New York-Tokyo*, 1983.

19. R. V. Kapanadze, On boundary value problems of the theory of elasticity for anisotropic homogeneous media. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **88**(1977), No. 2.

20. T. V. Burchuladze, and T. G. Gegelia Development of the potential method in the theory of elasticity. (Russian) *Metsniereba, Tbilisi,* 1985.

21. S. Prössdorf, Einige Klassen Singulärer Gleichungen. *Akademie-Verlag, Berlin,* 1974.

(Received 1.07.1996)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia