

Mikheil Usanetashvili

**BOUNDARY VALUE PROBLEMS FOR
SOME CLASSES OF DEGENERATING
SECOND ORDER PARTIAL
DIFFERENTIAL EQUATIONS**

Abstract. The first boundary value problem is studied for second order general elliptic equations degenerating on the whole boundary. In accordance with the type of degeneration, the cases are distinguished where the whole boundary becomes free of boundary conditions. For a class of second order degenerating elliptic equations, a new approach is proposed which enables one to prove the correctness of the Dirichlet problem. For second order general elliptic equations degenerating on a part of the boundary, conditions are found guaranteeing the correctness of the problem with oblique derivative. For the solution of this problem, an a priori estimate is obtained. A boundary value problem of conjugation type is studied in weighted spaces for a class of degenerating second order hyperbolic systems with discontinuous coefficients. The problems with oblique derivative are also investigated for mixed type equations with a Lavrent'ev–Bitsadze operator as the principal part.

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Key words and Phrases. Degenerating elliptic, hyperbolic and mixed type equations, the Dirichlet problem, problem with oblique derivative, extremum principle, index of the problem.

რეზიუმე. ნაშრომში მეორე რიგის ზოგადი სახის მთელს საზღვარზე გადაგვარებადი ელიფსური ტიპის განტოლებისათვის შესწავლილია პირველი სასაზღვრო ამოცანა. იმისდა მიხედვით, თუ რა ტიპის გადაგვარებას აქვს ადგილი, გამოყოფილია შემთხვევები, როდესაც არის მთელი საზღვარი მთლიანად თავისუფლდება სასაზღვრო პირობისაგან. მეორე რიგის გადაგვარებად ელიფსურ სისტემათა ერთი კლასისათვის შემოთავაზებულია ახალი მიდგომა, რომელიც საშუალებას იძლევა დავამტკიცოთ დირიხლეს ამოცანის კორექტულობა. საზღვრის ნაწილზე გადაგვარებადი მეორე რიგის ზოგადი სახის ელიფსური განტოლებისათვის მოძებნილია პირობები, რომლებიც უზრუნველყოფენ დახრილწარმოებულნი ამოცანის კორექტულობას. ამ ამოცანის ამოხსნისათვის მიღებულია აპრიორული შეფასება. მეორე რიგის წყვეტილკოეფიციენტებიანი ჰიპერბოლურ სისტემათა ერთი კლასისათვის შესწავლილია შეუღლების ტიპის ერთი სასაზღვრო ამოცანა წონიან სივრცეებში, აგრეთვე გამოკვლეულია დახრილწარმოებულნი ამოცანები შერეული ტიპის განტოლებისათვის, რომლის მთავარ ნაწილს წარმოადგენს ლავრენტიევ–ბიწადის ოპერატორი.

INTRODUCTION

In the theory of partial differential equations, under a degenerating equation is usually understood an equation which changes its type on the closure of the range of independent variables. Degeneration character may be so diverse that no successful classification of the types of degeneration has been described as yet.

Degenerating partial differential equations, in particular degenerating elliptic, hyperbolic and mixed type equations are encountered in solving many important problems in the membrane theory of shells of alternating curvature, the theory of infinitely small deformation of surfaces, in transonic and supersonic gas mechanics, in the theory of magnetohydrodynamic flows with passage over critical velocities, and in other divisions of mechanics.

Individual special classes of equations not coinciding with the well-studied equations of elliptic and hyperbolic type have been considered for a long time, for example, in Picone's work [60] published about 80 years ago.

Tricomi's work [62], as well as further investigation of equations of mixed type, evoked great interest in the study of elliptic and hyperbolic equations which degenerate on the boundary of a domain.

Boundary value problems for mixed type equations lead to new mixed boundary value problems for elliptic equations not satisfying the condition of uniform ellipticity, in particular, to boundary value problems for elliptic equations degenerating on a part of the boundary.

In a certain sense, Keldysh's fundamental work [40] has become a turning-point in the theory of degenerating elliptic equations. Having considered a specific second order equation in a hemisphere whose plane part of the boundary is a characteristic manifold, M. V. Keldysh has shown that under certain conditions imposed on the lower terms of the equation one should, when searching for a smooth solution of the equation, remove boundary conditions on the manifold of degeneration. Thus in his work M. V. Keldysh has clearly shown that the statement of the boundary value problems for degenerating elliptic equations depends actually on the behavior of the lower terms of the equation in the vicinity of a degenerating manifold. This work has stimulated further investigation in the direction indicated by him.

In 1956, in his summarizing report at the 3rd All-Union Congress of Mathematicians, A. V. Bitsadze [8] placed emphasis upon the importance of the subsequent study of various new problems for degenerating elliptic equations. In particular, it has been indicated that in the cases where the Dirichlet problem is not always solvable, one can naturally replace the condition of boundedness of a solution in the vicinity of the boundary of degeneration by a boundary condition which is satisfied by some weight function. Later on, these questions turned out to be topical for many specialists.

The next stage in the development of the theory of degenerating elliptic equations starts from the works of G. Fichera [22] and O. A. Oleinik [58] who proved the existence and uniqueness of a generalized solution of the

Dirichlet problem for second order equations with nonnegative characteristic form. Noteworthy is also an approach to the investigation of boundary value problems for degenerating elliptic equations on the basis of the theory of embedding of weighted functional spaces, suggested by M. I. Vishik and L. D. Kudryavtsev which was subsequently developed by their pupils and followers.

As for the theory of initial, initial-boundary value and the Goursat and Darboux problems for degenerating hyperbolic equations, it has a long-standing history. These problems were investigated in the works by Darboux [17], Tricomi [63, 64], Gellerstedt [25, 26], A. V. Bitsadze [9], A. M. Nakhushev [53, 54], etc.

Passage to the second order hyperbolic systems leads us to additional difficulties. A. V. Bitsadze [10] was the first who constructed examples of second order hyperbolic systems for which the corresponding homogeneous Goursat problem has infinitely many independent solutions. Of particular note in this direction are the works of S. S. Kharibegashvili [41, 42] in which he investigates different versions of the Goursat and Darboux problems for second order degenerating hyperbolic systems.

First fundamental investigation in the theory of equations equations of mixed type was carried on in the early 20s by Tricomi [65], and continued in the 30s by Cibrario [15, 16], Gellerstedt [27] and Holmgren [35].

The next, not less significant step in the development of the theory of mixed type equations was made by M. A. Lavrent'ev and A. V. Bitsadze [47], A. V. Bitsadze [11], K. I. Babenko [5], F. I. Frankl [23, 24] and others. In those works, besides a fundamental investigation of various essential problems of this theory, much attention was given to the practical importance of the problem of mixed type equations.

In the development of this theory, A.V. Bitsadze's investigations are of special interest. He has formulated and studied a wide class of boundary value problems both in two-dimensional and in spatial cases.

The present paper is devoted to the investigation of boundary value problems for degenerating elliptic and hyperbolic equations and systems, as well as for equations of mixed type.

The first boundary value problem for a general second order elliptic equation degenerating on the entire boundary is studied in Chapter I. Depending on the type of degeneration, we distinguish the cases where the boundary of the domain is wholly free from boundary conditions. Next, for a class of second order degenerating elliptic systems, we suggest a new approach enabling one to prove the unique solvability of the Dirichlet problem for these systems. Finally, for the second order elliptic equation of the general type degenerating on a part of the boundary, conditions are found which guarantee the correctness of the problem with oblique derivative. An a priori estimate of the solution of this problem is obtained.

In Chapter II, we deal with the boundary value problem of conjugation for a class of second order degenerating hyperbolic systems with discontinuous

coefficients. Conditions ensuring the unique solvability of this problem in the weight spaces determined by the character of degeneration of the system are also obtained.

Finally, in Chapter III, we investigate the problems with oblique derivative for mixed equations with Lavrent'ev–Bitsadze's operator in the principal part.

CHAPTER I
**BOUNDARY VALUE PROBLEMS
 FOR A SECOND ORDER LINEAR ELLIPTIC
 EQUATION OF THE DEGENERATING
 ON THE BOUNDARY OF A DOMAIN**

1. ON SOLVABILITY OF THE DIRICHLET PROBLEM FOR SECOND ORDER
 ELLIPTIC EQUATIONS WITH DEGENERATION ON THE WHOLE
 BOUNDARY

Let us consider an equation of the form

$$L(u) \equiv Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = f \quad (1.1)$$

in a bounded simply connected domain D of the plane of the variables x, y , where

$$A, B, C, a, b, c \in H^\alpha(\bar{D}), \quad 0 < \alpha < 1, \quad f \in C(\bar{D}) \cap H^\alpha(D), \quad c \leq 0. \quad (1.2)$$

In what follows, the equation (1.1) is assumed to be elliptic in D and degenerating on the boundary $\Gamma = \partial D$, i.e.,

$$(B^2 - AC)|_D < 0 \quad (1.3)$$

and

$$(B^2 - AC)|_{\partial D} = 0. \quad (1.4)$$

Obviously, due to the ellipticity of (1.1) in D , without loss of generality one may assume that $A|_D > 0$.

Let ∂D be given in terms of the equation $H(x, y) = 0$, where

$$H|_D > 0, \quad H \in C^2(\bar{D}), \quad H|_\Gamma = 0, \quad \nabla H|_\Gamma \neq 0.$$

The Dirichlet problem. *Find a regular in D solution $u \in C^2(D) \cap C(\bar{D})$ of the equation (1.1) by the boundary condition*

$$u|_\Gamma = \varphi, \quad \varphi \in C(\Gamma). \quad (1.5)$$

It should be noted that for some classes of degenerating elliptic equations, these questions have been considered in the works by M. I. Aliev [3,4], D. K. Gvazava [33,34], G. V. Jaiani [36–38], G. G.Devdariani [18, 19]. An approach to the investigation of boundary value problems for degenerating differential equations on the basis of the theory of embedding of weighted functional spaces has been first realized in the works by M. I. Vishik [73, 74] and L. D. Kudryavtsev [44, 45]. The results of these papers were later generalized and supplemented by S.M. Nikol'skiĭ [57], L. N. Lizorkin and N. V. Miroshin [48], V. T. Glushko [29–31], N. V. Miroshin [50], S. G.

Mikhlin [51], V. V. Grushin [32], S. N. Kruzhkov [43], A. A. Vasharin and P. I. Lizorkin [70] etc.

When investigating the Dirichlet problem, one should distinguish two cases

$$(AH_x^2 + 2BH_xH_y + CH_y^2)|_\Gamma \neq 0, \quad (1.6)$$

and

$$(AH_x^2 + 2BH_xH_y + CH_y^2)|_\Gamma = 0. \quad (1.7)$$

Remark. The equality (1.4) together with the condition (1.6) means that the equation (1.1) degenerates on $\Gamma = \partial D$ parabolically and at every point of the boundary, the tangent direction does not coincide with the characteristic one. The conditions (1.4) and (1.7) are equivalent to the fact that either the order of the equation degenerates at some points of the boundary or there takes place parabolic degeneration. In this case the characteristic direction coincides with the tangent one.

Owing to the condition (1.2), the uniqueness of the solution of the Dirichlet problem is obvious.

In order to prove the existence of the solution of the problem (1.1), (1.5), let us consider two auxiliary problems: 1. $L(u) = 0$, $u|_\Gamma = \varphi$; 2. $L(u) = f$, $u|_\Gamma = 0$. Below, when considering Problem 2, we will assume that

$$c \leq -k, \quad k = \text{const} > 0. \quad (1.8)$$

To prove the existence of a solution of Problem 1 in the case (1.6), we denote by D_h the domain consisting of the points of the domain D satisfying the condition $H(x, y) > h$, where h is a sufficiently small positive number. It is clear that for sufficiently small h , the domain D_h is simply connected. Let us take an arbitrary extension of the function φ to D of the class $C^\infty(D)$. Under our assumptions imposed both on the coefficients of the equation (1.1) and on the domain D , for sufficiently small h there exists a regular in D_h solution $u_h \in C(\bar{D}_h) \cap C^2(D_h)$ coinciding on the boundary D_h with the function φ , because of the extremum principle. Moreover, $|u_h(x, y)| \leq M$, $(x, y) \in \bar{D}_h$, where $M = \max_{\bar{D}} |\varphi|$. Consider a sequence of domains \bar{D}_{h_n} , $n = 0, 1, \dots$, where $h_0 > h_1 > \dots > h_n > \dots$ and $\lim_{n \rightarrow \infty} h_n = 0$. Since \bar{D}_{h_n} is a compactum and $\bar{D}_{h_n} \subset D_{h_p}$ for $p > n$, we may by virtue of the inequality $|u_{h_n}(x, y)| \leq M$ select from the sequence u_{h_n} in D_{h_0} a uniformly converging subsequence $u_{h_0}^0, u_{h_1}^0, \dots, u_{h_n}^0, \dots$ whose limit is also a solution of the class $C^2(D_{h_0})$ of the equation (1.1) [46]. Similarly, we may select from the sequence $u_{h_0}^0, u_{h_1}^0, \dots, u_{h_n}^0, \dots$ a uniformly converging on the compactum \bar{D}_{h_1} subsequence $u_{h_0}^1, u_{h_1}^1, \dots, u_{h_n}^1, \dots$, etc. Evidently, the diagonal subsequence $u_{h_0}^0, u_{h_1}^1, \dots, u_{h_n}^n, \dots$ uniformly converges on every compactum of the domain D , and its limit $u(x, y)$ is a regular solution of the class $C^2(D)$ of the equation (1.1).

As it is known, when the condition (1.8) is fulfilled, then for the solution of Problem 2, if it exists, the estimate [3]

$$\max_{\bar{D}} |u| \leq \max_{\bar{D}} \frac{|f|}{k}$$

is valid, whence repeating the above arguments, we obtain a solution $u(x, y)$ of the class $C^2(D)$ of the equation (1.1) as the limit of the sequence $u_{h_0}^0, u_{h_1}^1, \dots, u_{h_n}^n$ uniformly converging on every compactum of D . Clearly, the sum of solutions of Problems 1 and 2 will be a solution of problem (1.1), (1.5). Below we will restrict ourselves to the consideration of Problem 1.

As it is known [61], in order for the above constructed solution $u(x, y)$ of the equation (1.1) to satisfy the boundary condition (1.5), it suffices to construct at every point $Q(x_0, y_0)$ of the boundary of D a so-called barrier, i.e., a function $v(x, y)$ satisfying the following conditions: a) it is continuous in some neighborhood of the point $\omega_Q = \{P \in \bar{D} : |P - Q| \leq \varepsilon\}$; b) it equals zero at the point Q ; c) $v(x, y) > 0$ in $\omega_Q \setminus Q$; d) everywhere in this neighborhood $L(v) < 0$.

As a barrier, we consider the function

$$v(x, y) = (x - x_0)^2 + (y - y_0)^2 + H^\beta(x, y), \quad 0 < \beta = \text{const} < 1.$$

It is obvious that this function satisfies the conditions a), b) and c). Let us check the condition d). In the case (1.6), substituting in (1.1) the expression for $v(x, y)$, we get

$$\begin{aligned} L(v) = & \beta(\beta - 1)H^{\beta-1}(AH_x^2 + 2BH_xH_y + CH_y^2) + \\ & + \beta H^{\beta-1}(AH_{xx} + 2BH_{xy} + CH_{yy} + aH_x + bH_y) + 2A + 2C + \\ & + 2a(x - x_0) + 2b(y - y_0) + c((x - x_0)^2 + (y - y_0)^2) + cH^\beta, \end{aligned} \quad (1.9)$$

whence it immediately follows that for sufficiently small $H(x, y)$, the sign of $L(v)$ coincides with that of $\beta(\beta - 1)$, and by virtue of $0 < \beta < 1$, there exists a neighborhood of the point Q such that $L(v) < 0$.

When considering the case (1.7), we assume that in some neighborhood of the boundary Γ the representation

$$AH_x^2 + 2BH_xH_y + CH_y^2 = H^pG \quad (1.10)$$

holds, where $p = \text{const} > 0$, G is a positive, continuous and bounded in that neighborhood function.

Taking into account the representation (1.10), the expression (1.9) takes the form

$$\begin{aligned} L(v) = & \beta(\beta - 1)H^{\beta+p-2}G + \beta H^{\beta-1}[L(H) - cH] + \\ & + 2A + 2C + 2a(x - x_0) + 2b(y - y_0) + c((x - x_0)^2 + (y - y_0)^2) + cH^\beta. \end{aligned}$$

For $0 < p < 1$, the sign of $L(v)$ coincides with that of $\beta(\beta - 1)H^{p+\beta-2}$, i.e., $L(v) < 0$. If $p = 1$ and $(1 - IG^{-1})|_\Gamma > 0$, where $I = L(H) - cH$, then the sign of $L(v)$ coincides with that of $\beta[(\beta - 1)G + I]H^{\beta-1}$, and if we assume

that $\beta < (1 - IG^{-1})|_{\Gamma}$, then we will have $L(v) < 0$. For $1 < p < 2$ and $I|_{\Gamma} \leq 0$, we take $\beta < 2 - p$. Then $L(v) < 0$. In the case $p \geq 2$ and $I|_{\Gamma} < 0$, the sign of $L(v)$ coincides with that of $\beta H^{\beta-1}I$, i.e., $L(v) < 0$.

Thus the following theorems are valid (see [66, 67]).

Theorem 1.1. *Let (1.6) be fulfilled and $c < 0$ on \bar{D} . Then there exists a solution of the Dirichlet problem.*

Theorem 1.2. *Let (1.7) be fulfilled, $c < 0$ on \bar{D} and one of the following conditions is fulfilled: 1) $0 < p < 1$; 2) $p = 1$, $(1 - IG^{-1})|_{\Gamma} > 0$, $I = L(H) - cH$; 3) $1 < p < 2$, $I|_{\Gamma} \leq 0$; 4) $p \geq 2$, $I|_{\Gamma} < 0$. Then there exists a solution of the Dirichlet problem.*

Below we assume that (1.7), (1.10) and

$$H^{p-1}G - I \leq A_0H, \quad p \geq 1, \quad A_0 = \text{const} > 0,$$

hold. It can be easily verified that if the coefficients of the equation (1.1) are analytic, then the above inequality is fulfilled for sufficiently large $A_0 > 0$, when $(H^{p-1}G - I)|_{\Gamma} < 0$.

The following lemma holds [66, 67].

Lemma 1.1. *Let one of the following conditions be fulfilled: 1) $p = 1$, $IG^{-1}|_{\Gamma} \geq 1$; 2) $1 < p < 2$, $I|_{\Gamma} > 0$; 3) $p \geq 2$, $I|_{\Gamma} \geq 0$, and at every point $P_0 \in \bar{D}$ either $\Phi(P_0) = (Ag_x^2 + 2Bg_xg_y + Cg_y^2)(P_0) \neq 0$, where $g \in C^2(\bar{D})$, $g > 1$, or $\Phi(P_0) = 0$ and $c(P_0) < 0$. Then there exists a function $W(x, y)$ possessing the following properties: (a) $W(P) > 0$, $P \in \bar{D}$; (b) $\lim_{\rho(P, \partial D) \rightarrow 0} W(P) = +\infty$; (c) $L(W) < 0$, $P \in D$.*

Theorem 1.3. *Under the conditions of Lemma 1.1, the homogeneous equation $L(u) = 0$ in the class of bounded functions has only the trivial solution.*

Proof. For any positive ε , everywhere on the boundary $\Gamma = \partial D$ of D the inequality $\varepsilon W \pm u_0 \geq 0$ holds, where u_0 is any bounded solution and W is a function satisfying the conditions of Lemma 1.1. On the other hand, in the domain D the inequality $L(W) < 0$ is fulfilled. Therefore, due to the extremum principle, the inequality $|u_0| \leq \varepsilon W$ holds everywhere in the domain D . This implies that $u(x, y) \equiv 0$ in D because ε is taken to be arbitrary. ■

Corollary. *Every non-trivial solution of the equation $L(u) = 0$ is unbounded.*

Remark. If $c \equiv 0$, then the assertion of Theorem 1.3 is invalid, since $u = \text{const} \neq 0$ satisfies the equation $L(u) = 0$.

Theorem 1.4. *Let $c < 0$ in \bar{D} and the conditions of Theorem 1.3 be fulfilled. Then the equation (1.1) is uniquely solvable in the class of bounded functions.*

2. THE PROBLEM WITH OBLIQUE DERIVATIVE FOR A SECOND ORDER
ELLIPTIC EQUATION DEGENERATING ON A PORTION OF THE
BOUNDARY OF A DOMAIN

Consider an equation of the form

$$L(u) \equiv Au_{xx} + 2Bu_{xy} + Cu_{yy} + au_x + bu_y + cu = F \quad (2.1)$$

in a bounded simply connected domain D of the plane of the variables x, y , where $A, B, C \in C^{2,\alpha}(\bar{D})$, $a, b \in C^{1,\alpha}(\bar{D})$, $c \in C^{0,\alpha}(\bar{D})$, $F \in C(D)$, $0 < \alpha = \text{const} < 1$,

$$c \leq c_0 = \text{const} < 0. \quad (2.2)$$

Let $\partial D = \Gamma_1 \cup \Gamma_2 \cup P_1 \cup P_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, where Γ_1 and Γ_2 are open arcs with the ends at the points P_1 and P_2 . Note that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ belong to the class $C^{2,\alpha}$.

Below the equation (2.1) is assumed to be elliptic in the domain $D \cup \Gamma_1$ and degenerating on a part of the boundary $\bar{\Gamma}_2 \subset \partial D$, i.e.,

$$(B^2 - AC)|_{D \cup \Gamma_1} < 0 \quad (2.3)$$

and

$$(B^2 - AC)|_{\bar{\Gamma}_2} = 0. \quad (2.4)$$

Evidently, because the equation (2.1) is elliptic in D , we may without restriction of generality assume that $A|_{D \cup \Gamma_1} > 0$.

Let the equation Γ_i be given in terms of $H_i(x, y) = 0$, where $H_i|_D > 0$, $H_i \in C^2(\bar{D})$, $H_i|_{\Gamma_i} = 0$, $i = 1, 2$, $\nabla H_1|_{\bar{\Gamma}_1} \neq 0$, $\nabla H_2|_{\bar{\Gamma}_2} \neq 0$.

Below we assume that the points P_1 and P_2 are not cusps for the curve ∂D .

Problem with Oblique Derivative. Find a regular in the domain D solution $u \in C(\bar{D}) \cap C^1(D \cup \Gamma_1) \cap C^2(D)$ of the equation (2.1) satisfying the following boundary conditions:

$$\Lambda(u) \equiv \left(\frac{\partial u}{\partial l} + du \right) |_{\Gamma_1} = f_1, \quad (2.5)$$

$$u|_{\bar{\Gamma}_2} = f_2. \quad (2.6)$$

Here $\frac{\partial u}{\partial l}$ is the derivative with respect to the direction l forming an acute angle with the interior normal to the curve $\bar{\Gamma}_1$; $d, f \in C(\bar{\Gamma}_1)$ and $f_2 \in C(\bar{\Gamma}_2)$ are given functions, and

$$d \leq 0. \quad (2.7)$$

The components of the unit vector l are assumed to belong to the class $C(\bar{\Gamma}_1)$.

In investigating the problem (2.1), (2.5), (2.6), one should distinguish two cases:

$$(AH_{2x}^2 + 2BH_{2x}H_{2y} + CH_{2y}^2)|_{\bar{\Gamma}_2} \neq 0, \quad (2.8)$$

and

$$(AH_{2x}^2 + 2BH_{2x}H_{2y} + CH_{2y}^2)|_{\bar{\Gamma}_2} = 0. \quad (2.9)$$

Lemma 2.1. *For a solution u of the class $C(\bar{D}) \cap C^1(D \cup \Gamma_1) \cap C^2(D)$ of the problem (2.1), (2.5), (2.6), the following a priori estimate is valid:*

$$\|u\|_{C(\bar{D})} \leq C^* (\|f_1\|_{C(\Gamma_1)} + \|f_2\|_{C(\bar{\Gamma}_2)} + \|F\|_{C(D)}), \quad (2.10)$$

where C^* is a positive constant independent of u .

Proof. Since the direction l forms an acute angle with the interior normal to the curve $\bar{\Gamma}_1$ while the components of this vector are continuous on a closed arc $\bar{\Gamma}_1$, there exists δ_0 such that

$$\frac{\partial H_1}{\partial l}|_{\bar{\Gamma}_1} = (l \cdot \text{grad } H_1)|_{\bar{\Gamma}_1} \geq \delta_0 = \text{const} > 0. \quad (2.11)$$

Let

$$\mu = \max \left\{ \frac{\max_{\bar{\Gamma}_1} |f_1|}{|c_0| \delta_0} C_1 + \frac{\sup_D |F|}{|c_0|}; \frac{\max_{\bar{\Gamma}_1} |f_1|}{\delta_0} C_2 + \max_{\bar{\Gamma}_2} |f_2| \right\}, \quad (2.12)$$

where $C_1 = \max_D |L(H_1)|$, $C_2 = \max_D |H_1|$, and because of (2.2), $\min_D |c| \geq |c_0|$.

Consider the function $\omega = \mu - \lambda H_1 - u$, where $\lambda = \max_{\bar{\Gamma}_1} |f_1| / \delta_0$. Then by virtue of (2.7), (2.11), (2.12) and also the equality $H_1|_{\Gamma_1} = 0$, we have

$$\begin{aligned} \Lambda(\omega)|_{\Gamma_1} &= d\mu - \lambda \Lambda(H_1)|_{\Gamma_1} - \Lambda(u)|_{\Gamma_1} = \\ &= d\mu - \lambda \left(\frac{\partial H_1}{\partial l} + dH_1 \right)|_{\Gamma_1} - f_1 \leq \\ &\leq d\mu - \lambda \delta_0 - f_1 \leq 0. \end{aligned} \quad (2.13)$$

In the domain D

$$L(\omega) = \mu c - \lambda L(H_1) - L(u) = \mu c - \frac{\max_{\bar{\Gamma}_1} |f_1|}{\delta_0} L(H_1) - F,$$

whence because of (2.12) the inequality

$$L(\omega) \leq 0 \quad (2.14)$$

holds in D .

It follows from (2.12) that

$$\omega|_{\bar{\Gamma}_2} = \mu - \left(\max_{\bar{\Gamma}_1} |f_1| \cdot \frac{H_1}{\delta_0} + u \right) |_{\bar{\Gamma}_2} \geq 0. \quad (2.15)$$

Let us now show that in the domain \bar{D} the function $\omega \geq 0$. Indeed, by (2.2), (2.3), (2.14) and Hopf's extremum principle [11], the function ω would otherwise take at some point Q of the boundary of D its minimal negative value.

Next, consider two cases: 1) $Q \in \Gamma_1$; and 2) $Q \in \Gamma_2 \cup P_1 \cup P_2$. In the first case, according to Zaremba–Giraud's principle [11], at the point Q we have $\frac{\partial \omega}{\partial l} > 0$. Therefore $\Lambda(\omega) = \frac{\partial \omega}{\partial l} + d\omega > 0$ which because of (2.13) is impossible. In the second case, we have $\omega(Q) = \min_{\bar{D}} \omega < 0$ which contradicts (2.15). Thus $u \leq \mu$ in \bar{D} .

On the other hand, if u is a solution of the problem (2.1), (2.5), (2.6), then $-u$ will be a solution of the problem

$$\begin{aligned} L(-u) &= -F, \\ \Lambda(-u)|_{\Gamma_1} &= -f_1, \\ (-u)|_{\Gamma_2} &= -f_2, \end{aligned}$$

for which μ is given by the same expression (2.12) as for the problem (2.1), (2.5), (2.6). Therefore from the above reasoning we have either $-u \leq \mu$ or $-\mu \leq u$ in \bar{D} . Thus we have obtained that in the domain \bar{D}

$$|u| \leq \mu. \quad (2.16)$$

By virtue of (2.12) from (2.16), there immediately follows the a priori estimate (2.10) with a positive constant C^* not depending on u . ■

When considering the case (2.9), we will assume that along with the condition (2.4) the following representation holds in Γ_2 :

$$AH_{2x}^2 + 2BH_{2x}H_{2y} + CH_{2y}^2 = H_2^p G,$$

where $p = \text{const} > 0$, G is a positive, continuous and bounded in this neighborhood function. Moreover, we suppose that one of the following conditions is fulfilled: 1) $0 < p < 1$; 2) $p = 1$, $(1 - IG^{-1})|_{\bar{\Gamma}_2} > 0$, $I = L(H_2) - cH_2$; 3) $1 < p < 2$, $I|_{\bar{\Gamma}_2} \leq 0$; 4) $p \geq 2$, $I|_{\bar{\Gamma}_2} < 0$;

Then we have

Theorem 2.1. *Let at the points P_1 and P_2 the direction $l \in C^{1,\alpha}(\bar{\Gamma}_1)$ form with an interior normal to the curve $\bar{\Gamma}_2$ an obtuse angle. Then for any $F \in C^{0,\alpha}(\bar{D})$, $f_1 \in C^{1,\alpha}(\bar{\Gamma}_1)$ and $f_2 \in C(\bar{\Gamma}_2)$, there exists a unique solution $u(x, y)$ of the problem (2.1), (2.5), (2.6) of the class $C^{2,\alpha_1}(\bar{D} \setminus \bar{\Gamma}_2) \cap C(\bar{D})$, $0 < \alpha_1 < \alpha$.*

Proof. The uniqueness of the solution of the problem (2.1), (2.5), (2.6) in a more wide class of functions $C(\bar{D}) \cap C^1(D \cup \Gamma_1) \cap C^2(D)$ follows from the a priori estimate (2.10).

Let us prove the existence of the problem (2.1), (2.5), (2.6). To this end, we construct domains $D_h \subset D$ ($D_h \rightarrow D$ as $h \rightarrow +0$ and $D_{h_1} \subset D_h$, $\bar{\Gamma}_{1h_1} \subset \Gamma_{1h}$, $\Gamma_{2h_1} \subset D_h$, if $h_1 > h$) with smooth boundaries $S_h = \Gamma_{1h} \cup \Gamma_{2h} \cup P_{1h} \cup P_{2h}$ of the class $C^{2,\alpha}$, where Γ_{1h} and Γ_{2h} are open arcs with the ends P_{1h} and P_{2h} ; moreover, $P_{1h}, P_{2h} \in \Gamma_1$, $\Gamma_{1h} \subset \Gamma_1$, $\Gamma_{2h} \subset D$.

As is known, since $\bar{\Gamma}_i \in C^{2,\alpha}$, $i = 1, 2$, for an arbitrary function $f_2 \in C(\bar{\Gamma}_2)$ there exists its extension to \bar{D} (we denote it by \tilde{f}_2) such that $f_2 \in C^{2,\alpha}(\bar{D}_h)$ for any $h > 0$. In particular, the solution of the Dirichlet problem for the Laplace equation

$$\Delta u = 0, \quad u|_{\partial D} = \tilde{f}_2,$$

may serve as an example of such an extension, where $\tilde{f}_2|_{\bar{\Gamma}_2} = f_2$, $\tilde{f}_2|_{\Gamma_1} \in C^{2,\alpha}(\Gamma_1)$, $\tilde{f}_2|_{\bar{\Gamma}_1} \in C(\bar{\Gamma}_1)$.

It is known [7] that in the domain D_h there exists a solution of the boundary value problem

$$L(u_h) = F, \tag{2.17}$$

$$\left(\frac{\partial u_h}{\partial l} + du_h \right)_{\Gamma_{1h}} = f_1, \tag{2.18}$$

$$u_h|_{\Gamma_{2h}} = f_2, \tag{2.19}$$

which belongs to the class $C^{0,\alpha}(\bar{D}_h) \cap C^{2,\alpha}(D_h \cup \Gamma_{1h} \cup \Gamma_{2h})$.

Using the Green function $G_h(x, y; x_0, y_0)$, for the solution u_h of the problem (2.17), (2.18), (2.19) in D_h , we can write the following representation [7]

$$\begin{aligned} u_h(x, y) &= \int_{\Gamma_{1h}} G_h f_1 ds + \int_{\Gamma_{2h}} \frac{\partial G_h}{\partial \nu} f_2 ds + \\ &+ \int_{D_h} \int G_h(x, y; x_0, y_0) F(x_0, y_0) dx_0 dy_0, \end{aligned} \tag{2.20}$$

where ν is the unit vector of the conormal for the operator L .

By the representation (2.20) for the solution u_h in D_{h_1} with $h_1 > h$, we have

$$u_h(x, y) = \int_{\Gamma_{1h_1}} G_{h_1} f_1 ds + \int_{\Gamma_{2h_1}} \frac{\partial G_{h_1}}{\partial \nu} \psi_{hh_1} ds +$$

$$+ \int_{D_{h_1}} G_{h_1}(x, y; x_0, y_0) F(x_0, y_0) dx_0 dy_0, \quad (2.21)$$

where

$$\psi_{hh_1} = u_h|_{\Gamma_{2h_1}}. \quad (2.22)$$

Obviously,

$$\max_{\bar{D}} |f_2| \leq C_3 \max_{\bar{\Gamma}_2} |f_2| \quad (2.23)$$

for some $C_3 = \text{const} > 0$.

From the a priori estimate (2.10) and also by (2.23), for the solution u_h of the problem (2.17), (2.18), (2.19) in \bar{D}_h we have

$$\begin{aligned} |u_h| &\leq C^* (\|f_1\|_{C(\Gamma_{1h})} + \|f_2\|_{C(\bar{\Gamma}_{2h})} + \|F\|_{C(D_h)}) \leq \\ &\leq C^* (\|f_1\|_{C(\bar{\Gamma}_1)} + C_3 \|f_2\|_{C(\bar{\Gamma}_2)} + \|F\|_{C(D)}) = M \end{aligned} \quad (2.24)$$

with the same constant C^* as in (2.10) since $\max_{\bar{D}_h} |L(H_1)| \leq \max_{\bar{D}} |L(H_1)|$ and $\max_{\bar{D}_h} |H_1| \leq \max_{\bar{D}} |H_1|$.

Due to (2.22) and (2.24), we get

$$\begin{aligned} \|\psi_{hh_1}\|_{C(\bar{\Gamma}_{2h_1})} &= \|u_h\|_{C(\bar{\Gamma}_{2h_1})} \leq \\ &\leq C^* (\|f_1\|_{C(\bar{\Gamma}_1)} + C_3 \|f_2\|_{C(\bar{\Gamma}_2)} + \|F\|_{C(D)}). \end{aligned} \quad (2.25)$$

Let us consider the second integral operator in the right-hand side of (2.21) which acts by the formula

$$T\varphi = \int_{\Gamma_{1h_1}} \frac{\partial G_{h_1}}{\partial \nu} \varphi ds. \quad (2.26)$$

According to the results of [7], the operator T is continuous and acts from the space $C(\bar{\Gamma}_{2h_1})$ into the space $C^{2,\alpha}(\bar{D}_{h_2})$, where $h_2 > h_1$,

$$\|T\varphi\|_{C^{2,\alpha}(\bar{D}_{h_2})} \leq C_4 \|\varphi\|_{C(\bar{\Gamma}_{2h_1})}, \quad (2.27)$$

and C_4 is a positive constant not depending on φ .

From (2.22), (2.25) and (2.27), it follows

$$\begin{aligned} \|T\psi_{hh_1}\|_{C^{2,\alpha}(\bar{D}_{h_2})} &\leq \\ &\leq C_4 C^* (\|f_1\|_{C(\bar{\Gamma}_1)} + C_3 \|f_2\|_{C(\bar{\Gamma}_2)} + \|F\|_{C(D)}) \end{aligned} \quad (2.28)$$

for $h < h_1$. Since every bounded in $C^{2,\alpha}(\bar{D}_{h_2})$ set S is precompact in $C^{2,\alpha_1}(\bar{D}_{h_2})$ for $0 < \alpha_1 < \alpha$ [28], by virtue of (2.28) it follows from the representation (2.21) that one can select from the sequence $\{u_h\}_{0 < h < h_1}$ as $h \rightarrow 0$ a subsequence converging in the space $C^{2,\alpha_1}(\bar{D}_{h_2})$, $\alpha_1 < \alpha$. Turning now $h_2 \rightarrow 0$ ($h_2 > h_1 > h > 0$), we select exactly as in §1 a subsequence $u_{h_1^*}, u_{h_2^*}, \dots, u_{h_n^*}, \dots$ converging to a solution u of the equation (2.1) from

the space $C^{2,\alpha_1}(\bar{D} \setminus \bar{\Gamma}_2)$. It is evident that this solution satisfies the boundary condition (2.5), and because of (2.24) it is bounded in D . It remains to determine whether the function $u(x, y)$ equals f_2 on $\Gamma_2 \cup P_1 \cup P_2$.

This question, as is known [11], depends on the existence of a so-called barrier function $v(x, y)$ at every point of $\Gamma_2 \cup P_1 \cup P_2$.

When considering the case (2.9), we will assume that in some neighborhood of Γ_2 , the following representation holds:

$$AH_{2x}^2 + 2BH_{2x}H_{2y} + CH_{2y}^2 = H_2^p G,$$

where $p = \text{const} > 0$, $G > 0$. Then, as is shown in §1, in the case (2.8) as well as in the case (2.9), when one of the conditions 1) $0 < p < 1$; 2) $p = 1$, $(1 - IG^{-1})|_{\bar{\Gamma}_2} > 0$, $I = L(H_2) - cH_2$; 3) $1 < p < 2$, $I|_{\bar{\Gamma}_2} \leq 0$; 4) $p \geq 2$, $I|_{\bar{\Gamma}_2} < 0$; is fulfilled, the function

$$v(x, y) = (x - x_0)^2 + (y - y_0)^2 + H_2^\beta, \quad 0 < \beta < 1,$$

may be a barrier in some neighborhood $P(x_0, y_0) \in \bar{\Gamma}_2$ of the point σ_p . By the definition, the barrier function $v(x, y)$ possesses the following properties: a) it is continuous in $\bar{\sigma}_P$; b) it equals zero at the point P ; c) $v(x, y) > 0$ in $\bar{\sigma}_P \setminus P$; d) it satisfies the condition $L(v) < 0$ everywhere in this neighborhood.

The fact that at every point $P \in \Gamma_2$ the solution $u(x, y)$ takes in the above considered cases the value f_2 can be proved in exactly the same way as in the case of the Dirichlet problem in §1.

It remains to clarify whether the function $u(x, y)$ admits the values f_2 at the end points $P_i(x_i, y_i)$ ($i = 1, 2$) of the arc Γ_2 . ■

Lemma 2.2. *If at the point P_i ($i = 1, 2$) the direction l makes with the interior to the curve $\bar{\Gamma}_2$ normal an obtuse angle, i.e.,*

$$(l \cdot \text{grad } H_2)(P_i) < 0, \quad (2.29)$$

then

$$\lim_{P \rightarrow P_i, P \in \bar{\sigma}_{P_i} \cap \Gamma_1} \Lambda(v_i) = -\infty, \quad (2.30)$$

where $v_i(x, y) = (x - x_i)^2 + (y - y_i)^2 + H_2^\beta$, $0 < \beta < 1$, $l = (l_1, l_2)$.

Proof. We have

$$\Lambda(v_i) = 2l_1(x - x_i) + 2l_2(y - y_i) + \beta H_2^{\beta-1}(l \cdot \text{grad } H_2) + dv_i. \quad (2.31)$$

Since for $0 < \beta < 1$

$$\lim_{P \rightarrow P_i, P \in \bar{\sigma}_{P_i} \cap \Gamma_1} H_2^{\beta-1} = +\infty,$$

from (2.31) by virtue of (2.29) it follows (2.30). Thus the lemma is complete.

Further, owing to the continuity of the function $f_2(P)$, making the neighborhood σ_{P_i} smaller, we will arrive for a given positive number ε at the inequality

$$f_2(P_i) - \varepsilon \leq f_2(P) \leq f_2(P_i) + \varepsilon, \quad P \in \sigma_{P_i}. \quad (2.32)$$

Let us consider two functions:

$$\begin{aligned} \psi_i(P) &= f_2(P_i) + \varepsilon + kv_1(P), \\ \varphi_i(P) &= f_2(P_i) - \varepsilon - k_1v_i(P), \end{aligned} \quad (2.33)$$

where k_1 and k are arbitrary positive numbers.

Since $\lim_{P \rightarrow P_i} (L(v_i))(P) = -\infty$, we have

$$(L(\psi_i))(P) < \max_D |F|, \quad (L(\varphi_i))(P) > \max_D |F|, \quad P \in \sigma_{P_i}, \quad (2.34)$$

if the neighborhood σ_{P_i} is small enough.

In the neighborhood σ_{P_i} , because of (2.32) and (2.33) we have

$$\psi_i(P) \geq f_2(P). \quad (2.35)$$

Denote by $\omega_h \subset D$ the domain which is bounded by the curves $\gamma_1 = \bar{\sigma}_{P_i} \cap \bar{\Gamma}_{2h}$, $\gamma_2 = \bar{\sigma}_{P_i} \cap \bar{\Gamma}_{1h}$ and $\gamma_3 = (\partial\sigma_{P_i} \cap \bar{D}_h) \setminus (\gamma_1 \cup \gamma_2)$. Since $v_i \in C(\bar{\gamma}_3)$ and $v_i|_{\bar{\gamma}_3} > 0$, we have $v_i|_{\bar{\gamma}_3} \geq \sigma_0 = \text{const} > 0$. Therefore the number k in the first formula (2.33) may be chosen such that

$$\psi_i|_{\bar{\gamma}_3} > M, \quad (2.36)$$

where M is taken from the estimate (2.24).

By (2.30), we have

$$\Lambda(\psi_i - u_h)|_{\gamma_2} < 0 \quad (2.37)$$

if the domain σ_{P_i} is small enough.

It follows from (2.19) and (2.35) that

$$(\psi_i - u_h)|_{\gamma_1} \geq 0. \quad (2.38)$$

Analogously, (2.24) and (2.36) imply that

$$(\psi_i - u_h)|_{\bar{\gamma}_3} \geq 0. \quad (2.39)$$

Next, from (2.34) we obtain that in ω_h

$$L(\psi_i - u_h) < 0. \quad (2.40)$$

Let us now show that $(\psi_i - u_h) \geq 0$ in the domain ω_h . Really, because of (2.2), (2.40) and the Hopf extremum principle, the function $(\psi_i - u_h)$ would otherwise take at some point Q of the boundary $\gamma_1 \cup \gamma_2 \cup \gamma_3$ of ω_h its minimal negative value. By (2.38) and (2.39), the point Q does not belong to $\gamma_1 \cup \bar{\gamma}_3$. But according to Zaremba–Giraud’s principle, this point because

of (2.37) cannot likewise belong to γ_2 . The obtained contradiction shows that

$$u_h(P) \leq \psi_i(P), \quad P \in \omega_h. \quad (2.41)$$

Similarly, the number k_1 in the second formula (2.33) may be chosen in such a way that the inequality

$$\varphi_i(P) \leq u_h(P), \quad P \in \omega_h \quad (2.42)$$

would hold.

Now, on the basis of (2.41) and (2.42), we can conclude that in ω_h either

$$\varphi_i(P) \leq u_h(P) \leq \psi_i(P), \quad (i = 1, 2)$$

or

$$\begin{aligned} |u_h(P) - f_2(P_i)| &\leq \varepsilon + k_0 v_i(P), \quad P \in \omega_h, \\ k_0 &= \max(k, k_1). \end{aligned}$$

Passing in this inequality to the limit as $h \rightarrow 0$, we obtain

$$|u(P) - f_2(P_i)| \leq \varepsilon + k_0 v_i(P), \quad P \in \sigma_{P_i}. \quad (2.43)$$

According to the properties of the barrier v_i , there exists $\delta = \delta(\varepsilon) > 0$ such that for $\|P - P_i\| < \delta$ and $P \in \sigma_{P_i}$ we have

$$k_0 v_i(p) < \varepsilon$$

which, because of (2.43), implies that

$$|u(P) - f_2(P_i)| < 2\varepsilon$$

for $\|P - P_i\| < \delta$, $P \in \sigma_{P_i}$. ■

In this direction, one should note the works of S. Zaremba [76], O. A. Oleinik [58, 59], A. D. Vedenskaya [71], and others.

3. THE DIRICHLET PROBLEM FOR SECOND ORDER DEGENERATING ELLIPTIC SYSTEMS

Consider the systems of the form

$$L_1(u) \equiv y^m u_{xx} + u_{yy} + au_x + bu_y + cu = 0, \quad m > 0, \quad (3.1)$$

and

$$L_2(u) \equiv u_{xx} + y^m u_{yy} + au_x + bu_y + cu = 0, \quad m > 0, \quad (3.2)$$

in a simply connected domain D bounded by a segment AB of the x -axis of the line of their degeneration and by a smooth arc σ lying in the half plane $y > 0$ and ending at the points $A(0, 0)$ and $B(1, 0)$. Here a and b are

scalar functions of the class $C^1(\bar{D})$, and $c(x)$ is a given negatively definite $(n \times n)$ -matrix of the class $C^1(\bar{D})$, $n > 1$, that is,

$$(u, cu) \leq c_0(u, u), \quad c_0 = \text{const} < 0, \quad (3.3)$$

$u = (u_1, u_2, \dots, u_n)$ is an unknown n -dimensional vector and (\cdot, \cdot) is the scalar product.

The vector $u(x, y)$ of the class $C^{2,0}(D)$ satisfying the system (3.1) (or (3.2)) in D is referred to as a regular solution of this system.

The Dirichlet Problem. *Find in the domain D a regular solution of the equation (3.1) (or (3.2)) which is continuous in a closed domain \bar{D} and satisfies the boundary condition*

$$u|_{\Gamma} = f, \quad \Gamma = \partial D, \quad (3.4)$$

where $f = (f_1, f_2, \dots, f_n)$ is a given, continuous on Γ vector function.

The following extremum principle holds [12]: when the condition (3.3) is fulfilled, the norm

$$R(x, y) = \left(\sum_{i=1}^n |u_i(x, y)|^2 \right)^{\frac{1}{2}}$$

of a regular in D solution $u = (u_1, u_2, \dots, u_n)$ of the system (3.1) (or (3.2)) cannot reach a nonzero relative maximum at any point $P \in D$.

The uniqueness of the solution of the Dirichlet problem for the system (3.1) (or (3.2)) follows from the above-quoted extremum principle.

To construct a solution of the Dirichlet problem for the system (3.1) (or (3.2)), we take an arbitrary continuous extension of the function f to D and construct an increasing, as $h \rightarrow 0$, sequence of domains $D_h \subset D$ with the smooth boundaries. For all points of the domain D_h and of its boundary, we have $y > 0$. The boundary ∂D_h of D_h coincides with Γ for $y > h$, and beyond some neighborhood of the points A and B runs along the straight line $y = h$.

Let $u_h(x, y)$ be a solution of the Dirichlet problem for the system (3.1) (or (3.2)) admitting the value f on the boundary of the domain D_h . The solution $u_h(x, y)$ in D_h , as is known, does exist and is unique because the system (3.1) (or (3.2)) in the domain \bar{D}_h does not degenerate [12]. By virtue of (3.3), in D_h the inequality $\|u_h\| \leq M$ is valid, where $M = \max \|f(x, y)\|$ in \bar{D} . Let us show that the set of functions $\{u_h(x, y)\}$ is compact in D . Indeed, let h_0 be an arbitrarily fixed value of h . The set $\{u_h(x, y)\}$ for $h \leq h_0$ will be uniformly bounded in D_{h_0} ,

$$\|u_h(x, y)\| \leq M. \quad (3.5)$$

Owing to the extremum principle formulated above for the system (3.1) (or (3.2)), there exists in the domain D_{h_0} a Green function $G_{h_0}(x, y; \xi, \eta)$ of

the Dirichlet problem for the system (3.1) (or (3.2)) and there takes place the following representation [11, 61]:

$$u_h(x, y) = \int_{\partial D_{h_0}} u_h(s) \frac{\partial G_{h_0}(x, y; \xi, \eta)}{\partial \nu} ds \quad (h < h_0), \quad (3.6)$$

where ν is a conormal direction. It follows from (3.5) and (3.6) that the set of functions $\{u_h\}$ is equicontinuous in D_{h_0} . By Arzela's theorem [46], one can select from this set a subsequence uniformly converging to a function $u(x, y)$ which because of (3.6) is a solution of the equation (3.1) (or (3.2)) in D .

To prove that the solution u equals f at every point $Q(x_0, 0) \in AB$, we construct a so-called barrier function $v(x, y)$ satisfying the following properties:

- a) $v(x, y)$ is continuous in some neighborhood σ_{x_0} of the point Q ;
- b) $v(x, y) > 0$ at all points of σ_{x_0} with the exclusion of Q where it vanishes;
- c) $L_1^0(v) \equiv y^m v_{xx} + v_{yy} + av_x + bv_y < 0$ everywhere in the neighborhood σ_{x_0} .

Let us show that we can take as a barrier the function [11]

$$v(x, y) = (x - x_0)^2 + y^\beta, \quad 0 < \beta < 1. \quad (3.7)$$

Indeed, the function $v(x, y)$ obviously satisfies the conditions a) and b). Let us check the condition c). Substituting the expression $v(x, y)$ in $L_1^0(v)$, we get

$$L_1^0(v) = 2y^m + \beta(\beta - 1)y^{\beta-2} + 2(x - x_0)a + \beta by^{\beta-1},$$

whence because of $0 < \beta < 1$ it immediately follows that there exists a neighborhood σ_{x_0} of the point Q at which $L_1^0(v) < 0$.

Given a positive number ε , one can find, due to the continuity of the function f , a semicircular neighborhood $\sigma'_x \subset \sigma_{x_0}$ of the point Q at which the inequality

$$\|f(P) - f(Q)\| \leq \varepsilon, \quad P \in \sigma'_x, \quad 0 < \varepsilon < 1, \quad (3.8)$$

holds.

Consider two functions:

$$v_1(P) = \varepsilon + Kv(P), \quad K > 0, \quad \text{and} \quad u_h^*(P) = u_h(P) - f(Q),$$

where $P \in \sigma'_{x_0}$. In the domain $\omega_h = \sigma'_{x_0} \cap D_h$, where h is a sufficiently small positive number, we have

$$\sup_{(x, y) \in \omega_h} L_1^0(v) \leq \alpha_0 = \text{const} < 0, \quad (3.9)$$

$$L_1^0(u_h^*) = L_1^0(u_h), \quad (3.10)$$

$$\|u_h^*\|_{C(\omega_h)} \leq 2M, \quad M = \max_{\bar{D}} \|f\|. \quad (3.11)$$

Assume $g(P) = (u_h^*(P), u_h^*(P))$, $P \in \omega_h$. By virtue of (3.3), (3.9)–(3.11), for sufficiently large K we have

$$\begin{aligned}
L_1^0(v - g) &= L_1^0(\varepsilon) + KL_1^0(v) - L_1^0(g) = KL_1^0(v) - \\
&- 2y^m[(u_{hx}^*, u_{hx}^*) + (u_h^*, u_{hxx}^*)] - 2[(u_{hy}^*, u_{hy}^*) + (u_h^*, u_{hyy}^*)] - \\
&- 2a(u_h^*, u_{hx}^*) - 2b(u_h^*, u_{hy}^*) = KL_1^0(v) - 2[y^m(u_{hx}^*, u_{hx}^*) + \\
&+ (u_{hy}^*, u_{hy}^*)] - 2(u_h^*, y^m u_{hxx}^* + u_{hyy}^* + au_{hx}^* + bu_{hy}^*) \leq \\
\leq KL_1^0(v) - 2(u_h^*, L_1^0(u_h^*)) &= KL_1^0(v) - 2(u_h(P) - f(Q), L_1^0(u_h)) = \\
= KL_1^0(v) - 2(u_h(P) - f(Q), -cu_h) &= KL_1^0(v) + 2(u_h, cu_h) - \\
&- 2(f(Q), cu_h) \leq K\alpha_0 - 2(f(Q), cu_h) \leq \\
\leq K\alpha_0 + 2\|f(Q)\|\|c\|\|u_h\| &\leq K\alpha_0 + 2M^2\|c\| < 0. \tag{3.12}
\end{aligned}$$

Let us now clarify what sign has $v_1 - g$ on the boundary of the domain ω_h , $\partial\omega_h = \gamma_h \cup \gamma_{1h}$, where $\gamma_{1h} = \partial\omega_h \cap \partial D_h$, $\gamma_h = \partial\omega_h \setminus \gamma_{1h}$. We have

$$\left. \begin{aligned}
g|_{\gamma_{1h}} &= (u_h(P) - f(Q), u_h(P) - f(Q))|_{\gamma_{1h}} = \\
&= \|f(P) - f(Q)\|^2 < \varepsilon^2 < \varepsilon, \\
v_1|_{\gamma_{1h}} &= \varepsilon + Kv(P)|_{\gamma_{1h}} \geq \varepsilon + K \min_{\gamma_{1h}} v(P), \\
g|_{\gamma_h} &\leq 4M^2.
\end{aligned} \right\} \tag{3.13}$$

By (3.13), for sufficiently large K we have

$$\begin{aligned}
(v_1 - g)|_{\gamma_{1h}} &= (\varepsilon + Kv - g)|_{\gamma_{1h}} \geq \varepsilon + K \min_{P \in \gamma_{1h}} v - \varepsilon = K \min_{P \in \gamma_{1h}} v > 0, \\
(v_1 - g)|_{\gamma_h} &= (\varepsilon + Kv - g)|_{\gamma_h} \geq \varepsilon + K \min_{P \in \gamma_h} v - 4M^2 > 0.
\end{aligned}$$

According to the extremum principle [12], (3.12) and (3.13) result in

$$(v_1 - g)|_{\omega_h} \geq 0,$$

whence $g \leq v_1$ over all domain ω_h , that is,

$$(u_h(P) - f(Q), u_h(P) - f(Q)) \leq \varepsilon + Kv(P). \tag{3.14}$$

Passing in the inequality (3.14) to the limit as $h \rightarrow 0$, we obtain

$$(u(P) - f(Q), u(P) - f(Q)) \leq \varepsilon + Kv(P), \quad P \in \sigma'_{x_0}. \tag{3.15}$$

Due to the property of the barrier v , there exists $\delta = \delta(\varepsilon) > 0$ such that for $\|P - Q\| < \delta$ and $P \in \sigma'_{x_0}$ we have

$$Kv(P) < \varepsilon. \tag{3.16}$$

The inequalities (3.15) and (3.16) imply

$$\|u(P) - f(Q)\| \leq \sqrt{2\varepsilon}$$

for $\|P - Q\| < \delta$.

Thus the following theorem is valid.

Theorem 3.1. *The Dirichlet problem (3.1), (3.4) has always a unique solution.*

Problem E. *Find a regular in the domain D solution of the system (3.2) which remains bounded as $y \rightarrow 0$ and coincides with a prescribed continuous function f only on the curve σ .*

Lemma 3.1. *If there exists a positive in $D \cup \sigma$ function $W(x, y)$ uniformly tending to infinity as $y \rightarrow 0$ and satisfying the inequality $L_2^0(W) < 0$, then the solution of Problem E is unique.*

Proof. Let $u(x, y)$ be a solution of the system (3.2) equal to zero on σ . Consider the expression

$$L_2^0(\varepsilon W - (u, u)) = \varepsilon L_2^0(W) - L_2^0((u, u)).$$

Calculating $L_2^0((u, u))$, because of (3.3) we arrive at

$$\begin{aligned} L_2^0((u, u)) &= 2[(u_x, u_x) + (u, u_{xx})] + 2y^m[(u_y, u_y) + (u, u_{yy})] + \\ &+ 2a(u, u_x) + 2b(u, u_y) = 2[(u_x, u_x) + y^m(u_y, u_y)] + \\ &+ 2[u, u_{xx} + y^m u_{yy} + au_x + bu_y] = \\ &= 2[(u_x, u_x) + y^m(u_y, u_y)] + 2(u, -cu) \geq 0. \end{aligned} \quad (3.17)$$

It follows from the conditions of the lemma and also from (3.17) that

$$L_2^0(\varepsilon W - (u, u)) < 0. \quad (3.18)$$

Owing to (3.18) and the extremum principle, the function $\varepsilon W - (u, u)$ is unable to have in D a negative minimum and, since its values on the boundary are positive, everywhere in D we have $(u, u) \leq \varepsilon W$. This, due to the fact that $\varepsilon > 0$ is arbitrary, implies that $\|u\| = 0$. ■

Lemma 3.2. *Under the conditions of Lemma 3.1, for every continuous on σ data there exists a solution of Problem E.*

The proof of this lemma is carried out exactly in the same way as that of Theorem 3.1.

Theorem 3.2. *If m and $b(x, 0)$ satisfy one of the conditions*

- 1) $0 < m < 1$;
- 2) $m = 1$, $b(x, 0) < 1$;
- 3) $1 < m < 2$, $b(x, 0) \leq 0$;
- 4) $m \geq 2$, $b(x, 0) < 0$,

then there exists a unique solution of the Dirichlet problem (3.2), (3.4).

Theorem 3.3. *If m and $b(x, 0)$ satisfy one of the conditions*

- 1) $m = 1$, $b(x, 0) \geq 1$;
- 2) $1 < m < 2$, $b(x, 0) > 0$;
- 3) $m \geq 2$, $b(x, 0) \geq 0$,

then there exists a unique solution of Problem E.

To prove the theorems it should be noted that one has to take (3.7) for all cases when we state the existence of the Dirichlet problem and, by Lemma 3.1, the function [61]

$$W(x, y) = -\log y - (x - \delta)^2 + K, \quad K > 0, \quad (x - \delta) > 1,$$

for all cases when we state the uniqueness of the solution of Problem *E*.

The works carried out in this direction by A. V. Bitsadze [13], V. P. Didenko [20, 21], E. A. Baderko [6] and others are noteworthy.

CHAPTER II
**BOUNDARY VALUE PROBLEM OF CONJUGATION
 TYPE FOR DEGENERATING SECOND ORDER HYPERBOLIC
 SYSTEMS WITH DISCONTINUOUS COEFFICIENTS**

4. STATEMENT OF THE PROBLEM

In the plane of the variables x, y , let us consider a system of linear differential equations

$$L(u) \equiv \begin{cases} y^m A_1 u_{xx}^+ + 2y^{\frac{m}{2}} B_1 u_{xy}^+ + C_1 u_{yy}^+ + a_1 u_x^+ + \\ \quad + b_1 u_y^+ + c_1 u^+ = F_1, & x > 0, \\ y^m A_2 u_{xx}^- + 2y^{\frac{m}{2}} B_2 u_{xy}^- + C_2 u_{yy}^- + a_2 u_x^- + \\ \quad + b_2 u_y^- + c_2 u^- = F_2, & x < 0, \end{cases} \quad (4.1)$$

where $A_i, B_i, C_i, a_i, b_i, c_i$ ($i = 1, 2$) are given real $(n \times n)$ -matrices, F_i ($i = 1, 2$) is a given n -dimensional vector, and u^\pm is an unknown one, $m = \text{const} > 0, n > 1$.

In what follows, A_i, B_i, C_i ($i = 1, 2$) are assumed to be constant matrices, $\det C_i \neq 0$ ($i = 1, 2$), and the polynomials $P_1(\lambda) = \det(A_1 + 2B_1\lambda + C_1\lambda^2)$, $P_2(\mu) = \det(A_2 + 2B_2\mu + C_2\mu^2)$ are assumed to have only simple real roots $\lambda_1^*, \lambda_2^*, \dots, \lambda_{2n}^*$ and $\mu_1^*, \mu_2^*, \dots, \mu_{2n}^*$ satisfying

$$\begin{aligned} \lambda_1^* < \lambda_2^* < \dots < \lambda_n^* < 0 < \lambda_{n+1}^* < \lambda_{n+2}^* < \dots < \lambda_{2n}^*, \\ \mu_1^* < \mu_2^* < \dots < \mu_n^* < 0 < \mu_{n+1}^* < \mu_{n+2}^* < \dots < \mu_{2n}^*. \end{aligned} \quad (4.2)$$

In this case, the system (4.1) is strictly hyperbolic for $y > 0$ and parabolically degenerates for $y = 0$. Under these conditions, the values $y^{\frac{m}{2}} \lambda_1^*, \dots, y^{\frac{m}{2}} \lambda_{2n}^*$ and $y^{\frac{m}{2}} \mu_1^*, \dots, y^{\frac{m}{2}} \mu_{2n}^*$ are respectively the roots of the characteristic polynomials

$$\begin{aligned} p_1(y, \lambda^*) &= \det(y^m A_1 + 2y^{\frac{m}{2}} B_1 \lambda^* + C_1 \lambda^{*2}), \\ p_2(y, \mu^*) &= \det(y^m A_2 + 2y^{\frac{m}{2}} B_2 \mu^* + C_2 \mu^{*2}) \end{aligned}$$

of the system (4.1), while the characteristics of the systems (4.1) passing through the point $P(x_0, y_0)$, $y_0 > 0$, satisfy the equations

$$\begin{aligned} x + \frac{2\lambda_i^*}{m+2} y^{\frac{m+2}{2}} &= x_0 + \frac{2\lambda_i^*}{m+2} y_0^{\frac{m+2}{2}}, \quad x_0 > 0, \quad i = 1, 2, \dots, 2n, \\ x + \frac{2\mu_j^*}{m+2} y^{\frac{m+2}{2}} &= x_0 + \frac{2\mu_j^*}{m+2} y_0^{\frac{m+2}{2}}, \quad x_0 < 0, \quad j = 1, 2, \dots, 2n. \end{aligned}$$

Let D be a finite domain lying in the upper half-plane $y > 0$ and bounded by two characteristics of the system (4.1) going out of the origin $O(0, 0)$

$$\gamma_1 : x + \frac{2\lambda_n^*}{m+2} y^{\frac{m+2}{2}} = 0, \quad \gamma_2 : x + \frac{2\mu_{n+1}^*}{m+2} y^{\frac{m+2}{2}} = 0,$$

and by two characteristics going out of the point $O_1(0, y_0)$

$$\begin{aligned}\gamma_3 : x + \frac{2\lambda_{2n}^*}{m+2}y^{\frac{m+2}{2}} &= \frac{2\lambda_{2n}^*}{m+2}y_0^{\frac{m+2}{2}}, \\ \gamma_4 : x + \frac{2\mu_1^*}{m+2}y^{\frac{m+2}{2}} &= \frac{2\mu_1^*}{m+2}y_0^{\frac{m+2}{2}},\end{aligned}$$

where $y_0 > 0$ is an arbitrary fixed number. Denote by P_1 and P_2 the points of intersection of the characteristics γ_1 and γ_2 with γ_3 and γ_4 , respectively. By $D^+ \subset D$ we denote the domain bounded by the curves γ_1, γ_3 and the straight line $x = 0$, and by $D^- \subset D$ the domain bounded by curves γ_2, γ_4 and the straight line $x = 0$.

Consider the characteristic problem formulated as follows: Find a regular solution

$$u(x, y) = \begin{cases} u^+(x, y), & (x, y) \in D^+, \\ u^-(x, y), & (x, y) \in D^- \end{cases}$$

of the system (4.1) satisfying both the boundary conditions

$$\left(y^{\frac{m}{2}} M_1 \frac{\partial u^+}{\partial x} + N_1 \frac{\partial u^+}{\partial y} + S_1 u^+ \right) \Big|_{OP_1} = f_1, \quad (4.3)$$

$$\left(y^{\frac{m}{2}} M_2 \frac{\partial u^-}{\partial x} + N_2 \frac{\partial u^-}{\partial y} + S_2 u^- \right) \Big|_{OP_2} = f_2, \quad (4.4)$$

and the conditions of conjugation on OO_1

$$u^+(0, y) - \Lambda_1 u^-(0, y) = g_1(y), \quad 0 \leq y \leq y_0, \quad (4.5)$$

$$u_x^+(0, y) - \Lambda_2 u_x^-(0, y) = g_2(y), \quad 0 \leq y \leq y_0, \quad (4.6)$$

where M_i, N_i, S_i, Λ_i ($i = 1, 2$) are given real $(n \times n)$ -matrices; moreover for the sake of simplicity, Λ_i ($i = 1, 2$) are assumed to be constant matrices, and f_i, g_i ($i = 1, 2$) are given real n -dimensional vectors.

Below we assume that $a_1, b_1, c_1, F_1 \in C^1(D^+)$, $a_2, b_2, c_2, F_2 \in C^1(D^-)$, $M_i, N_i, S_i, f_i \in C^1(OP_i)$ ($i = 1, 2$), $g_i \in C^1(OO_1)$ ($i = 1, 2$) and, moreover,

in D^+ and D^-

$$\begin{aligned}
& \sup_{\bar{D}^+ \setminus O} \|y^{1-\frac{m}{2}} a_1\| < \infty, \quad \sup_{\bar{D}^+ \setminus O} \|y^{1-\frac{m}{2}} a_{1x}\| < \infty, \\
& \sup_{D^- \setminus O} \|y^{1-\frac{m}{2}} a_2\| < \infty, \quad \sup_{D^- \setminus O} \|y^{1-\frac{m}{2}} a_{2x}\| < \infty, \\
& \sup_{\bar{D}^+ \setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} F_1\| < \infty, \quad \sup_{\bar{D}^+ \setminus O} \|y^{-(\alpha-2)} F_{1x}\| < \infty, \\
& \sup_{D^- \setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} F_2\| < \infty, \quad \sup_{D^- \setminus O} \|y^{-(\alpha-2)} F_{2x}\| < \infty, \quad (4.7) \\
& f_i(0) = g_i(0) = 0 \quad (i = 1, 2), \quad \alpha = \text{const} > 0, \\
& \sup_{OP_i \setminus O} \|y^{-(\alpha+\frac{m}{2})} f_i\| < \infty, \quad \sup_{OP_i \setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} f'_i\| < \infty, \\
& \sup_{OO_i \setminus O} \|y^{-(\alpha+\frac{m}{2})} g_i\| < \infty, \quad \sup_{OO_i \setminus O} \|y^{-(\alpha+\frac{m}{2}-1)} g'_i\| < \infty, \quad i = 1, 2.
\end{aligned}$$

Note also that the conditions

$$\sup_{\bar{D}^\pm \setminus O} \|y_i^{1-\frac{m}{2}}\| < \infty, \quad i = 1, 2,$$

imposed on the lowest coefficients of the system (4.1) are the analogues of the well-known Gellerstedt's condition for one equation.

The solution of the problem (4.1), (4.3)–(4.6) will be sought in the class

$$\left\{ u^\pm \in C^2(\bar{D}^\pm) : u^\pm(0, 0) = 0, \sup_{\bar{D}^+ \setminus O} \|y^{-\alpha} u_x^+\| < \infty, \sup_{\bar{D}^- \setminus O} \|y^{-\alpha} u_x^-\| < \infty, \right. \\
\left. \sup_{\bar{D}^+ \setminus O} \|y^{-(\alpha+\frac{m}{2})} u_y^+\| < \infty, \sup_{\bar{D}^- \setminus O} \|y^{-(\alpha+\frac{m}{2})} u_y^-\| < \infty \right\}. \quad (4.8)$$

It should be noted that some variants of characteristic problems for second order hyperbolic systems with parabolic degeneration have been studied by S.S.Kharibegashvili [41, 42], while for systems of the form

$$K(y)u_{xx} - Eu_{yy} + au_x + bu_y + cu = F$$

by M. Meredov [49]. For one second order hyperbolic equation with parabolic degeneration of the form

$$y^m u_{xx} - u_{yy} + au_x + bu_y + cu = f$$

in a quadrangle bounded by the characteristics going out of the points $O(0, 0)$ and $P(0, 1)$, the characteristic problem with boundary conditions on pieces of characteristics going out of the origin $O(0, 0)$ has been investigated by A. Sh. Agababyan and A. B. Nersesyan [1]. In the case of a triangular domain bounded by the segment $[0, 1]$ of the x -axis and by pieces

of characteristics going out of the points $O(0, 0)$ and $Q(1, 0)$, the characteristic problem for the equation

$$y^2 u_{xx} - u_{yy} + au_x = 0$$

is considered in T. Sh. Kalmenov's work [39]. The works of V. N. Vragov [75] and B. A. Bubnov [14] are also worth mentioning in which, in particular, the characteristic problem was treated in domains containing a piece of the line of degeneration. The case where OP_1 is a segment of the x -axis and OP_2 is a piece of a characteristic for one hyperbolic equation with parabolic degeneration is studied in their works by V. N. Vragov [75] and A. M. Nakhushev [55, 56].

Let us renumber $\lambda_i^*, \mu_i^*, i = 1, 2, \dots, 2n$, from (4.2) as follows:

$$\begin{aligned} \lambda_1 &= \lambda_{n+1}^*, \quad \lambda_2 = \lambda_{n+2}^*, \dots, \lambda_n = \lambda_{2n}^*, \quad \lambda_{n+1} = \lambda_1^*, \dots, \lambda_{2n} = \lambda_n^*, \\ \mu_1 &= \mu_{n+1}^*, \quad \mu_2 = \mu_{n+2}^*, \dots, \mu_n = \mu_{2n}^*, \quad \mu_{n+1} = \mu_1^*, \dots, \mu_{2n} = \mu_n^*. \end{aligned}$$

Since the roots $\lambda_1, \lambda_2, \dots, \lambda_{2n}$ and $\mu_1, \mu_2, \dots, \mu_{2n}$ of the polynomials $p_1(\lambda)$ and $p_2(\lambda)$ are simple, we have

$$\begin{aligned} \dim \text{Ker}(A_1 + 2B_1\lambda_i + C_1\lambda_i^2) &= 1, \\ \dim \text{Ker}(A_2 + 2B_2\mu_j + C_2\mu_j^2) &= 1, \quad 1 \leq i, \quad j \leq 2n. \end{aligned}$$

Denote by ν_i and ν_j^* vectors, satisfy $\nu_i \in \text{Ker}(A_1 + 2B_1\lambda_i + C_1\lambda_i^2)$, $\|\nu_i\| \neq 0$, $\nu_j^* \in \text{Ker}(A_2 + 2B_2\mu_j + C_2\mu_j^2)$, $\|\nu_j^*\| \neq 0$, $1 \leq i, j \leq 2n$, where $\|\cdot\|$ denotes the norm in R^n .

5. SOME STRUCTURAL PROPERTIES OF THE HYPERBOLIC SYSTEM (4.1)

We introduce into consideration the $(2n \times 2n)$ -matrices

$$\begin{aligned} A_0 &= \begin{vmatrix} 0 & -E \\ C_1^{-1}A_1 & 2C_1^{-1}B_1 \end{vmatrix}, \quad \tilde{A}_0 = \begin{vmatrix} 0 & -E \\ y^m C_1^{-1}A_1 & 2y^{\frac{m}{2}} C_1^{-1}B_1 \end{vmatrix}, \\ A_0^* &= \begin{vmatrix} 0 & -E \\ C_2^{-1}A_2 & 2C_2^{-1}B_2 \end{vmatrix}, \quad \tilde{A}_0^* = \begin{vmatrix} 0 & -E \\ y^m C_2^{-1}A_2 & 2y^{\frac{m}{2}} C_2^{-1}B_2 \end{vmatrix}, \\ K &= \begin{pmatrix} \nu_1 & \dots & \nu_{2n} \\ \lambda_1 \nu_1 & \dots & \lambda_{2n} \nu_{2n} \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} y^{-\frac{m}{2}} \nu_1 & \dots & y^{-\frac{m}{2}} \nu_{2n} \\ \lambda_1 \nu_1 & \dots & \lambda_{2n} \nu_{2n} \end{pmatrix}, \\ K^* &= \begin{pmatrix} \nu_1^* & \dots & \nu_{2n}^* \\ \mu_1 \nu_1^* & \dots & \mu_{2n} \nu_{2n}^* \end{pmatrix}, \quad \tilde{K}^* = \begin{pmatrix} y^{-\frac{m}{2}} \nu_1^* & \dots & y^{-\frac{m}{2}} \nu_{2n}^* \\ \mu_1 \nu_1^* & \dots & \mu_{2n} \nu_{2n}^* \end{pmatrix}, \end{aligned}$$

where E is the unit $(n \times n)$ -matrix.

It can be easily shown that

$$\begin{aligned} K^{-1}A_0K &= D_1, & \tilde{K}^{-1}\tilde{A}_0\tilde{K} &= \tilde{D}_1, \\ K^{*-1}A_0^*K^* &= D_1^*, & \tilde{K}^{*-1}\tilde{A}_0^*\tilde{K}^* &= \tilde{D}_1^*. \end{aligned} \tag{5.1}$$

Here $D_1 = \text{diag}(-\lambda_1, \dots, -\lambda_{2n})$, $\tilde{D}_1 = \text{diag}(-y^{-\frac{m}{2}}\lambda_1, \dots, -y^{-\frac{m}{2}}\lambda_{2n})$, $D_1^* = \text{diag}(-\mu_1, \dots, -\mu_{2n})$, $\tilde{D}_1^* = \text{diag}(-y^{-\frac{m}{2}}\mu_1, \dots, -y^{-\frac{m}{2}}\mu_{2n})$.

Suppose

$$\begin{aligned} \Gamma_i &= (M_i, N_i), \quad i = 1, 2, \\ K &= \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = (V_1, V_2), \quad K^{-1} = (K_1^0, K_2^0), \\ K^* &= \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} = (V_1^*, V_2^*), \quad K^{*-1} = (K_1^{*0}, K_2^{*0}), \end{aligned} \quad (5.2)$$

where K_1, K_2, K_1^*, K_2^* are matrices of the order $(n \times 2n)$ and $V_1, V_2, V_1^*, V_2^*, K_1^0, K_2^0, K_1^{*0}, K_2^{*0}$ are $(2n \times n)$ -matrices.

From (5.2) it directly follows that

$$\begin{aligned} \tilde{K} &= \begin{pmatrix} y^{-\frac{m}{2}}K_1 \\ K_2 \end{pmatrix}, \quad \tilde{K}^{-1} = (y^{\frac{m}{2}}K_1^0, K_2^0), \\ \tilde{K}^* &= \begin{pmatrix} y^{-\frac{m}{2}}K_1^* \\ K_2^* \end{pmatrix}, \quad \tilde{K}^{*-1} = (y^{\frac{m}{2}}K_1^{*0}, K_2^{*0}). \end{aligned} \quad (5.3)$$

By (5.2) and (5.3), we have

$$\begin{aligned} \tilde{K}_y &= -\frac{m}{2} \begin{pmatrix} y^{-\frac{m}{2}-1}K_1 \\ 0 \end{pmatrix}, \quad \tilde{K}^{-1}\tilde{K}_y = -\frac{m}{2y}K_1^0 \times K_1, \\ \tilde{K}_y^* &= -\frac{m}{2} \begin{pmatrix} y^{-\frac{m}{2}-1}K_1^* \\ 0 \end{pmatrix}, \quad \tilde{K}^{*-1}\tilde{K}_y^* = -\frac{m}{2y}K_1^{*0} \times K_1^*. \end{aligned} \quad (5.4)$$

If

$$B_0 = \left\| \begin{array}{cc} 0 & 0 \\ C_1^{-1}a_1 & C_1^{-1}b_1 \end{array} \right\|, \quad B_0^* = \left\| \begin{array}{cc} 0 & 0 \\ C_2^{-1}a_2 & C_2^{-1}b_2 \end{array} \right\|,$$

then obviously,

$$\tilde{K}^{-1}B_0\tilde{K} = \frac{1}{y}\tilde{B}_0 + \tilde{B}_1, \quad \tilde{K}^{*-1}B_0^*\tilde{K}^* = \frac{1}{y}\tilde{B}_0^* + \tilde{B}_1^*,$$

where

$$\begin{aligned} \tilde{B}_0 &= y^{1-\frac{m}{2}}K_2^0C_1^{-1}a_1K_1, \quad \tilde{B}_1 = K_2^0C_1^{-1}b_1K_2, \\ \tilde{B}_0^* &= y^{1-\frac{m}{2}}K_2^{*0}C_2^{-1}a_2K_1^*, \quad \tilde{B}_1^* = K_2^{*0}C_2^{-1}b_2K_2^*. \end{aligned}$$

Since by assumption

$$\begin{aligned} \sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{1-\frac{m}{2}}a_1\| &< \infty, \quad \sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{1-\frac{m}{2}}a_{1x}\| < \infty, \\ \sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{1-\frac{m}{2}}a_2\| &< \infty, \quad \sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{1-\frac{m}{2}}a_{2x}\| < \infty, \end{aligned}$$

we have

$$\sup_{\bar{D}^+ \setminus \mathcal{O}} \|\tilde{B}_0\| = \sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{1-\frac{m}{2}}K_2^0C_1^{-1}a_1K_1\| < \infty, \quad (5.5)$$

$$\sup_{\bar{D}^+ \setminus \mathcal{O}} \|\tilde{B}_{0x}\| = \sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{1-\frac{m}{2}} K_2^0 C_1^{-1} a_{1x} K_1\| < \infty, \quad (5.6)$$

$$\sup_{\bar{D}^- \setminus \mathcal{O}} \|\tilde{B}_0^*\| = \sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{1-\frac{m}{2}} K_2^{*0} C_2^{-1} a_2 K_1^*\| < \infty, \quad (5.7)$$

$$\sup_{\bar{D}^- \setminus \mathcal{O}} \|\tilde{B}_{0x}^*\| = \sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{1-\frac{m}{2}} K_2^{*0} C_2^{-1} a_{2x} K_1^*\| < \infty. \quad (5.8)$$

6. REDUCTION OF THE PROBLEM (4.1), (4.3)–(4.6) TO A SYSTEM OF INTEGRAL FUNCTIONAL EQUATIONS AND ITS INVESTIGATION

It can be easily verified that in the class (4.8) the problem (4.1), (4.3)–(4.6) can be equivalently rewritten in the form

$$v_y^+ + \tilde{A}_0 v_x^+ + B_0 v^+ + C_0 u^{0+} = F_0, \quad (6.1)$$

$$\left(-y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^+ = -y^{\frac{m}{2}} \lambda_1 v_1^+ + v_2^+, \quad (6.2)$$

$$\left. \begin{aligned} & (y^{\frac{m}{2}} M_1 v_1^+ + N_1 v_2^+ + S_1 u^+) \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\right) = f_1(y), \\ & \quad 0 \leq y \leq d_1, \\ & \left[\left(-y^{\frac{m}{2}} \lambda_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^+ - (-y^{\frac{m}{2}} \lambda_{2n} v_1^+ + v_2^+) \right] \times \\ & \quad \times \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\right) = 0, \end{aligned} \right\} \quad (6.3)$$

$$(v_2^+ - \Lambda_1 v_2^-)(0, y) = g_1'(y), \quad 0 \leq y \leq y_0, \quad (6.4)$$

$$v_y^- + \tilde{A}_0^* v_x^- + B_0^* v^- + C_0^* u^{0-} = F_0^*, \quad (6.5)$$

$$\left(-y^{\frac{m}{2}} \mu_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^- = -y^{\frac{m}{2}} \mu_{2n} v_1^- + v_2^-, \quad (6.6)$$

$$\left. \begin{aligned} & (y^{\frac{m}{2}} M_2 v_1^- + N_2 v_2^- + S_2 u^-) \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y\right) = f_2(y), \\ & \quad 0 \leq y \leq d_2, \\ & \left[\left(-y^{\frac{m}{2}} \mu_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^- - (-y^{\frac{m}{2}} \mu_1 v_1^- + v_2^-) \right] \times \\ & \quad \times \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y\right) = 0, \end{aligned} \right\} \quad (6.7)$$

$$(v_1^+ - \Lambda_2 v_1^-)(0, y) = g_2(y), \quad 0 \leq y \leq y_0. \quad (6.8)$$

Here d_i is the ordinate of the point $P_i \in \gamma_i$, $i = 1, 2$,

$$C_0 = \left\| \begin{array}{cc} 0 & 0 \\ 0 & C_1^{-1} c_1 \end{array} \right\|, \quad C_0^* = \left\| \begin{array}{cc} 0 & 0 \\ 0 & C_2^{-1} c_2 \end{array} \right\|,$$

$$u^{0+} = (0, u^+), \quad u^{0-} = (0, u^-), \quad F_0 = (0, C_1^{-1} F_1), \quad F_0^* = (0, C_2^{-1} F_2),$$

$$v_1^+ = u_x^+, \quad v_1^- = u_x^-, \quad v_2^+ = u_y^+, \quad v_2^- = u_y^-, \quad v^+ = (v_1^+, v_2^+), \quad v^- = (v_1^-, v_2^-),$$

$$\begin{aligned}
v_i^+ &\in C^1(\bar{D}^+), \quad i = 1, 2, \quad v_i^- \in C^1(\bar{D}^-), \quad i = 1, 2, \\
\sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{-\alpha} v_1^+\| &< \infty, \quad \sup_{\bar{D}^+ \setminus \mathcal{O}} \|y^{-(\alpha + \frac{m}{2})} v_2^+\| < \infty, \\
\sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{-\alpha} v_1^-\| &< \infty, \quad \sup_{\bar{D}^- \setminus \mathcal{O}} \|y^{-(\alpha + \frac{m}{2})} v_2^-\| < \infty.
\end{aligned} \tag{6.9}$$

As a result of the substitution of the unknown functions $v^+ = \tilde{K}W^+$ and $v^- = \tilde{K}^*W^-$, by virtue of (5.1) instead of (6.1)–(6.8) we will have

$$W_y^+ + \tilde{D}_1 W_x^+ = B_2 W^+ + C_2 u^{0+} + F_1^*, \tag{6.10}$$

$$\left(-y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^+ = \left(-y^{\frac{m}{2}} \lambda_1 \tilde{K}_1 + \tilde{K}_2\right) W^+, \tag{6.11}$$

$$\left. \begin{aligned}
&\left[\left(y^{\frac{m}{2}} M_1 \tilde{K}_1 + N_1 \tilde{K}_2\right) W^+ + S_1 u^+\right] \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\right) = f_1(y), \\
&\quad 0 \leq y \leq d_1, \\
&\left[\left(-y^{\frac{m}{2}} \lambda_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^+ - \left(-y^{\frac{m}{2}} \lambda_{2n} \tilde{K}_1 + \tilde{K}_2\right) W^+\right] \times \\
&\quad \times \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y\right) = 0, \quad 0 \leq y \leq d_2,
\end{aligned} \right\} \tag{6.12}$$

$$(\tilde{K}_2 W^+ - \Lambda_1 \tilde{K}_2^* W^-)(0, y) = g_1'(y), \quad 0 \leq y \leq y_0, \tag{6.13}$$

$$W_y^- + \tilde{D}_1^* W_x^- = B_2^* W^- + C_2^* u^{0-} + F_2^*, \tag{6.14}$$

$$\left(-y^{\frac{m}{2}} \mu_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^- = \left(-y^{\frac{m}{2}} \mu_{2n} \tilde{K}_1^* + \tilde{K}_2^*\right) W^-, \tag{6.15}$$

$$\left. \begin{aligned}
&\left[\left(y^{\frac{m}{2}} M_2 \tilde{K}_1^* + N_2 \tilde{K}_2^*\right) W^- + S_2 u^-\right] \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y\right) = f_2(y), \\
&\quad 0 \leq y \leq d_2, \\
&\left[\left(-y^{\frac{m}{2}} \mu_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u^- - \left(-y^{\frac{m}{2}} \mu_1 \tilde{K}_1^* + \tilde{K}_2^*\right) W^-\right] \times \\
&\quad \times \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y\right) = 0, \quad 0 \leq y \leq d_2,
\end{aligned} \right\} \tag{6.16}$$

$$(\tilde{K}_1 W^+ - \Lambda_2 \tilde{K}_1^* W^-)(0, y) = g_2(y), \quad 0 \leq y \leq y_0, \tag{6.17}$$

where $B_2 = -\tilde{K}^{-1} \tilde{K}_y - \tilde{K}^{-1} B_0 \tilde{K}$, $C_2 = -\tilde{K}^{-1} C_0$, $F_1^* = \tilde{K}^{-1} F_0$, $B_2^* = -\tilde{K}^{*-1} \tilde{K}_y^* - \tilde{K}^{*-1} B_0 \tilde{K}^*$, $C_2^* = \tilde{K}^{*-1} C_0^*$, $F_2^* = \tilde{K}^{*-1} F_0^*$; $\tilde{K}_1(\tilde{K}_1^*)$ and $\tilde{K}_2(\tilde{K}_2^*)$ are the $n \times 2n$ -matrices composed respectively of the first and the last n rows of the matrix $\tilde{K}(\tilde{K}^*)$.

By (5.2)–(5.4), we have

$$\begin{aligned}
\tilde{K}_1 &= y^{-\frac{m}{2}} K_1, \quad \tilde{K}_2 = K_2, \quad B_2 = \frac{m}{2y} K_1^0 \times K_1 - \frac{1}{y} \tilde{B}_0 - \tilde{B}_1, \\
y^{\frac{m}{2}} M_1 \tilde{K}_1 + N_1 \tilde{K}_2 &= M_1 K_1 + N_1 K_2 = \Gamma_1 \times K, \quad \Gamma_1 = (M_1, N_1), \\
-y^{\frac{m}{2}} \lambda_{2n} \tilde{K}_1 + \tilde{K}_2 &= -\lambda_{2n} K_1 + K_2, \\
\tilde{K}_1^* &= y^{-\frac{m}{2}} K_1^*, \quad \tilde{K}_2^* = K_2^*, \quad B_2^* = \frac{m}{2y} K_1^{*0} \times K_1^* - \frac{1}{y} \tilde{B}_0^* - \tilde{B}_1^*, \quad (6.18) \\
y^{\frac{m}{2}} M_2 \tilde{K}_1^* + N_2 \tilde{K}_2^* &= M_2 K_1^* + N_2 K_2^* = \Gamma_2 \times K^*, \quad \Gamma_2 = (M_2, N_2), \\
-y^{\frac{m}{2}} \mu_1 \tilde{K}_1^* + \tilde{K}_2^* &= -\mu_1 K_1^* + K_2^*.
\end{aligned}$$

Taking into account (6.18), we rewrite the problem (6.1)–(6.17) as

$$W_y^+ + \tilde{D}_1 W_x^+ = \frac{1}{y} (B_3 W^+ + y C_2 u^{0+}) + F_1^*, \quad (6.19)$$

$$\left(-y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^+ = (-\lambda_1 K_1 + K_2) W^+, \quad (6.20)$$

$$\left. \begin{aligned}
& [(\Gamma_1 \times K) W^+ + S_1 u^+] \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y \right) = f_1(y), \\
& \quad \quad \quad 0 \leq y \leq d_1, \\
& \left[\left(-y^{\frac{m}{2}} \lambda_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^+ - (-\lambda_{2n} K_1 + K_2) W^+ \right] \times \\
& \quad \quad \quad \times \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y \right) = 0, \quad 0 \leq y \leq d_1,
\end{aligned} \right\} \quad (6.21)$$

$$(\tilde{K}_2 W^+ - \Lambda_1 \tilde{K}_2^* W^-)(0, y) = g_1'(y), \quad 0 \leq y \leq y_0, \quad (6.22)$$

$$W_y^- + \tilde{D}_1^* W_x^- = \frac{1}{y} (B_3^* W^- + y C_2^* u^{0-}) + F_2^+, \quad (6.23)$$

$$\left(-y^{\frac{m}{2}} \mu_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^- = (-\mu_{2n} K_1^* + K_2^*) W^-, \quad (6.24)$$

$$\left. \begin{aligned}
& [(\Gamma_2 \times K^*) W^- + S_2 u^-] \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y \right) = f_2(y), \\
& \quad \quad \quad 0 \leq y \leq d_2, \\
& \left[\left(-y^{\frac{m}{2}} \mu_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^- - (-\mu_1 K_1^* + K_2^*) W^- \right] \times \\
& \quad \quad \quad \times \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y \right) = 0, \quad 0 \leq y \leq d_2,
\end{aligned} \right\} \quad (6.25)$$

$$(\tilde{K}_1 W^+ - \Lambda_2 \tilde{K}_1^* W^-)(0, y) = g_2(y), \quad 0 \leq y \leq y_0, \quad (6.26)$$

where $B_3 = \frac{m}{2} K_1^0 K_1 - B_0 - y B_1$, $B_3^* = \frac{m}{2} K_1^{*0} K_1^* - B_0^* - y B_1^*$; moreover,

by (5.5)–(5.8) we obtain

$$\begin{aligned} \sup_{\bar{D}^+ \setminus O} \|B_3\| < \infty, & \quad \sup_{\bar{D}^+ \setminus O} \|B_{3x}\| < \infty, \\ \sup_{\bar{D}^- \setminus O} \|B_3^*\| < \infty, & \quad \sup_{\bar{D}^- \setminus O} \|B_{3x}^*\| < \infty. \end{aligned} \quad (6.27)$$

It follows from (6.27) that

$$\begin{aligned} v_1^+ &= y^{-\frac{m}{2}} K_1 W^+, \quad v_2^+ = K_2 W^+, \quad W^+ = y^{\frac{m}{2}} K_1^0 v_1^+ + K_2^0 v_2^+, \\ v_1^- &= y^{-\frac{m}{2}} K_1^* W^-, \quad v_2^- = K_2^* W^-, \quad W^- = y^{\frac{m}{2}} K_1^{*0} v_1^- + K_2^{*0} v_2^-. \end{aligned}$$

Therefore the conditions $v_i^+ \in C^1(\bar{D}^+)$ and $v_i^- \in C^1(\bar{D}^-)$, $i = 1, 2$, as well as

$$\begin{aligned} \sup_{\bar{D}^+ \setminus 0} \|y^{-\alpha} v_1^+\| < \infty, & \quad \sup_{\bar{D}^+ \setminus 0} \|y^{-(\alpha + \frac{m}{2})} v_2^+\| < \infty, \\ \sup_{\bar{D}^- \setminus 0} \|y^{-\alpha} v_1^-\| < \infty, & \quad \sup_{\bar{D}^- \setminus 0} \|y^{-(\alpha + \frac{m}{2})} v_2^-\| < \infty \end{aligned}$$

are fulfilled if and only if

$$\begin{aligned} W^+ &\in C^1(\bar{D}^+), \quad \sup_{\bar{D}^+ \setminus O} \|y^{-(\alpha + \frac{m}{2})} W^+\| < \infty, \\ W^- &\in C^1(\bar{D}^-), \quad \sup_{\bar{D}^- \setminus O} \|y^{-(\alpha + \frac{m}{2})} W^-\| < \infty. \end{aligned}$$

Let

$$\begin{aligned} L_i(x_0, y_0) : \quad x &= z_i(x_0, y_0, t) \equiv x_0 + \frac{2\lambda_i}{m+2} y_0^{\frac{m+2}{2}} - \frac{2\lambda_i}{m+2} t^{\frac{m+2}{2}}, \\ y &= t, \quad (x_0, y_0) \in \bar{D}^+, \quad 1 \leq i \leq 2n, \\ L_j^*(x_0, y_0) : \quad x &= z_j^*(x_0, y_0, t) \equiv x_0 + \frac{2\mu_j}{m+2} y_0^{\frac{m+2}{2}} - \frac{2\mu_j}{m+2} t^{\frac{m+2}{2}}, \\ y &= t, \quad (x_0, y_0) \in \bar{D}^-, \quad 1 \leq j \leq 2n, \end{aligned}$$

be parametric representations of the characteristic curves passing through the point (x_0, y_0) . Denote by $\omega_i(x, y)$ the ordinate of the point of intersection of the characteristic $L_i(x, y)$ with the curve γ_1 for $1 \leq i \leq n$ and with the straight line $x = 0$ for $n < i \leq 2n$. Similarly, let $\omega_i^*(x, y)$ be the ordinate of the point of intersection of the characteristic $L_i^*(x, y)$ with the curve γ_2 for $n < i \leq 2n$ and with the straight line $x = 0$ for $1 \leq i \leq n$. From our construction of the functions $\omega_i(x, y)$ and $\omega_i^*(x, y)$ and from the inequalities (4.2), it follows that

$$\begin{aligned} 0 &\leq \omega_i(x, y) \leq y, \quad (x, y) \in D^+, \quad i = 1, 2, \dots, 2n, \\ 0 &\leq \omega_i^*(x, y) \leq y, \quad (x, y) \in D^-, \quad i = 1, 2, \dots, 2n. \end{aligned} \quad (6.28)$$

It is not difficult to verify that

$$\begin{aligned}
\omega_i|_{OP_1} &= \begin{cases} y, & i = 1, 2, \dots, n, \\ \tau_{1i}y, & i = n+1, \dots, 2n-1, \\ 0, & i = 2n, \end{cases} \\
\omega_i|_{OO_1} &= \begin{cases} \tau_{2i}y, & i = 1, 2, \dots, n, \\ y, & i = n+1, \dots, 2n, \end{cases} \\
\omega_i^*|_{OP_2} &= \begin{cases} 0, & i = 1, \\ \tau_{4i}y, & i = 2, \dots, n, \\ y, & i = n+1, \dots, 2n, \end{cases} \\
\omega_i^*|_{OO_1} &= \begin{cases} y, & i = 1, 2, \dots, n, \\ \tau_{3i}y, & i = n+1, \dots, 2n. \end{cases}
\end{aligned} \tag{6.29}$$

The constant numbers τ_{ij} here satisfy

$$0 < \tau_{ij} < 1, \quad 1 \leq i, j \leq 2n. \tag{6.30}$$

Assume

$$\begin{aligned}
\varphi_i &= W_i^+|_{OP_1} = W_i^+ \left(-\frac{2\lambda_{2n}}{m+2} y^{\frac{m+2}{2}}, y \right), \quad 0 \leq y \leq d_1, \quad i = 1, 2, \dots, n, \\
\psi_i &= W_i^+|_{OO_1} = W_i^+(0, y), \quad 0 \leq y \leq y_0, \quad i = n+1, \dots, 2n, \\
\varphi_i^* &= W_i^-|_{OO_1} = W_i^-(0, y), \quad 0 \leq y \leq y_0, \quad i = 1, 2, \dots, n, \\
\psi_i^* &= W_i^-|_{OP_2} = W_i^- \left(-\frac{2\mu_1}{m+2} y^{\frac{m+2}{2}}, y \right), \quad 0 \leq y \leq d_2, \quad i = n+1, \dots, 2n.
\end{aligned}$$

Since $\alpha > 0$, it is evident that $\varphi_i = W_i^+(0, 0) = 0$, $i = 1, 2, \dots, n$, $\psi_i(0) = W_i^+(0, 0) = 0$, $i = n+1, \dots, 2n$, $\varphi_i^* = W_i^-(0, 0) = 0$, $i = 1, \dots, n$, $\psi_i^*(0) = W_i^-(0, 0) = 0$, $i = n+1, \dots, 2n$.

Integrating the i -th equation of the system (6.19) along the i -th characteristic $L_i(x, y)$ from the point $P(x, y) \in \bar{D}^+$ to the point of intersection of $L_i(x, y)$ with the curve γ_1 for $i \leq n$ and with the straight line $x = 0$ for $i > n$, we arrive at

$$\begin{aligned}
W_i^+(x, y) &= \varphi_i(\omega_i(x, y)) + \int_{\omega_i(x, y)}^y \frac{1}{t} \left(\sum_{j=1}^{2n} B_{3ij} W_j^+ + \right. \\
&+ \left. \sum_{j=1}^n t C_{2ij} u_j^+ \right) (z_i(x, y, t), t) dt + \tilde{F}_{1i}(x, y), \quad i = 1, \dots, n, \\
W_i^+(x, y) &= \psi_i(\omega_i(x, y)) + \int_{\omega_i(x, y)}^y \frac{1}{t} \left(\sum_{j=1}^{2n} B_{3ij} W_j^+ + \right.
\end{aligned}$$

$$+ \sum_{j=1}^n tC_{2ij}u_j^+ \Big) (z_i(x, y, t), t) dt + \tilde{F}_{1i}(x, y), \quad i = n+1, \dots, 2n, \quad (6.31)$$

where

$$\tilde{F}_{1i}(x, y) = \int_{\omega_i(x, y)}^y F_{1i}^*(z_i(x, y, t), t) dt, \quad i = 1, \dots, 2n.$$

Integration of the equation (6.20) along the characteristic $L_1(x, y)$ results in

$$u^+(x, y) = g_1^*(\omega_1(x, y)) + \int_{\omega_1(x, y)}^y (-\lambda_1 K_1 + K_2) W^+(z_1(x, y, t), t) dt, \quad (6.32)$$

where

$$\begin{aligned} g_1^*(\omega_1(x, y)) &= u^+ \left(-\frac{2\lambda_1}{m+2} \omega_1^{\frac{m+2}{2}}(x, y), \omega_1(x, y) \right) = \\ &= \int_0^{\omega_1(x, y)} \left(-y^{\frac{m}{2}} \lambda_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^+ \left(-\frac{2\lambda_1}{m+2} t^{\frac{m+2}{2}}, t \right) dt = \\ &= \int_0^{\omega_1(x, y)} (-\lambda_1 K_1 + K_2) W^+ \left(-\frac{2\lambda_1}{m+2} t^{\frac{m+2}{2}}, t \right) dt. \end{aligned}$$

Analogously, integrating the j -th equation of the system (6.23) along the j -th characteristic $L_j^*(x, y)$ from the point $P^*(x, y) \in \bar{D}^-$ to the point of intersection of $L_j^*(x, y)$ with the curve γ_2 for $j > n$ and with the straight line $x = 0$ for $j \leq n$, we obtain

$$\begin{aligned} W_i^-(x, y) &= \varphi_i^*(\omega_i^*(x, y)) + \int_{\omega_i^*(x, y)}^y \frac{1}{t} \left(\sum_{j=1}^{2n} B_{3ij}^* W_j^- + \right. \\ &+ \left. \sum_{j=1}^n tC_{2ij}^* u_j^- \right) (z_i^*(x, y, t), t) dt + \tilde{F}_{2i}(x, y), \quad i = 1, \dots, n, \\ W_i^-(x, y) &= \psi_i^*(\omega_i^*(x, y)) + \int_{\omega_i^*(x, y)}^y \frac{1}{t} \left(\sum_{j=1}^{2n} B_{3ij}^* W_j^- + \right. \\ &+ \left. \sum_{j=1}^n tC_{2ij}^* u_j^- \right) (z_i^*(x, y, t), t) dt + \tilde{F}_{2i}(x, y), \quad i = n+1, \dots, 2n, \quad (6.33) \end{aligned}$$

where

$$\tilde{F}_{2i}(x, y) = \int_{\omega_i^*(x, y)}^y F_{2i}^*(z_i^*(x, y, t), t) dt, \quad i = 1, \dots, 2n.$$

Integration of the equation (6.24) along the characteristic $L_{2n}^*(x, y)$ yields

$$\begin{aligned} u^-(x, y) &= g_2^*(\omega_{2n}^*(x, y)) + \\ &+ \int_{\omega_{2n}^*(x, y)}^y (-\mu_{2n}K_1^* + K_2^*)W^-(z_{2n}^*(x, y, t), t) dt, \end{aligned} \quad (6.34)$$

where

$$\begin{aligned} g_2^*(\omega_{2n}^*(x, y)) &= u^- \left(-\frac{2\mu_{2n}}{m+2} \omega_{2n}^{*\frac{m+2}{2}}(x, y), \omega_{2n}^*(x, y) \right) = \\ &= \int_0^{\omega_{2n}^*(x, y)} \left(-y^{\frac{m}{2}} \mu_{2n} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) u^- \left(-\frac{2\mu_{2n}}{m+2} t^{\frac{m+2}{2}}, t \right) dt = \\ &= \int_0^{\omega_{2n}^*(x, y)} (-\mu_{2n}K_1^* + K_2^*)W^- \left(-\frac{2\mu_{2n}}{m+2} t^{\frac{m+2}{2}}, t \right) dt. \end{aligned}$$

We rewrite the system of equations (6.31) and (6.33) in terms of one equation

$$\begin{aligned} W^+(x, y) &= \tilde{\varphi}(x, y) + \\ &+ \sum_{i=1}^{2n} \int_{\omega_i(x, y)}^y \frac{1}{t} (B_{4i}W^+ + C_{3i}u^+) (z_i(x, y, t), t) dt + \tilde{F}_1(x, y), \end{aligned} \quad (6.35)$$

$$\begin{aligned} W^-(x, y) &= \tilde{\varphi}^*(x, y) + \\ &+ \sum_{i=1}^{2n} \int_{\omega_i^*(x, y)}^y \frac{1}{t} (B_{4i}^*W^- + C_{3i}^*u^-) (z_i^*(x, y, t), t) dt + \tilde{F}_2(x, y), \end{aligned} \quad (6.36)$$

where B_{4i} , B_{4i}^* and C_{3i} , C_{3i}^* are well-defined matrices of orders $(2n \times 2n)$ and $(2n \times n)$, respectively, and

$$\begin{aligned} \tilde{\varphi}(x, y) &= (\varphi_1(\omega_1(x, y)), \dots, \varphi_n(\omega_n(x, y)), \\ &\quad \psi_{n+1}(\omega_{n+1}(x, y)), \dots, \psi_{2n}(\omega_{2n}(x, y))), \\ \tilde{\varphi}^*(x, y) &= (\varphi_1^*(\omega_1^*(x, y)), \dots, \varphi_n^*(\omega_n^*(x, y)), \\ &\quad \psi_{n+1}^*(\omega_{n+1}^*(x, y)), \dots, \psi_{2n}^*(\omega_{2n}^*(x, y))). \end{aligned}$$

Substituting the expressions for u^+ , u^- and W^+ , W^- from (6.32), (6.34) and (6.35), (6.36) into the boundary conditions (6.21) and (6.25), we obtain

$$G_0^1 \varphi(y) + \sum_{i=n+1}^{2n-1} G_i^1(y) \psi(\tau_{1i} y) + [T_1(W^+, u^+)](y) = f_3(y), \quad 0 \leq y \leq d_1, \quad (6.37)$$

$$G_0^2 \psi^*(y) + \sum_{j=2}^n G_j^2(y) \varphi^*(\tau_{4j} y) + [T_2(W^-, u^-)](y) = f_4(y), \quad 0 \leq y \leq d_2, \quad (6.38)$$

where G_i^1 , G_j^2 are well-defined constant $(n \times n)$ -matrices, f_3 and f_4 are functions defined by means of $f_1, f_2, \tilde{F}_1, \tilde{F}_2$, and T_1, T_2 are linear integral operators; $\varphi = (\varphi_1, \dots, \varphi_n)$, $\psi = (\psi_{n+1}, \dots, \psi_{2n})$, $\varphi^* = (\varphi_1^*, \dots, \varphi_n^*)$, $\psi^* = (\psi_{n+1}^*, \dots, \psi_{2n}^*)$.

As is easily seen,

$$W^+|_{OO_1} = G_1 \psi(y) + \sum_{i=2}^n G_i^3 \varphi(\tau_{2i} y) + [T_3(W^+, u^+)](y), \quad (6.39)$$

$$W^-|_{OO_1} = G_2 \varphi^*(y) + \sum_{i=n+1}^{2n-1} G_i^4 \psi^*(\tau_{3i} y) + [T_4(W^-, u^-)](y), \quad (6.40)$$

where

$$G_1 = \begin{pmatrix} 0 \\ E \end{pmatrix}, \quad G_2 = \begin{pmatrix} E \\ 0 \end{pmatrix},$$

E is the unit $(n \times n)$ -matrix, G_i^3, G_i^4 are well-defined matrices, and T_3 and T_4 are linear integral operators.

Substituting the expressions (6.39) and (6.40) for $W^\pm|_{OO_1}$ into the conditions of conjugation (6.22) and (6.26), we get

$$G_0^3 \psi(y) + G_0^4 \varphi^*(y) + \sum_{i=2}^n \tilde{K}_2 G_i^3 \varphi(\tau_{2i} y) + \sum_{i=n+1}^{2n-1} (-\Lambda_1 K_2^*) G_i^4 \psi^*(\tau_{3i} y) + [T_5(u^+, W^+, u^-, W^-)](y) = g_1'(y), \quad 0 \leq y \leq y_0, \quad (6.41)$$

$$G_0^5 \psi(y) + G_0^6 \varphi^*(y) + \sum_{i=2}^n \tilde{K}_1 G_i^3 \varphi(\tau_{2i} y) + \sum_{i=n+1}^{2n-1} (-\Lambda_2 \tilde{K}_1^*) G_i^3 \psi^*(\tau_{3i} y) + [T_6(u^+, W^+, u^-, W^-)](y) = g_2(y), \quad 0 \leq y \leq y_0, \quad (6.42)$$

where $G_0^3, G_0^4, G_0^5, G_0^6$ are well-defined $(n \times n)$ -matrices, and T_5, T_6 are linear integral operators.

Remark 1. It is easy to see that the matrices G_0^i , $i = 1, \dots, 6$, from (6.37), (6.38) and (6.41), (6.42) are representable in terms of

$$\begin{aligned} G_0^1 &= \Gamma_1 \times V_1, \quad G_0^2 = \Gamma_2 \times V_2^*, \quad G_0^3 = \tilde{K}_2 G_1, \quad G_0^4 = -\Lambda_1 \tilde{K}_2^* G_2, \\ G_0^5 &= \tilde{K}_1 G_1, \quad G_0^6 = -\Lambda_2 \tilde{K}_1^* G_2. \end{aligned}$$

Therefore the $(2n \times 2n)$ -matrix

$$\begin{vmatrix} G_0^3 & G_0^4 \\ G_0^5 & G_0^6 \end{vmatrix}$$

and the $(n \times n)$ -matrices G_0^1 and G_0^2 are invertible in the domain of their definition if and only if

$$\det \begin{vmatrix} \tilde{K}_2 G_1 & -\Lambda_1 \tilde{K}_2^* G_2 \\ \tilde{K}_1 G_1 & -\Lambda_2 \tilde{K}_1^* G_2 \end{vmatrix} \neq 0, \quad (x, y) \in OO_1, \quad (6.43)$$

$$\det(\Gamma_1 \times V_1)(x, y) \neq 0, \quad (x, y) \in OP_1, \quad (6.44)$$

$$\det(\Gamma_2 \times V_2^*)(x, y) \neq 0, \quad (x, y) \in OP_2. \quad (6.45)$$

Remark 2. It is clear from the above arguments that in the class (4.8) the problem (4.1), (4.3)–(4.6) is equivalent to the system of integral differential equations (6.31)–(6.36), (6.41), (6.42) with respect to the unknown functions u^+ , u^- , W^+ , W^- , φ , ψ , φ^* , ψ^* , where $u^\pm \in C^2(\bar{D}^\pm)$, $u^\pm(0, 0) = 0$,

$$\sup_{\bar{D}^\pm \setminus O} \|y^{-\alpha} u_x^\pm\| < \infty, \quad \sup_{\bar{D}^\pm \setminus O} \|y^{-(\alpha + \frac{m}{2})} u_y^\pm\| < \infty, \quad W^\pm \in C^1(D^\pm),$$

$$\sup_{\bar{D}^\pm \setminus O} \|y^{-(\alpha + \frac{m}{2})} W^\pm\| < \infty,$$

$$\varphi \in C^1[0, d_1], \quad \psi \in C^1[0, y_0], \quad \varphi^* \in C^1[a, y_0], \quad \psi^* \in C^1[0, d_2].$$

$$\sup_{0 < y \leq d_1} \|y^{-(\alpha + \frac{m}{2})} \varphi\| < \infty, \quad \sup_{0 < y \leq y_0} \|y^{-(\alpha + \frac{m}{2})} \psi\| < \infty,$$

$$\sup_{0 < y \leq y_0} \|y^{-(\alpha + \frac{m}{2})} \varphi^*\| < \infty, \quad \sup_{0 < y \leq d_2} \|y^{-(\alpha + \frac{m}{2})} \psi^*\| < \infty.$$

Remark 3. By virtue of (6.28) and (6.30), the integral operators in the left-hand sides of the equations (6.31)–(6.36), (6.41) and (6.42) are of Volterra structure.

Bearing in mind the above remarks in solving the system of integral differential equations (6.31)–(6.36), (6.41), (6.42) by the method of successive approximations and using the scheme suggested in [41], we arrive at the following

Theorem 6.1. *Let the conditions (6.43)–(6.45) be fulfilled. Then there exists a positive number α_0 depending only on the coefficients of the system (4.1) and on the boundary conditions (4.3)–(4.6), such that for $\alpha > \alpha_0$ the problem (4.1), (4.3)–(4.6) is uniquely solvable in the class (4.8).*

CHAPTER III
**PROBLEM WITH OBLIQUE DERIVATIVE
 FOR THE EQUATION OF MIXED TYPE**

7. STATEMENT OF THE PROBLEM AND ITS INVESTIGATION IN THE
 ELLIPTIC PART OF THE DOMAIN

In the plane of the variables x, y , let us consider a mixed type equation

$$\operatorname{sgn} y u_{xx} + u_{yy} + au_x + bu_y + cu = 0, \quad (7.1)$$

where a, b, c are given entire analytic functions of their arguments taking real values for real x, y , and u is an unknown real function.

Let D be a singly connected domain in the plane of the variables x, y which is bounded by a curve σ of the class C^2 with the ends at the points $C_1(0, 0)$ and $C_2(1, 0)$ and lying in the upper half-plane $y > 0$, and by characteristics $CC_1 : y = -x$, $CC_2 : y = x - 1$, $C = (\frac{1}{2}, -\frac{1}{2})$ of the equation (7.1).

Consider **Problem A** which is formulated as follows: it is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y)$ is a solution of the equation (7.1) for $y \neq 0$ in the domain D ; 2) it is continuous in the closed domain D and has continuous first derivatives everywhere in this domain, except maybe at the points C_1 and C_2 near which $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ tend to infinity with an order less than 1; 3) it satisfies the boundary conditions

$$(p_1 u_x + q_1 u_y + \lambda_1 u)|_\sigma = \varphi, \quad (7.2)$$

$$(p_2 u_x + q_2 u_y + \lambda_2 u)|_{CC_1} = \psi, \quad (7.3)$$

where p_i, q_i, λ_i ($i = 1, 2$), φ, ψ are given real functions satisfying the Hölder condition.

Let D^+ and D^- be respectively the elliptic and the hyperbolic parts of the domain D . Below we assume that $\partial D^+ \in C^{2,h}$, $\varphi, p_i, q_i \in C^{1,h}$ ($i = 1, 2$), $\psi \in C^{2,h}$, $0 < h < 1$.

Instead of the real variables x and y , $(x, y) \in D^+$, we introduce the complex variables $z = x + iy$, $\bar{z} = x - iy$. Then the equation (7.1) takes the form

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + A(z, \bar{z}) \frac{\partial u}{\partial z} + \overline{A(z, \bar{z})} \frac{\partial u}{\partial \bar{z}} + C(z, \bar{z}) u = 0, \quad (7.4)$$

where

$$A(z, \bar{z}) = a \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) + ib \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right),$$

$$4C(z, \bar{z}) = \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right).$$

Introduce the functions

$$\alpha(z, \bar{z}) = \exp \left[- \int_0^{\bar{z}} (z, \bar{t}) d\bar{z} \right], \quad \beta(z, \bar{z}, t) = \int_0^{\bar{z}} V(z, \bar{z}; t, \bar{t}) d\bar{t},$$

where $V(z, \bar{z}; t, \bar{t})$ is a function uniquely defined as follows: 1) $V(z, \bar{z}; t, \bar{t})$ is a solution of the differential equation (7.4); 2)

$$V(z, \bar{t}; t, \bar{t}) = \gamma(t, \bar{t}) \exp \left[- \int_t^z \bar{A}(t_1, t) dt_1 \right], \quad (7.5)$$

$$V(t, \bar{z}; t, \bar{t}) = \gamma(t, \bar{t}) \exp \left[- \int_{\bar{t}}^{\bar{z}} A(t, \bar{t}_1) d\bar{t}_1 \right], \quad (7.6)$$

where

$$\begin{aligned} -\gamma(z, \bar{z}) &= \frac{\partial \alpha(z, \bar{z})}{\partial z \partial \bar{z}} + A(z, \bar{z}) \frac{\partial \alpha(z, \bar{z})}{\partial z} + \\ &+ \overline{A(z, \bar{z})} \frac{\partial \alpha(z, \bar{z})}{\partial \bar{z}} + C(z, \bar{z}) \alpha(z, \bar{z}). \end{aligned}$$

If we take into account the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i \frac{\partial}{\partial z} - i \frac{\partial}{\partial \bar{z}},$$

then the boundary condition (7.2) takes the form

$$H(s) \frac{\partial u}{\partial t} + \overline{H(s)} \frac{\partial u}{\partial \bar{t}} + \lambda(s) u = \varphi(s), \quad t \in \sigma, \quad (7.7)$$

where $H(s) = p_1(s) + iq_1(s)$, s is an arc abscissa on σ .

Let us make use of a general representation of regular in D^+ solutions of the equation (7.1) expressed in terms of the analytic functions [72]

$$u(x, y) = \operatorname{Re} \left\{ \alpha(z, \bar{z}) \omega(z) + \int_{P_0}^z \beta(z, \bar{z}, t) \omega(t) dt \right\}, \quad (7.8)$$

where $\omega(z)$ is an arbitrary analytic in D^+ function satisfying the condition $\operatorname{Im} \omega(P_0) = 0$, $P_0 \in D^+$, and $\alpha(z, \bar{z})$ and $\beta(z, \bar{z}, t)$ are entire functions of their arguments defined by the formulas (7.5), (7.6).

I. N. Vekua [72] has proved that if $\omega(z) \in C^{1,h}(\bar{D}^+)$ is an analytic in a singly connected domain D^+ function satisfying the condition $\operatorname{Im} \omega(P_0) = 0$, then there exists a unique real function $\mu(t) \in C^{0,h}$ such that the following formula holds:

$$\omega(z) = \int_{\partial D^+} \mu(t) \log e \left(1 - \frac{z}{t} \right) ds_t, \quad (7.9)$$

where ds_t is the arc element of the boundary ∂D^+ and under $\log\left(1 - \frac{z}{t}\right)$, $z \in D^+$, $t \in D^+$ is understood the branch of the function which equals zero for $z = 0$.

Supposing that the analytic function $\omega(z)$ appearing in the formula (7.8) has the continuous first derivative in \bar{D}^+ and substituting the expression (7.8) in the boundary condition (7.7), we obtain

$$\operatorname{Re} \left\{ M(t)\omega'(t) + N(t)\omega(t) + \int_0^t Q(t, t_1)\omega(t_1)dt_1 \right\} = \varphi(s), \quad t \in \sigma, \quad (7.10)$$

where

$$\begin{aligned} M(t) &= \alpha(t, \bar{t})H(t), \\ N(t) &= \alpha(t, \bar{t})C(t) + \beta(t, \bar{t}, t)H(t) + \frac{\partial\alpha(t, \bar{t})}{\partial t}H(t) + \frac{\partial\alpha(t, \bar{t})}{\partial \bar{t}_1}\overline{H(t)}, \\ Q(t, t_1) &= \beta(t, \bar{t}, t_1)C(t) + \frac{\partial\beta(t, \bar{t}, t_1)}{\partial t}H(t) + \frac{\partial\beta(t, \bar{t}, t_1)}{\partial \bar{t}}\overline{H(t)}. \end{aligned}$$

It is easy to verify that the limiting values of the functions $\omega(z)$ and $\omega'(z)$, as z tends to the point t , $t \in \partial D^+$, are given by the formulas

$$\begin{aligned} \omega(t) &= \int_{\partial D^+} \mu(t_1) \log e \left(1 - \frac{t}{t_1}\right) ds_1, \\ \omega'(t) &= -\pi i \bar{t}' \mu(t) - \int_{\partial D^+} \frac{\mu(t_1) ds_1}{t_1 - t}. \end{aligned}$$

Substituting them in the boundary condition (7.10), after some transformations, we get the integral equation

$$\alpha(s)\mu(s) + \int_{\partial D^+} K(s, s_1)\mu(s_1)ds_1 = \varphi(s), \quad (7.11)$$

where

$$\begin{aligned} \alpha(s) &= \operatorname{Re} [-\pi i \bar{t}' M(t)], \\ K(s, s_1) &= \operatorname{Re} \left[N(t) \log e \left(1 - \frac{t}{t_1}\right) - \frac{M(t)}{t_1 - t} + Q^*(t, t_1) \right], \\ \operatorname{Re} \left[N(t) \log e \left(1 - \frac{t}{t_1}\right) \right] &= n_1 \log e \left| 1 - \frac{t}{t_1} \right| - n_2 \arg \left(1 - \frac{t}{t_1}\right), \\ N(t) &= n_1(t) + i n_2(t), \\ \operatorname{Re} \left[\frac{M(t)}{t_1 - t} \right] &= \frac{1}{2} \frac{\overline{M(t)} + M(t) e^{2i\theta(t, t_1)}}{t_1 - t}, \\ \theta(t, t_1) &= \arg(t_1 - t), \end{aligned}$$

$$\operatorname{Re}[Q^*(t, t_1)] = \operatorname{Re} \left[Q(t, t_1)\Omega(t, t_1) - \int_0^t \Omega(t_2, t) \frac{\partial Q(t, t_2)}{\partial t_2} dt_2 \right] = P(t, t_1),$$

$$\Omega(t, t_1) = \int_0^t \log e \left(1 - \frac{t_2}{t_1} \right) dt_2 = (t - t_1) \log \left(1 - \frac{t}{t_1} \right).$$

We can easily give (7.11) the following form

$$\alpha(t)\mu(t) - \int_{\partial D^+} \frac{K_1(t, t_1)}{t_1 - t} \mu(t_1) dt_1 = \varphi(t), \quad t \in \partial D^+ \setminus AB, \quad (7.12)$$

with

$$2K_1(t, t_1) = \overline{M(t)}\bar{t}_1 + M(t)\bar{t}_1 e^{2i\theta(t, t_1)} - 2(t_1 - t)\bar{t}_1 \left[n_1(t) \log e \left| 1 - \frac{t}{t_1} \right| - n_2(t) \arg \left(1 - \frac{t}{t_1} \right) + P(t, t_1) \right],$$

or

$$\alpha_1(t)\mu(t) + \beta_1(t) \int_{\partial D^+} \frac{\mu(t_1) dt_1}{t_1 - t} + \int_{\partial D^+} K(t, t_1)\mu(t_1) dt_1 = \varphi(t),$$

$$t \in \partial D^+ \setminus AB. \quad (7.13)$$

Here

$$\alpha_1(t) = \operatorname{Re}[-\pi i \bar{t}' M(t)], \quad \beta_1(t) = \operatorname{Im}(-i(p_1(t) + iq_1(t))\alpha(t, \bar{t})\bar{t}'),$$

$$K(t, t_1) = \frac{K_1(t, t_1) - K_1(t, t)}{t_1 - t}.$$

8. INVESTIGATION OF THE PROBLEM (7.1), (7.2), (7.3) IN THE HYPERBOLIC PART OF THE DOMAIN

Denote by $R(x, y; x_1, y_1)$ the Riemann function which by definition is a solution of the so-called conjugate equation [46]

$$R_{xy} - (aR)_x - (bR)_y + cR = 0 \quad (8.1)$$

which takes on the characteristics $x = x_1, y = y_1$ the values

$$R(x_1, y; x_1, y_1) = \exp \left(\int_{y_1}^y a(x_1, \eta) d\eta \right), \quad (8.2)$$

$$R(x, y_1; x_1, y_1) = \exp \left(\int_{x_1}^x b(\xi, y_1) d\xi \right), \quad (8.3)$$

where (x_1, y_1) is an arbitrarily fixed point in the domain D^- .

By (8.1), (8.2) and (8.3), the function $R(x, y; x_1, y_1)$ satisfies the integral equation

$$\begin{aligned} R(x, y; x_1, y_1) - \int_{x_1}^x b(\xi, \eta) R(\xi, \eta; x_1, y_1) d\xi - \\ - \int_{y_1}^y a(x, \eta) R(x, \eta; x_1, y_1) d\eta + \\ + \int_{x_1}^x d\xi \int_{y_1}^y c(\xi, \eta) R(\xi, \eta; x_1, y_1) d\eta = 1. \end{aligned} \quad (8.4)$$

As is known, the equation (8.4) has a unique solution $R(x, y; x_1, y_1)$ which, as it is easily verified, possesses the following continuous derivatives:

$$\begin{aligned} \partial_{x,y}^{i,j} \partial_{x_1,y_1}^{i_1,j_1} R(x, y; x_1, y_1) \in C(\bar{D}^- \times \bar{D}^-), \\ 0 \leq i + j \leq 1, 0 \leq i_1 + j_1 \leq 2, \partial_{x,y}^{i,j} = \partial^{i+j} / \partial x^i \partial y^j. \end{aligned} \quad (8.5)$$

The equalities (8.2) and (8.3) imply that

$$\left. \begin{aligned} \frac{\partial R(x_1, y; x_1, y_1)}{\partial y} - a(x_1, y) R(x_1, y; x_1, y_1) &= 0, \\ \frac{\partial R(x, y_1; x_1, y_1)}{\partial x} - b(x, y_1) R(x, y_1; x_1, y_1) &= 0, \\ R(x_1, y_1; x_1, y_1) &= 1, \\ \frac{\partial R(x, y; x, y_1)}{\partial y_1} + a(x, y_1) R(x, y; x, y_1) &= 0, \\ \frac{\partial R(x, y; x_1, y)}{\partial x_1} + b(x_1, y) R(x, y; x_1, y) &= 0, \\ R(x, y; x, y) &= 1. \end{aligned} \right\} \quad (8.6)$$

By virtue of (8.4) and (8.5), every solution $u(x, y)$ of the equation (7.1) of the class $C^2(\bar{D}^-)$ can be represented in the domain D^- in the form [46]

$$\begin{aligned} u(x, y) = \frac{1}{2} [\tau(x+y) R(x+y, -x-y; x-y, -x-y) + \\ + \tau(x-y) R(x-y, -x+y; x-y, -x-y)] + \\ + \int_{x+y}^{x-y} \left[-\frac{1}{2} R(t, -t; x-y, -x-y) \nu(t) + \right. \\ \left. + (a(t, -t) + b(t, -t)) R(t, -t; x-y, -x-y) \tau(t) - \right. \\ \left. - \frac{1}{2} (R_x(t, -t; x-y, -x-y) + \right. \end{aligned}$$

$$+R_y^{\sim}(t, -t; x - y, -x - y)\tau(t) \Big] dt \quad (8.7)$$

as the solution of the Cauchy problem

$$u(x, 0) = \tau(x), \quad u_y(x, 0) = \nu(x),$$

where $R(\tilde{x}, \tilde{y}; \xi, \eta)$ is the Riemann function for the equation

$$\begin{aligned} u_{\tilde{x}\tilde{y}} + \tilde{a}u_{\tilde{x}} + \tilde{b}u_{\tilde{y}} + \tilde{c}u &= 0, \quad \tilde{x} = x - y, \quad \tilde{y} = -x - y, \\ \tilde{a}(\tilde{x}, \tilde{y}) &= \frac{a - b}{4} \left(\frac{\tilde{x} - \tilde{y}}{2}, \frac{-\tilde{x} - \tilde{y}}{2} \right), \\ \tilde{b}(\tilde{x}, \tilde{y}) &= -\frac{a + b}{4} \left(\frac{\tilde{x} - \tilde{y}}{2}, \frac{-\tilde{x} - \tilde{y}}{2} \right), \\ \tilde{c}(\tilde{x}, \tilde{y}) &= -\frac{1}{4}c \left(\frac{\tilde{x} - \tilde{y}}{2}, \frac{-\tilde{x} - \tilde{y}}{2} \right). \end{aligned}$$

From (8.7), we have

$$\begin{aligned} u_x(x, y) &= \frac{1}{2}[\tau'(x + y)R(x + y, -x - y; x - y, -x - y) + \\ &+ \tau(x + y)\{R_x - R_y + R_\xi - R_\eta\}(x + y, -x - y; x - y, -x - y) + \\ &+ \tau'(x - y)R(x - y, -x + y; x - y, -x - y) + \\ &+ \tau(x - y)\{R_x^{\sim} - R_y^{\sim} + R_\xi - R_\eta\}(x - y, -x + y; x - y, -x - y)] + \\ &+ \left[-\frac{1}{2}R(x - y, -x + y; x - y, -x - y)\nu(x - y) + \right. \\ &\quad \left. + (a(x - y, -x + y) + b(x - y, -x + y)) \cdot \right. \\ &\quad \left. \cdot R(x - y, -x + y; x - y, -x - y)\tau(x - y) \right] - \\ &\quad - \frac{1}{2}(R_x^{\sim}(x - y, -x + y; x - y, -x - y) + \\ &\quad + R_y^{\sim}(x - y, -x + y; x - y, -x - y))\tau(x - y) - \\ &- \left[-\frac{1}{2}R(x + y, -x - y; x - y, -x - y)\nu(x + y) + (a(x + y, -x - y) + \right. \\ &\quad \left. + b(x + y, -x - y))R(x + y, -x - y; x - y, -x - y)\tau(x + y) \right] + \\ &\quad + \frac{1}{2}(R_x^{\sim}(x + y, -x - y; x - y, -x - y) + \\ &\quad + R_y^{\sim}(x + y, -x - y; x - y, -x - y))\tau(x + y) + \\ &\quad + \int_{x+y}^{x-y} \left[-\frac{1}{2}(R_\xi - R_\eta)(t, -t; x - y, -x - y)\nu(t) + \right. \\ &\quad \left. + (a(t, -t) + b(t, -t))(R_\xi - R_\eta)(t, -t; x - y, -x - y)\tau(t) - \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}((R_{x\xi}^{\sim} - R_{x\eta}^{\sim})(t, -t; x - y, -x - y) + \\
& + (R_{y\xi}^{\sim} - R_{y\eta}^{\sim})(t, -t; x - y, -x - y))\tau(t) \Big] dt. \quad (8.8)
\end{aligned}$$

It follows from (8.8) that for $y = -x$, $x \in [0, \frac{1}{2}]$, i.e., on the characteristic CC_1 we have

$$\begin{aligned}
u_x|_{CC_1} &= u_x(x, -x) = \frac{1}{2}[\tau'(0)R(0, 0; 2x, 0) + \\
& + \tau(0)\{R_x^{\sim} - R_y^{\sim} - R_\xi - R_\eta\}(0, 0; 2x, 0) + \tau'(2x)R(2x, -2x; 2x, 0) + \\
& + \tau(2x)\{R_x^{\sim} - R_y^{\sim} + R_\xi - R_\eta\}(2x, -2x; 2x, 0)] + \\
& + \left[-\frac{1}{2}R(2x, -2x; 2x, 0)\nu(2x) + (a(2x, -2x) + \right. \\
& \quad \left. + b(2x, -2x))R(2x, -2x; 2x, 0)\tau(2x) \right] - \\
& - \frac{1}{2}(R_x^{\sim}(2x, -2x; 2x, 0) + R_y^{\sim}(2x, -2x; 2x, 0))\tau(2x) - \\
& - \left[-\frac{1}{2}R(0, 0; 2x, 0)\nu(0) + (a(0, 0) + b(0, 0))R(0, 0; 2x, 0)\tau(0) \right] + \\
& \quad + \frac{1}{2}(R_x^{\sim}(0, 0; 2x, 0) + R_y^{\sim}(0, 0; 2x, 0))\tau(0) + \\
& \quad + \int_0^{2x} \left[-\frac{1}{2}(R_\xi - R_\eta)(t, -t; 2x, 0)\nu(t) + (a(t, -t) + \right. \\
& \quad \left. + b(t, -t))(R_\xi - R_\eta)(t, -t; 2x, 0)\tau(t) - \right. \\
& \quad \left. - \frac{1}{2}((R_{x\xi}^{\sim} - R_{x\eta}^{\sim})(t, -t; 2x, 0) + (R_{y\xi}^{\sim} - R_{y\eta}^{\sim})(t, -t; 2x, 0))\tau(t) \right] dt. \quad (8.9)
\end{aligned}$$

Similarly, from (8.7) and $y = -x$, $x \in [0, \frac{1}{2}]$, we get

$$\begin{aligned}
u_y|_{CC_1} &= u_y(x, -x) = \frac{1}{2}[\tau'(0)R(0, 0; 2x, 0) + \\
& + \tau(0)\{R_x^{\sim} - R_y^{\sim} - R_\xi - R_\eta\}(0, 0; 2x, 0) - \tau'(2x)R(2x, -2x; 2x, 0) + \\
& + \tau(2x)\{-R_x^{\sim} + R_y^{\sim} - R_\xi - R_\eta\}(2x, -2x; 2x, 0)] - \\
& - \left[-\frac{1}{2}R(2x, -2x; 2x, 0)\nu(2x) + \right. \\
& \quad \left. + (a(2x, -2x) + b(2x, -2x))R(2x, -2x; 2x, 0)\tau(2x) - \right. \\
& \quad \left. - \frac{1}{2}(R_x^{\sim}(2x, -2x; 2x, 0) + R_y^{\sim}(2x, -2x; 2x, 0))\tau(2x) \right] - \\
& - \left[-\frac{1}{2}R(0, 0; 2x, 0)\nu(0) + (a(0, 0) + b(0, 0))R(0, 0; 2x, 0)\tau(0) - \right. \\
& \quad \left. - \frac{1}{2}(R_x^{\sim}(0, 0; 2x, 0) + R_y^{\sim}(0, 0; 2x, 0))\tau(0) \right] +
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{2x} \left[-\frac{1}{2}(R_\xi - R_\eta)(t, -t; 2x, 0)\nu(t) + (a(t, -t) + \right. \\
& + b(t, -t))(-R_\xi - R_\eta)(t, -t; 2x, 0)\tau(t) - \frac{1}{2}(-R_{x\xi}^\sim - R_{x\eta}^\sim - \\
& \left. + R_{y\xi}^\sim - R_{y\eta}^\sim)(t, -t; 2x, 0)\tau(t) \right] dt. \tag{8.10}
\end{aligned}$$

Substitution of (8.9) and (8.10) in (7.3) yields

$$\begin{aligned}
(p_2 u_x + q_2 u_y + \lambda_2 x)|_{C C_1} &= \frac{1}{2} p_2 R(2x, -2x; 2x, 0) \tau'(2x) - \\
-\frac{1}{2} p_2 R(2x, -2x; 2x, 0) \nu(2x) &- \frac{1}{2} q_2 R(2x, -2x; 2x, 0) \tau'(2x) + \\
+\frac{1}{2} q_2 R(2x, -2x; 2x, 0) \nu(2x) &+ [T_1(\tau, \nu)](x) = \\
&= \psi(x), \quad 0 \leq x \leq \frac{1}{2}, \tag{8.11}
\end{aligned}$$

where $T_1(\tau, \nu)$ is a well-defined linear operator acting by the formula

$$\begin{aligned}
[T_1(\tau, \nu)](x) &= -\frac{1}{2} p_2 [\tau'(0) R(0, 0; 2x, 0) + \\
&+ \tau(0) \{R_x^\sim - R_y^\sim - R_\xi - R_\eta\}(0, 0; 2x, 0) + \\
&+ \tau(2x) \{R_x^\sim - R_y^\sim + R_\xi - R_\eta\}(2x, -2x; 2x, 0)] + \\
&+ p_2 (a(2x, -2x) + b(2x, -2x)) R(2x, -2x; 2x, 0) \tau(2x) - \\
&- \frac{1}{2} p_2 (R_x^\sim(2x, -2x; 2x, 0) + R_y^\sim(2x, -2x; 2x, 0) \tau(2x) - \\
-p_2 \left[-\frac{1}{2} R(0, 0; 2x, 0) \nu(0) + (a(0, 0) + b(0, 0)) R(0, 0; 2x, 0) \tau(0) + \right. \\
&+ \left. \frac{1}{2} (R_x^\sim(0, 0; 2x, 0) + R_y^\sim(0, 0; 2x, 0)) \tau(0) \right] + \\
&+ p_2 \int_0^{2x} \left[-\frac{1}{2} (R_\xi - R_\eta)(t, -t; 2x, 0) \nu(t) + \right. \\
&+ (a(t, -t) + b(t, -t)) (R_\xi - R_\eta)(t, -t; 2x, 0) \tau(t) - \\
&- \frac{1}{2} ((R_{x\xi}^\sim - R_{x\eta}^\sim)(t, -t; 2x, 0) + (R_{y\xi}^\sim - R_{y\eta}^\sim)(t, -t; 2x, 0)) \tau(t) \left. \right] dt + \\
&+ \frac{1}{2} q_2 \left[\tau'(0) R(0, 0; 2x, 0) + \tau(0) \{R_x^\sim - R_y^\sim - R_\xi - R_\eta\}(0, 0; 2x, 0) + \right. \\
&+ \tau(2x) \{-R_x^\sim + R_y^\sim - R_\xi - R_\eta\}(2x, -2x; 2x, 0) \left. \right] - \\
&- q_2 \left[(a(2x, -2x) + b(2x, -2x)) R(2x, -2x; 2x, 0) \tau(2x) - \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(R_{\tilde{x}}(2x, -2x; 2x, 0) + R_{\tilde{y}}(2x, -2x; 2x, 0))\tau(2x) \Big] - \\
& -q_2 \left[-\frac{1}{2}R(0, 0; 2x, 0)\nu(0) + (a(0, 0) + b(0, 0))R(0, 0; 2x, 0)\tau(0) - \right. \\
& \quad \left. -\frac{1}{2}(R_{\tilde{x}}(0, 0; 2x, 0) + R_{\tilde{y}}(0, 0; 2x, 0))\tau(0) \right] + \\
& \quad +q_2 \int_0^x \left[-\frac{1}{2}(R_{\xi} - R_{\eta})(t, -t; 2x, 0)\nu(t) + \right. \\
& \quad + (a(t, -t) + b(t, -t))(-R_{\xi} - R_{\eta})(t, -t; 2x, 0)\tau(t) - \\
& \quad \left. -\frac{1}{2}(R_{\tilde{x}\xi} - R_{\tilde{x}\eta} - R_{\tilde{y}\xi} - R_{\tilde{y}\eta})(t, -t; 2x, 0)\tau(t) \right] dt + \\
& \quad +\frac{1}{2}\lambda_2 [\tau(0)R(0, 0; 2x, 0) + \tau(2x)R(2x, -2x; 2x, 0)] + \\
& +\lambda_2 \int_0^{2x} \left[-\frac{1}{2}R(t, -t; 2x, 0)\nu(t) + (a(t, -t) + b(t, -t))R(t, -t; 2x, 0)\tau(t) - \right. \\
& \quad \left. -\frac{1}{2}(R_{\tilde{x}}(t, -t; 2x, 0) + R_{\tilde{y}}(t, -t; 2x, 0))\tau(t) \right] dt.
\end{aligned}$$

As a result of not complicated transformations of the expression (8.11), we arrive at

$$\begin{aligned}
& \frac{1}{2}(p_2 - q_2)R(x, -x; x, 0)\tau'(x) - \frac{1}{2}(p_2 - q_2)R(x, -x; x, 0)\nu(x) + \\
& \quad + [T_1(\tau, \nu)]\left(\frac{x}{2}\right) = \psi\left(\frac{x}{2}\right), \quad 0 \leq x \leq 1. \tag{8.12}
\end{aligned}$$

Since $R(x, -x; x, 0) \neq 0$, under our assumption that $(p_2 - q_2)|_{CC_1} \neq 0$ and $p_1(0) \neq 0$, we divide (8.12) by $\frac{(p_2 - q_2)}{2p_1(0)}R(x, -x; x, 0)$ and obtain

$$p_1(0)\tau'(x) - p_1(0)\nu(x) + [T_2(\tau, \nu)](x) = \tilde{\psi}(x), \tag{8.13}$$

where

$$\begin{aligned}
T_2(\tau, \nu) &= \frac{2p_1(0)(T_1(\tau, \nu)]\left(\frac{x}{2}\right)}{(p_2 - q_2)R(x, -x; x, 0)}; \\
\tilde{\psi}(x) &= \frac{2p_1(0)\psi\left(\frac{x}{2}\right)}{(p_2 - q_2)R(x, -x; x, 0)}.
\end{aligned}$$

Applying the general representation of regular in D^+ solutions, we can get [68, 69]

$$\tau'(t) = \tilde{\alpha}_1(t)\mu(t) + \tilde{\beta}_1(t) \int_{\partial D^+} \frac{\mu(t_1)}{t_1 - t} dt_1 + [K_1(\mu)](t), \tag{8.14}$$

$$\nu(t) = \tilde{\alpha}_2(t)\mu(t) + \tilde{\beta}_2(t) \int_{\partial D^+} \frac{\mu(t_1)dt_1}{t_1 - t} + [K_2(\mu)](t), \quad (8.15)$$

where

$$\begin{aligned} \tilde{\alpha}_1(t) &= \operatorname{Re}(-\pi\alpha(t, \bar{t})\bar{t}'), & \tilde{\beta}_1(t) &= \operatorname{Im}(-i\alpha(t, \bar{t})\bar{t}'), \\ \tilde{\alpha}_2(t) &= \operatorname{Re}(\pi\alpha(t, \bar{t})\bar{t}'), & \tilde{\beta}_2(t) &= \operatorname{Im}(\alpha(t, \bar{t})\bar{t}'), \end{aligned}$$

Here $K_1(\mu)$ and $K_2(\mu)$ are well-defined integral operators.

Substituting (8.14) and (8.15) in (8.13), we obtain

$$\alpha_3(t)\mu(t) + \beta_3(t) \int_{\partial D^+} \frac{\mu(t_1)dt_1}{t_1 - t} + [K_3(\mu)](t) = \tilde{\psi}(t), \quad t \in (0, 1), \quad (8.16)$$

where

$$\begin{aligned} \alpha_3(t) &= \operatorname{Re}[-\pi(1+i)\alpha(t, \bar{t})\bar{t}'p_1(0)], \\ \beta_3(t) &= \operatorname{Im}[-(1+i)\alpha(t, \bar{t})\bar{t}'p_1(0)]. \end{aligned}$$

Here $K_3(\mu)$ is a linear integral operator.

9. INVESTIGATION OF THE PROBLEM (7.1), (7.2), (7.3) IN A MIXED DOMAIN

We rewrite the equations (7.13) and (8.16) in the form of one singular equation on the whole boundary ∂D^+ ,

$$\alpha_4(t)\mu(t) + \beta_4(t) \int_{\partial D^+} \frac{\mu(t_1)dt_1}{t_1 - t} + [K_4(\mu)](t) = f_1(t), \quad (9.1)$$

where

$$\begin{aligned} \alpha_4(t) &= \begin{cases} \alpha_1(t), & t \in \partial D^+ \setminus C_1C_2, \\ \alpha_3(t), & t \in C_1C_2, \end{cases} & \beta_4(t) &= \begin{cases} \beta_1(t), & t \in \partial D^+ \setminus C_1C_2, \\ \beta_3(t), & t \in C_1C_2, \end{cases} \\ f(t) &= \begin{cases} \varphi(t), & t \in \partial D^+ \setminus C_1C_2, \\ \tilde{\psi}(t), & t \in C_1C_2, \end{cases} \end{aligned}$$

and $K_4(\mu)$ is a well-defined, compact linear integral operator.

Remark. The coefficients $\alpha_4(t)$ and $\beta_4(t)$ below are assumed to be continuous at the point $t = 0$, i.e., $p_1(0) + q_1(0) = 0$.

A solution $\mu(t)$ of the singular integral equation (9.1) is sought in the space $H^*(\partial D^+)$, the point $C_2(1, 0)$ being the node of the curve ∂D^+ [52].

Under the assumption that

$$H(t) = (p_1 + iq_1)(t) \neq 0, \quad t \in \sigma,$$

we put

$$\omega(t) = \frac{\alpha_4(t) - i\pi\beta_4(t)}{\alpha_4(t) + i\pi\beta_4(t)} = \begin{cases} \frac{\pi i t' \overline{H(t)\alpha(t, \bar{t})}}{-\pi i t' H(t)\alpha(t, \bar{t})}, & t \in \sigma, \\ \frac{-\pi(1-i)t'\alpha(t, \bar{t})p_1(0)}{-\pi(1+i)t'\alpha(t, \bar{t})p_1(0)}, & t \in \partial D^+ \setminus \sigma. \end{cases}$$

The index κ of the singular integral equation (9.1) in the class $H^*(\partial D^+)$ is defined as follows [52]: denote by $\arg \omega_-(t)$ and $\arg \omega_+(t)$ continuous branches of the argument of the function $\omega(t)$ respectively on $\partial D^+ \setminus \sigma$ and σ .

Let

$$\begin{aligned} d &= \frac{1}{2\pi} \arg \omega_-(C_2) - \frac{1}{2\pi} \arg \omega_+(C_2) = \\ &= \frac{1}{2\pi} \left(2 \arg(1-i) + 2 \arg t' + 2 \arg \overline{\alpha(t, \bar{t})} \right) (C_2) - \\ &\quad - \frac{1}{2\pi} \left(2 \arg i + 2 \arg t' + 2 \arg \overline{H(t)} + 2 \arg \overline{\alpha(t, \bar{t})} \right) (C_2) = \\ &= \frac{1}{2\pi} (2 \arg(1-i) - 2 \arg i) - \frac{1}{\pi} \arg \overline{H(C_2)} = -\frac{3}{4} - \frac{1}{\pi} \arg \overline{H(C_2)}. \end{aligned}$$

The index κ is defined by the formula

$$\kappa = \begin{cases} -[d] - 1, & \text{if } d \notin Z, \\ -d, & \text{if } d \in Z, \end{cases} \quad (9.2)$$

where Z is the set of all integers and $[d]$ is the integer part of a number d .

Thus the following theorem is valid.

Theorem 9.1. *Let the conditions*

$$\begin{aligned} H(t) &= (p_1 + iq_1)(t) \neq 0, \quad t \in \sigma, \\ p_2(t) - q_2(t) &\neq 0, \quad t \in CC_1, \\ p_1(0) &\neq 0, \quad p_1(0) + q_1(0) = 0, \quad \varphi(0) = \psi(0) \end{aligned} \quad (9.3)$$

be fulfilled. Then the problem (7.1), (7.2), (7.3) is Noetherian and its index is given by the formula (9.2).

Below we consider **Problem B** which is formulated as follows: it is required to determine a function $u(x, y)$ possessing the following properties:

- 1) $u(x, y)$ is a solution of the equation (7.1) in the domain D for $y \neq 0$;
- 2) it is continuous in the closed domain \bar{D} and has continuous first derivatives everywhere in the same domain except maybe at the points C_1 and C_2 near which u_x and u_y tend to infinity with an order less than 1;
- 3) it satisfies the boundary conditions

$$(pu_x + qu_y + \lambda u)|_\sigma = \varphi, \quad (9.4)$$

$$u|_{CC_1} = \psi, \quad (9.5)$$

where $\varphi, p, q, \lambda \in C^{1,h}$, $\psi \in C^{2,h}$, $0 < h < 1$.

Theorem 9.2. *Let $H|_\sigma \neq 0$ and let the direction $l = (p, q)$ form with the exterior to the curve σ normal an acute angle, i.e., $\cos(\widehat{l, \widehat{N}})|_\sigma > 0$ and*

$$c|_D \leq 0, \quad (9.6)$$

$$\lambda|_\sigma \geq 0, \quad (9.7)$$

$$[a^2 - b^2 + 4c - 2(a_x + b_y + a_y + b_x)]|_{D^-} \geq 0, \quad (a + b)|_{D^-} \geq 0. \quad (9.8)$$

Then Problem B fails to have more than one solution.

Proof. Let us show that the corresponding to (7.1), (9.4), (9.5) homogeneous problem has only the trivial solution. Let $u_0(x, y)$ be a nonzero solution of the corresponding homogeneous problem. As far as $u|_{CC_1} = 0$, $u_0(x, y)$ is not constant. By (9.6), (9.7), $\cos(\widehat{l, \widehat{N}})|_\sigma > 0$ and by the Hopf and Zaremba–Giraud principles, the function $u_0(x, y)$ cannot reach in $D^+ \cup \sigma$ a positive maximum and a negative minimum [11]. Since the conditions (9.8) are fulfilled, by virtue of the extremum principle for hyperbolic equations [2], the function $u_0(x, y)$ takes its positive maximum and negative minimum on the segment C_1C_2 . Suppose, for example, that the function $u_0(x, y)$ at the point $P(x_0, 0)$, $0 < x_0 < 1$, reaches its positive maximum. Then on the one hand

$$\frac{\partial u_0}{\partial y}(x_0, 0) = \lim_{y \rightarrow -0} \frac{u_0(x_0, y) - u_0(x_0, 0)}{y} \geq 0, \quad (9.9)$$

but on the other hand, because of the Zaremba–Giraud principle, for the domain D^+ at the point $P(x_0, 0) \in \partial D^+$ we have

$$\frac{\partial u}{\partial y}(x_0, 0) < 0,$$

which contradicts (9.9). The case of the negative minimum is considered analogously. ■

When considering the question on solvability of Problem B, we assume below that $c \equiv 0$, $\lambda \equiv 0$.

Having differentiated the condition (9.5) along the characteristic CC_1 , we obtain

$$(u_x - u_y)|_{CC_1} = \psi'. \quad (9.10)$$

The problem (7.1), (9.4), (9.10) is in fact Problem A under the conditions $c \equiv 0$, $p_1 = p$, $q_1 = q$, $\lambda_1 = 0$, $p_2 = 1$, $q_2 = -1$, $\lambda_2 = 0$, $\psi = \psi'$. Denote by κ_1 the index of the problem (7.1), (9.4), (9.10).

We have the following

Theorem 9.3. *Under the conditions $(p+iq)|_\sigma \neq 0$, $p(0) \neq 0$, $p(0)+q(0) = 0$, $c \equiv 0$, $\lambda \equiv 0$, $\kappa = 1$, $\varphi(0) = \psi'(0) = 0$, from the uniqueness of the solution of Problem B follows its existence.*

Proof. According to the definition of the index $\kappa_1 = d_1 - d_2$, where d_1 is the number of linearly independent solutions of the corresponding to (7.1), (9.4), (9.10) homogeneous problem and d_2 is the number of solvability conditions imposed on the right sides of the problem (7.1), (9.4), (9.10). It is obvious that under the conditions of Theorem 9.3, the function $u \equiv const$ is a solution of the corresponding to (7.1), (9.4), (9.10) homogeneous problem. Let us show that the homogeneous problem fails to have other solutions. Indeed, let \tilde{u} be a solution of the corresponding to (7.1), (9.4), (9.10) homogeneous problem. Then because of the fact that $\frac{\partial \tilde{u}}{\partial CC_1} = (\tilde{u}_x - \tilde{u}_y)|_{CC_1} = 0$ and hence $\tilde{u}|_{CC_1} = const$, the function $\tilde{u} - \tilde{u}(C_1)$ will be a solution of the homogeneous problem. The uniqueness of the solution of Problem B implies that either $\tilde{u} - \tilde{u}(C_1) \equiv 0$ or $\tilde{u} \equiv const$. Thus we have shown that $d_1 = 1$. But $\kappa_1 = 1$, therefore $d_2 = d_1 - \kappa_1 = 0$ and consequently, the nonhomogeneous problem (7.1), (9.4), (9.10) is undoubtedly solvable. Let \tilde{u} be its solution. Then the solution of Problem B has the form $u(P) = \tilde{u}(P) - \tilde{u}(C_1) + \psi(C_1)$, $P \in D$. Really, it is obvious that $u(P)$ satisfies both equations (7.1) and (9.4). It remains to verify the condition (9.5). As it is easily seen, $\frac{\partial \tilde{u}}{\partial CC_1} = \psi'$ implies that

$$\tilde{u}(P) - \tilde{u}(C_1) = \int_{C_1}^P \psi' ds = \psi(P) - \psi(C_1), \quad P \in CC_1.$$

Therefore for $P \in CC_1$ we have

$$u(P) = \tilde{u}(P) - \tilde{u}(C_1) + \psi(C_1) = \psi(P) - \psi(C_1) + \psi(C_1) = \psi(P),$$

which proves Theorem 9.3. ■

From Theorems 9.3 and 9.4 we have

Theorem 9.4. *Let (9.8) be fulfilled and $(p + iq)|_\sigma \neq 0$, $\cos(\widehat{l, N})|_\sigma > 0$, $p(0) \neq 0$, $p(0) + q(0) = 0$, $c \equiv 0$, $\lambda \equiv 0$, $\kappa_1 = 1$, $\varphi(0) = \psi'(0) = 0$. Then Problem B is uniquely solvable.*

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Author's address:
 Department of Mathematics (99)
 Georgian Technical University
 77, M. Kostava St., Tbilisi 380075
 Georgia