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THE UPPER ESTIMATE OF THE SPECTRAL RADIUS OF THE ISOTONIC OPERATOR IN THE SPACE OF CONTINUOUS FUNCTIONS

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For the isotonic compact integral operator

$$(Ax)(t) \stackrel{\text{def}}{=} \int_a^b K(t,s)x(s)ds \quad \big(K(t,s) \geq 0, \quad (t,s) \in [a,b] \times [a,b]\big)$$

in the space $\mathbb{C}[a,b]$ of continuous on $[a,b]$ functions the following assertion holds: the spectral radius $\rho(A)$ of $A : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ is less than 1 if and only if there exists a $v \in \mathbb{C}[a,b]$ such that

$$v(t) \geq 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad t \in [a,b].$$

Besides, the set of zeros of r is at most countable. This assertion plays an important role in the theory of differential equations. In the theory of functional differential equations, there arises the necessity in the estimate $\rho(A) < 1$ for the isotonic operator $A : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ which is not integral [1]. The above assertion is a corollary of G.G. Islamov's theorem [2, 3]. In accordance with this theorem, the inequality $\rho(A) < 1$ for a general isotonic compact linear operator $A : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ holds if and only if there exists a $v \in \mathbb{C}[a,b]$ such that

$$v(t) \geq 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad t \in [a,b],$$

the set of zeros of r being at most countable, and besides $r(t) > 0$ at some special points of $[a,b]$, the so-called "singular points".

The refusal from the compactness of A and the weakening of the demand concerning r became possible at the expense of some properties of A. We offer some development of the ideas proposed in [4].

Let $T \subset \mathbb{R}^1$ be a Lebesgue-measurable set, $\operatorname{mes} T \leq +\infty$, $\mathbb{C}$ be the Banach space of continuous bounded functions $x : T \to \mathbb{R}^1$, $\|x\|_{\mathbb{C}} = \sup\limits_{t \in T}|x(t)|$. Let further $\gamma : T \to \mathbb{R}^1$ be continuous, $\gamma(t) > 0$, $t \in T$, $\mathbb{C}^\gamma$ be a Banach space of the functions $x : T \to \mathbb{R}^1$ such that $\frac{x}{\gamma} \in \mathbb{C}$, $\|x\|_{\mathbb{C}^\gamma} = \sup\limits_{t \in T}\frac{|x(t)|}{\gamma(t)}$. The linear operator $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ is said to be isotonic, if $(Ax)(t) \geq 0$, $t \in T$, for any $x \in \mathbb{C}^\gamma$ such that $x(t) \geq 0$, $t \in T$.

Lemma. *Let $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ be linear, bounded and isotonic. $\rho(A) < 1$ if and only if there exists $v \in \mathbb{C}^\gamma$ such that*

$$\inf_{t \in T} \frac{v(t)}{\gamma(t)} > 0, \quad \inf_{t \in T} \frac{v(t) - (Av)(t)}{\gamma(t)} > 0.$$

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Note that for the case $T = [a, b]$, $\gamma(t) \equiv 1$ this assertion is well known.

Proof. The necessity is obtained by taking the solution of the equation $x - Ax = \gamma$ in the capacity of v.

To prove the sufficiency, let us introduce in the space $\mathbb{C}^\gamma$ a new norm $\|x\|_v = \sup\limits_{t \in T} \frac{|x(t)|}{v(t)}$. Then for the norm $\|A\|_v$ of A with respect to $\| \cdot \|_v$ we have $\|A\|_v = \|Av\|_v$. Since $\|Av\|_v < 1$, by the assertion we obtain $\rho(A) \leq \|A\|_v < 1$. ■

The demands concerning v and $r = v - Av$ might be weakened at the expense of additional assumptions on the properties of A. One of such properties is

Property M. We will say that a linear operator $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ has Property M, if $\inf\limits_{t \in T} \frac{(Ax)(t)}{\gamma(t)} > 0$ for any $x \in \mathbb{C}^\gamma$ such that $x(t) \geq 0$, $x(t) \not\equiv 0$, $t \in T$.

Theorem 1. *Let a linear bounded $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ have Property M. Let further there exist $v \in \mathbb{C}^\gamma$ such that*

$$\inf_{t \in T} \frac{v(t)}{\gamma(t)} > 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad r(t) \not\equiv 0, \quad t \in T.$$

Then $\rho(A) < 1$.

Proof. The proof is needed only in the case $\inf\limits_{t \in T} \frac{r(t)}{\gamma(t)} = 0$. Applying A to the both parts of the equality $v - Av = r$, we get $Av - A^2v = Ar$. From this and the inequality $v(t) - (Av)(t) \geq 0$ we have

$$r_1(t) \stackrel{\text{def}}{=} v(t) - (A^2v)(t) \geq (Ar)(t).$$

Consequently, $\inf\limits_{t \in T} \frac{r_1(t)}{\gamma(t)} > 0$. Because of Lemma, $\rho(A^2) < 1$. Thus

$$\rho(A) = \sqrt{\rho(A^2)} < 1. \ ■$$

Remark 1. It is impossible to weaken the condition of Lemma about v in the presence of Property M. Indeed, from $r(t) \geq 0$, there follow

$$\frac{v(t)}{\gamma(t)} \geq \frac{(Av)(t)}{\gamma(t)} \quad \text{and} \quad \inf_{t \in T} \frac{v(t)}{\gamma(t)} \geq \inf_{t \in T} \frac{(Av)(t)}{\gamma(t)} > 0,$$

if $v(t) \geq 0$, $v(t) \not\equiv 0$.

Property N. We will say that a linear operator $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ has Property N, if there exist a measurable set $\Delta \subset T$ and an element $\varphi \in \mathbb{C}^\gamma$ such that

$$\varphi(t) \geq 0, \quad \varphi(t) \not\equiv 0, \quad t \in T, \quad \inf_{t \in \Delta} \frac{\varphi(t) - 2(A\varphi)(t)}{\gamma(t)} > 0.$$

This property is common for some operators arising in studying multipoint boundary value problems and makes it possible to weaken the conditions of Lemma with respect to v as one can see by the following assertion.

Theorem 2. *Let a linear bounded isotonic $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ have Property N. Let further there exist $v \in \mathbb{C}^\gamma$ such that*

$$v(t) \geq 0, \quad t \in T, \quad \inf_{t \in T \setminus \Delta} \frac{v(t)}{\gamma(t)} > 0;$$

$$r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad t \in T, \quad \inf_{t \in T \setminus \Delta} \frac{r(t)}{\gamma(t)} > 0.$$

Then $\rho(A) < 1$.

The proof consists in constructing the bases of v and φ of a function satisfying the conditions of Lemma. Such will be the function $v_\varepsilon = v + \varepsilon(\varphi - a\varphi)$ with an $\varepsilon > 0$.

Property MN. We will say that a linear $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ has Property MN, if it has Property N and $\inf\limits_{t\in T\setminus\Delta} \frac{(Ax)(t)}{\gamma(t)} > 0$ for any $x \in \mathbb{C}^\gamma$ such that $x(t) \geq 0$, $x(t) \not\equiv 0$, $t \in T$.

Theorem 3. *Let a linear bounded isotonic $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ have Property MN. Let further there exist $v \in \mathbb{C}^\gamma$ such that*

$$v(t) \geq 0, \quad t \in T, \quad \inf_{t\in T\setminus\Delta} \frac{v(t)}{\gamma(t)} > 0;$$

$$r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad r(t) \not\equiv 0, \quad t \in T.$$

Then $\rho(A) < 1$.

The proof can be obtained by using the scheme of the proof of Theorem 1 and by replacing T by $T\setminus\Delta$ and substituting the reference to Theorem 2.

Remark 2. Due to Lemma, the conditions of Theorems 1, 2 and 3 with respect to v and r are necessary for the estimate $\rho(A) < 1$.

Corollary follows from Theorem 2 of [4].

Let $T = [a,b]$ and $A : \mathbb{C}^\gamma \to \mathbb{C}^\gamma$ be linear, bounded and isotonic. Let further the following conditions be satisfied: there exist the points $t_1, \ldots, t_k \in [a,b]$ such that $(Ax)(t_i) = 0$, $i = 1, \ldots, k$, for any $x \in \mathbb{C}^\gamma$. Then $\rho(A) < 1$ if and only if there exists $v \in \mathbb{C}^\gamma$ such that $v(t) > 0$ and $r(t) > 0$ for $t \in [a,b]\setminus\{t_1, \ldots, t_k\}$.

In this case, the operator A has Property N. Really, if we take as Δ the union of neighborhoods of the points $t_1, \ldots, t_k$ such that in these neighborhoods the inequality $\frac{(A\gamma)(t)}{\gamma(t)} \leq q < \frac{1}{2}$ holds, then

$$\inf_{t\in\Delta} \frac{\gamma(t) - 2(A\gamma)(t)}{\gamma(t)} > 0.$$

Example. Consider the boundary value problem

$$\begin{gathered} x^{(n)}(t) + \int\limits_a^b x(s)d_s r(t,s) = f(t), \quad n \geq 2, \quad t \in [a,b], \\ x^{(i)}(a) = 0, \quad i = 0, \ldots, n-2, \quad x(b) = 0 \end{gathered} \tag{1}$$

under the assumption that $r(t,\cdot)$ does not decrease on $[a,b]$ for almost all $t \in [a,b]$, $r(\cdot,s)$ is summable on $[a,b]$ for any $s \in [a,b]$ and $f(\cdot)$ is summable on $[a,b]$. A solution of (1) is understood to be a function x with absolutely continuous derivative of the $(n-1)$-th order which satisfy both the boundary value conditions and the equation almost everywhere on $[a,b]$.

We write

$$(Ax)(t) = -\int\limits_a^b G_0(t,s) \int\limits_a^b x(\tau)d_\tau r(s,\tau)ds, \tag{2}$$

$$g(t) = \int\limits_a^b G_0(t,s)f(s)ds,$$

where $G_0(t,s)$ is the Green function of the problem

$$x^{(n)}(t) = z(t), \quad x^{(i)}(a) = 0, \quad i = 0, \ldots, n-2, \quad x(b) = 0.$$

The operator $A : \mathbb{C}[a,b] \to \mathbb{C}[a,b]$ defined by (2) is isotonic since $G_0(t,s) < 0$ in the square $(a,b) \times (a,b)$. Besides, $(Ax)(a) = (Ax)(b) = 0$ for any $x \in \mathbb{C}[a,b]$. The function g and the values of A on continuous functions are functions with absolutely continuous derivative of the $(n-1)$-th order. Thus the equation

$$x = Ax + g$$

in the space $\mathbb{C}[a,b]$ is equivalent to the problem (1). Therefore the inequality $\rho(A) < 1$ guarantees unique solvability of the problem (1) for any summable f.

Let

$$v(t) = (t-a)^{n-1}(b-t) = -n! \int_a^b G_0(t,s)ds.$$

Then

$$r(t) = v(t) - (Av)(t) = -\int_a^b G_0(t,s)\Big[n! - \int_a^b (\tau - a)^{n-1}(b-\tau)d_\tau r(s,\tau)\Big]ds.$$

Thus $r(t) > 0$, $t \in (a,b)$, if almost everywhere on $[a,b]$

$$\int_a^b (\tau - a)^{n-1}(b-\tau)d_\tau r(t,\tau) \leq n! \tag{3}$$

and besides, the inequality is strict on a set of positive measure. Consequently, because of Corollary of Theorem 2 we have the estimate $\rho(A) < 1$.

The solution x of the problem (1) has the representation

$$x(t) = \int_a^b G(t,s)f(s)ds,$$

where $G(t,s)$ is the Green function of this problem [1]. From the equality

$$\int_a^b G(t,s)f(s)ds = g(t) + (Ag)(t) + (A^2 g)(t) + \cdots$$

it follows that $x(t)$ does not admit positive values if $f(t) \geq 0$. Therefore the inequality (3) guarantees the inequality $G(t,s) \leq 0$ in the square $(a,b) \times (a,b)$.

In the case of the equation with concentrated deviation of the argument

$$x^{(n)}(t) + p(t)x[h(t)] = f(t),$$
$$x(\xi) = 0, \quad \text{if} \quad \xi \notin [a,b],$$

under the assumption that $p(t)$ is bounded, $p(t) \geq 0$, and $h(t)$ is measurable, the inequality (3) takes the form

$$p(t)\sigma_h(t)[h(t) - a]^{n-1}[b - h(t)] \leq n!,$$

where

$$\sigma_h(t) = \begin{cases} 1, & \text{if} \quad h(t) \in [a,b], \\ 0, & \text{if} \quad h(t) \notin [a,b]. \end{cases}$$

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