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ON A CRITERION OF THE ABSENCE OF STRONGLY INCREASING SOLUTIONS OF THE EMDEN-FOWLER EQUATION

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For the Emden-Fowler type equation

$$u^{(n)} = p(t)|u(t)|^{\lambda} \operatorname{sign} u(t), \quad p(t) \geq 0, \quad t > 0, \quad \lambda > 1, \quad n \geq 2, \tag{1}$$

with a locally Lebesgue integrable function $p(t)$, different from zero on a set of positive measure in any neighborhood of $+\infty$, the problem on conditions of existence of strongly increasing solutions

$$(-1)^m u^{(i)}(t) > 0, \quad i = \overline{0, n-1}, \quad t \geq a, \quad \lim_{t \to +\infty} |u^{(n-1)}(t)| = +\infty, \tag{2}$$

where $m \in \{0, 1\}$, is considered.

I. T. Kiguradze and G. G. Kvinikadze [1, Th. 16.12] established that the condition

$$J(+\infty) < +\infty, \quad J(t) \equiv \int_a^t p(\tau)\tau^{(n-1)\lambda} d\tau \tag{3}$$

guarantee the solvability of the problem (1), (2). N. A. Izobov offered an approach [2] that allows to build necessary conditions of solvability of the named problem, and, as I. T. Kiguradze [1, Th. 16.13] has noticed, an asymptotic estimate of its solutions.

In the present offered note, contiguous to the works [3, 4] and their continuation, the above approach is much modified, which allows to raise considerably the efficiency of the necessary conditions of existence of strongly increasing solutions of the equation (1), to obtain with its help asymptotic estimates of such solutions, and to study deeper the problem on the necessity of the condition (3).

Introduce the notation. Let $u(t)$ be a certain determined on a half-axis $t > a > 0$ solution of the problem (1), (2), $0 < \varphi(t)$ be any function with a range of definition $D_\varphi \subset [a, +\infty)$. Put $v_i(t) = |u^{(i)}(t)|t^{i+1-n}$, $v_{\varphi,i}(t) = v_i(t)\varphi(t)$, $i = \overline{0, n}$, $t \in D_\varphi$; $A_\varphi(t) = \{x \in D_\varphi : (x - t)\varphi(x) > 0\}$, $A_\varphi = A_\varphi(a)$.

Characteristic, from our point of view, properties of strongly increasing solutions are given in the following simple

Lemma 1. *Let $u(t)$ be a solution of the problem (1), (2), $\varphi(t) > 0$ be any nondecreasing on the set A_p function satisfying $0 < t\dot{\varphi}(t)/\varphi(t) \leq 1$ for all $t \in A_{\dot{\varphi}} \bigcup A_p$. Then the functions $v_i(t)$, $i = \overline{0, n-1}$, do not decrease, beginning from a moment $t_u > a$, and for all $t \in A_p \bigcap A_{\dot{\varphi}}(t_u)$ and $i = \overline{0, n-2}$*

$$\dot{v}_{\varphi,n-1}(t) > t^{-1} v_{\varphi,n}(t), \quad \dot{v}_{\varphi,i}(t)/v_{\varphi,i+1}(t) > \dot{\varphi}(t)/((n-i-1)\varphi(t)) \tag{4}$$

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Proof. Let $u(t) > 0$ (the case $u(t) < 0$, is reduced to this one) be a strongly increasing solution of the equation (1) defined for $t > a$. Denote $R_i(t) = \dot{v}_i(t)t^{n-i} = tu^{(i+1)}(t) - (n-i-1)u^{(i)}(t)$, $i = \overline{0, n-1}$, $t > a$. It is easy to see that $R_{n-1}(t) = tu^{(n)}(t)$, and also, taking into account (2),

$$R_{n-2}(t) \geq tu^{(n-1)}(t) - (u^{(n-2)}(a) + (t-a)u^{(n-1)}(t)) > \\ > au^{(n-1)}(t)/2 \tag{5}$$

for all $t > t_{n-2}$ provided the moment t_{n-2} is chosen, such that

$$au^{(n-1)}(t_{n-2}) > 2u^{(n-2)}(a).$$

For the other values $i < n-2$, the fulfilment of the inequalities $R_i(t) > 0$ for large t will be proved by induction, the checking of the inequalities (5) being its first step. So, we will assume that for a natural $i < n-2$ and positive constants c and t_i the inequality $R_i(t) > cu^{(i+1)}(t)$ is fulfilled for all $t > t_i$. Then in view of the obvious equality $\dot{R}_{i-1}(t) = R_i(t)$ we have

$$R_{i-1}(t) = R_{i-1}(t_i) + \int\limits_{t_i}^{t} R_i(\tau)d\tau > \\ > R_{i-1}(t_i) + c(u^{(i)}(t) - u^{(i)}(t_i)) > cu^{(i)}(t)/2 > 0$$

for all $t > t_{i-1}$, where a moment $t_{i-1} > t_i$ is chosen such that $cu^{(i)}(t_{i-1})/2 > cu^{(i)}(t_i) - R_{i-1}(t_i)$. Hence, for all $t > t_u = t_0$ the inequalities $R_i(t) > 0$, $i = \overline{0, n-2}$, $R_{n-1}(t) \geq 0$, are fulfilled ensuring the nonnegativity of the first derivatives of the functions $v_i(t)$ for the same values of the argument. From here it follows that the functions $v_i(t)$ do not decrease on the half-axis $t > t_u$.

We will proceed to the proof of the inequalities (4). For any non-negative $i < n-2$ and $t \in A_{\dot{\varphi}} \bigcap A_p(t_u)$ we have

$$\dot{v}_{\varphi,i}(t)t^{n-i} \equiv R_{\varphi,i}(t) = \varphi(t)(R_i(t) + t\dot{\varphi}(t)u^{(i)}(t)/\varphi(t)) = \\ = \varphi(t)\left(\frac{t^2\dot{\varphi}(t)u^{(i+1)}(t)}{(n-i-1)\varphi(t)} + \left(1 - \frac{t\dot{\varphi}(t)}{\varphi(t)}\right)R_i(t)\right) > \frac{t^2\dot{\varphi}(t)u^{(i+1)}(t)}{n-i-1},$$

whence it follows (4).

The case $i = n-2$ is considered much easier:

$$\dot{v}_{\varphi,n-2}(t) = \frac{1}{t^2}(t\dot{\varphi}(t)u^{(n-2)}(t) + \varphi(t)R_{n-2}(t)) = \frac{\varphi(t)}{t^2}\left(\frac{tu^{(n-2)}(t)\dot{\varphi}(t)}{\varphi(t)} + \right. \\ \left. + R_{n-2}(t)\right) > \varphi(t)u^{n-1}(t) + (1 - t\dot{\varphi}(t)/\varphi(t))R_{n-2}(t) > v_{\varphi,n-1}(t)\dot{\varphi}(t)/\varphi(t).$$

To complete the proof, it remains to notice that the first inequality in (4) is evident. ■

The basic result of this article is the following

Theorem 1. *Let $u(t)$ be a solution of the problem* (1), (2), *$\varphi(t) > 0$ be any nondecreasing on the set A_p function satisfying $t\dot{\varphi}(t)/\varphi(t) \leq 1$ for all $t \in A_{\dot{\varphi}} \bigcup A_p$. Then for any numbers $\mu \in (0, 1/n)$ and $\varepsilon > 0$ we have*

$$\lim_{t \to +\infty} F_{\mu,\varepsilon}(\varphi(t)) = 0 \tag{6}$$

and, begincning from a moment $t_u > a$, the estimate

$$u(t) < \gamma t^{n-1}[F_{\mu,\varepsilon}(\varphi(t))]^{1/(1-\lambda)\mu} \tag{7}$$

where

$$F_{\mu,\varepsilon}(\varphi(t)) = \varphi^{\varepsilon}(t) \int_{t}^{+\infty} (p(\tau)\tau^{(n-1)\lambda})^{\mu} \varphi^{-\varepsilon}(\tau)(\dot{\varphi}(\tau)/\varphi(\tau))^{1-\mu} d\tau$$

and $\gamma > 0$ *is a constant dependent on the initial values of* u *and on* n, λ, μ.

Proof. Assume that the problem (1), (2) has a solution $u(t)$, which, not losing generality, we will assume to be positive. Choose any satisfying the conditions of the theorem function $\varphi(t)$ and numbers μ and ε, and define a sequence $\{\mu_i\}_{i=1}^{n}$ by $\mu_i = \mu + (n-i)\delta$, $i = \overline{1,n}$, where $\delta = 2(1-n\mu)/(n(n-1)) > 0$. Taking a number $\varepsilon_1 < n^{-1}\min\{\varepsilon, n\delta, (\lambda-1)\mu\}$, we obviously have

$$\lambda\mu_n - \mu_1 > \varepsilon_1, \quad \mu_i - \mu_{i+1} > \varepsilon_1, \quad i = \overline{1, n-1}. \tag{8}$$

For the first derivative of the auxiliary function $\omega_{\varphi}(t) = \prod_{i=0}^{n-1} v_{\varphi,i}(t)$, by virtue of (8) and Lemma 1 for all $t > t_u$ we have

$$\frac{\dot{\omega}_{\varphi}(t)}{\omega_{\varphi}(t)} = \sum_{i=0}^{n-1} \frac{\dot{v}_{\varphi,i}(t)}{v_{\varphi,i}(t)} > \frac{v_{\varphi,n}(t)}{v_{\varphi,n-1}(t)} + \sum_{i=0}^{n-2} \frac{v_{\varphi,i+1}(t)\dot{\varphi}(t)}{(n-i-1)v_{\varphi,i}(t)\varphi(t)} \geq$$
$$\geq \left(\frac{v_{\varphi,n}(t)}{v_{\varphi,n-1}(t)}\right)^{\mu} \prod_{i=0}^{n-2} \left(\frac{v_{\varphi,i+1}(t)\dot{\varphi}(t)}{(n-i-1)v_{\varphi,i}(t)\varphi(t)}\right)^{\mu_{i+1}} =$$
$$= c\left(\frac{\dot{\varphi}(t)}{\varphi(t)}\right)^{1-\nu} \left(\frac{v_{\varphi,n}(t)}{t}\right)^{\mu} v_{\varphi,0}^{-\mu_1}(t) \prod_{i=1}^{n-1} v_{\varphi,i}^{\mu_i-\mu_{i+1}}(t),$$

where c is a positive constant dependent only on n, λ, μ. From here it easily follows

$$\dot{\omega}_{\varphi}(t)\omega_{\varphi}^{-1-\varepsilon_1}(t)\varphi^{n\varepsilon_1-\varepsilon}(t) \geq c(\dot{\varphi}(t)/\varphi(t))^{1-\mu} \times$$
$$\times \varphi^{-\varepsilon}(t)(p(t)t^{(n-1)\lambda})^{\mu} v_0^{\lambda\mu-\mu_1-\varepsilon_1}(t) \prod_{i=1}^{n-1} v_i^{\mu_i-\mu_{i+1}-\varepsilon_1}(t).$$

Integrating this inequality from t to $+\infty$, by virtue of (8), monotonicity the of functions $\varphi(t)$, $v_i(t)$, $i = \overline{1, n-1}$, and the following from the proof of Lemma 1 inequalities $v_i(t) > v_0(t)$, $t > t_u$, $i = \overline{1, n-1}$, gives the equality (6) and the estimate (7). ■

Results of the works [3, 4] are consequences of Theorem 1 whith $\varphi(t) = t^{\varepsilon}$.

The necessary condition (6), due to a wide arbitrariness in the choice of the function $\varphi(t)$, allows to solve some remaining till now open problems, connected with the necessity of the sufficient condition (3) of solvability of the problem (1), (2). For example, it appeared, that for a wide class of functions $p(t)$ (in particular, those having a power majorant), the condition (3) is necessary and sufficient for the problem under consideration to have a solution.

The classical case $p(t) < ct^{-1-(n-1)\lambda}$, $t > a$, partially investigated in [5], is completely considered in

Theorem 2. *If a function* $p(t)$ *satisfies* $p(t) < ct^{-1-(n-1)\lambda}$, $t > a$, *and* $J(+\infty) = +\infty$, *then the problem* (1), (2) *has no solution.*

Proof. Assume that the conditions of the theorem are fulfilled. Then the function $\varphi(t) = J(t)$ monotonically increases on a the half-axis $t>a$ and $t\dot{\varphi}(t)/\varphi(t)=p(t)t^{1+(n-1)\lambda}/J(t) < c/J(t) < 1$ for $t > b$, where $b > a$ is such that $J(b) > c$. Therefore $t\dot{\varphi}(t)/\varphi(t) \leq 1$ and it is enough to be convinced that the condition (6) is not fulfilled. Really,

$$\lim_{t\to\infty} J^{\varepsilon}(t) \int_t^{+\infty} \frac{(J'(\tau))^{\mu}}{J^{\varepsilon}(\tau)} \left(\frac{J'(\tau)}{J(\tau)}\right)^{1-\mu} d\tau = \lim_{t\to+\infty} J^{\varepsilon}(t) \int_t^{+\infty} \frac{dJ(\tau)}{J^{1-\mu+\varepsilon}(\tau)} = +\infty$$

provided $\varepsilon \leq \mu$. ■

Thus, the sufficient condition (3) of solvability of the problem (1), (2) becomes necessary and sufficient provided in a neighborhood of $+\infty$ $p(t) < ct^{-1-(n-1)\lambda}$.

A more common situation is considered in

Theorem 3. *If there is a function* $\psi : A_p \to]0,1[$ *such that* $J_{\psi}(+\infty) = +\infty$, $J'_{\psi}(t)/J_{\psi}(t) < 1/\delta t$, $\delta > 0$, $t \in A_p$, *where* $J_{\psi}(t) \equiv \int_a^t p(\tau)\tau^{(n-1)\lambda} \times \psi(\tau)d\tau$, *then the problem* (1), (2) *is unsolvable.*

Proof. Define $\varphi(t) = J^{\delta}_{\psi}(t)$. Then $\dot{\varphi}(t)/\varphi(t) = \delta J'_{\psi}(t)/J_{\psi}(t) < 1/t$ for all $t \in A_{\psi}$ and, if $\mu \neq \varepsilon\delta$, $F_{\mu,\varepsilon}(\varphi(t)) = cJ^{\varepsilon\delta}_{\psi}(t) \int_t^{+\infty} J'_{\psi}(\tau)\psi^{-\mu}(\tau)\times \times J_{\psi}^{-1+\mu-\varepsilon\delta}(\tau)d\tau > cJ^{\varepsilon\delta}_{\psi}(t)$, $\int_t^{+\infty} J_{\psi}^{-1+\mu-\varepsilon\delta}(\tau)d[J_{\psi}(\tau)] > cJ^{\mu}_{\psi}(t) \uparrow +\infty$ as it $t \uparrow +\infty$, whence it is clear that in the considered case the condition (6) of Theorem 1 is not fulfilled and, hence, the problem (1), (2) is unsolvable. ■

One of the possible even more general situations is considered in

Theorem 4. *If there are functions* $\psi(t)$ *and* $f(t)$ *satisfying at large* t

$$\psi(t) > 1 > f(t) > 0, \quad P_{f,\psi}(t) \equiv \frac{J'_f(t)}{\psi(J_f(t))} < \frac{1}{t}, \quad \int_a^{+\infty} P_{f,\psi}(\tau)d\tau = +\infty,$$

the problem (1), (2) *is unsolvable.*

Proof. Assume that the conditions of the theorem are fulfilled and show that in this case the condition (6) of Theorem 1 is not fulfilled for $\varphi(t) = \exp\left(\int_1^t dJ_{p,f}(\tau)\big/\psi(J_{p,f}(\tau))\right)$. Really, in this case $\dot{\varphi}(t)/\varphi(t) = J'_{p,f}(t)/J_{p,f}(t) < 1/t$ for all $t \in A_p$ and the estimate

$$F_{\mu,\varepsilon}(\varphi(t)) = \varphi^{\varepsilon}(t) \int_t^{+\infty} \left(\frac{\psi(J_{p,f}(\tau))}{f(\tau)}\right)^{\mu} \frac{J'_{p,f}(\tau)d\tau}{\psi(J_{p,f}(\tau))\varphi^{\varepsilon}(\tau)} >$$

$$> \varphi^{\varepsilon}(t) \int_t^{+\infty} \frac{d\varphi(\tau)}{\varphi^{1+\varepsilon}(\tau)} = \varepsilon^{-1} > 0$$

is true for all $\mu \in (0, 1/n)\varepsilon > 0$ and all rather large values of t. From here it follows that the condition (6) is not fulfilled and, by Theorem 1, the problem (1), (2) is unsolvable. ■

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