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THE PROBLEM OF OPTIMAL CONTROL FOR NONLINEAR SYSTEMS WITH VARIABLE STRUCTURE, DELAYS AND PIECEWISE CONTINUOUS PREHISTORY

Abstract. Necessary conditions both for the optimality of controls and initial functions are proved in the form of an integral maximum principle and conditions of transversality for nonlinear systems with a variable structure, delays and a piecewise continuous initial function in the case where values of the initial function (prehistory) and of the trajectory at a non-fixed initial moment and at a moment of variation of the structure do not, generally speaking, coincide.

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რეზიუმე. არაწრფივი დაგვიანებულ არგუმენტიანი ცვლადსტრუქტურიანი სისტემისათვის უბან-უბან უწყვეტი საწყისი ფუნქციით დამტკიცებულია მართვის და საწყისი ფუნქციის ოპტიმალურობის აუცილებელი პირობები ინტეგრალური მაქსიმუმის პრინციპისა და ტრანსვერსალობის პირობების სახით იმ შემთხვევაში, როცა საწყისი ფუნქციის (წინაისტორიის) და ტრაექტორის მნიშვნელობები არაფიქსირებულ საწყის მომენტში და სტრუქტურის ცვლილების მომენტში, საზოგადოდ, არ ემთხვევიან ერთმანეთს.

1. Introduction

Necessary conditions of optimality for the problem given below (see 2) are derived from a necessary condition of criticality [1,2,3] the basis of which forms the notion of quasi-convex set introduced by R.V. Gamkrelidze [4].

Variation of the structure of a system means that the system at some beforehand unknown moment may go over from one law of movement to another. After variation of the structure, the values of the initial function of the system depend on its previous state. This joins them into a single system with variable structure.

In conclusion, it should be noted that some particular cases of the problem under consideration have been studied in [5].

2. Statement of the Problem. Necessary Conditions of Optimality

Let $O_i \subset R^{n_i}$, $G_i \subset R^{r_i}$, $i = 1,\dots,m$, be open sets, R^n be an n-dimensional Euclidean space, $J = [a,b]$ be a finite interval; let the functions $f_i : J \times O_i^2 \times G_i^2 \to R^{n_i}$, $i = 1,\dots,m$, be continuously differentiable with respect to $(x_i, z_i) \in O_i^2$, $i = 1,\dots,m$, respectively, let $\tau_i : R^1 \to R^1$, $\theta_i : R^1 \to R^1$, $i = 1,\dots,m$, be continuously differentiable functions satisfying the conditions $\tau_i(t) \le t$, $\dot{\tau}_i(t) > 0$, $\theta_i(t) < t$, $\dot{\theta}_i(t) > 0$; let $q^k(t_1,\dots,t_{m+1},x_{10},x_1,\dots,x_{m0},x_m)$, $k = 0,\dots,l$, be scalar functions continuously differentiable in all arguments $t_i \in J$, $i = 1,\dots,m+1$, $(x_{i0},x_i) \in O_i^2$, $i = 1,\dots,m$; let the functions $g_i : J \times O_{i-1} \to O_i$, $i = 2,\dots,m$ be continuous and continuously differentiable with respect to $x_{i-1} \in O_{i-1}$, $i = 1,\dots,m$; respectively, let $\Delta_i = \Delta(J_{i1}; N_i)$ be the set of piecewise continuous functions $\varphi_i : J_{i1} \to N_i$ with a finite number of points of discontinuity, $J_{i1}[\tau_i(a),b]$, $N_i \subset O_i$ be a convex bounded set, $\|\varphi_i\| = \sup\{\varphi_i(t)|^t \in J_{i1}\}$; let Ω_i be the set of measurable functions $u_i : J_{i2} \to U_i$ satisfying the condition: $clu_i(J_{i2})$-is a compactum lying in G_i, $J_{i2} = [\theta_i(a),b]$, $U_i \subset G_i$ is an arbitrary set.

Consider the sets

$$A_i = J^{1+i} \times \prod_{p=1}^{i} O_p \times \prod_{p=1}^{i} \Delta_i \times \prod_{p=1}^{i} \Omega_p, \quad i = 1,\dots,m,$$

with the elements $\sigma_i = (t_1,\dots,t_{i+1},x_{10},\dots,x_{i0},\varphi_1,\dots,\varphi_i,u_1,\dots,u_i)$, $i = 1,\dots,m$; $\prod_{p=1}^{i} O_p = O_1 \times \cdots \times O_i$, respectively.

To every element $\sigma_m \in A_m$, $t_i < t_{i+1}$, $i = 1,\dots,m$, we assign the differential equation of variable structure

$$\dot{x}_i(t) = f_i(t, x_i(t), x_i(\tau_i(t)), u_i(t), u_i(\theta_i(t))), \quad t \in [t_i, t_{i+1}], \tag{1_i}$$

$$x_i(t) = \varphi_i(t) + g_i(t, x_{i-1}(t)), t \in [\tau_i(t_i), t_i), x_i(t_i) = x_{i0}, \tag{2_i}$$

$$i = 1,\dots,m.$$

Here and in the sequel we assume that $g_1 = 0$, that is, $x_1(t) = \varphi_1(t), t \in [\tau_1(t_1), t_1)$.

Definition 1. The set of functions $\{x_i(t) = x_i(t;\sigma_i), t \in [\tau_i(t_i), t_{i+1}], i = 1,\dots,m\}$, is said to be a solution of the equation with variable structure which correspond to an element $\sigma_m \in A_m$, if the function $x_i(t) \in O_i$ on the interval $[\tau_i(t_i), t_i]$ satisfies the condition (2_i), while on the interval $[t_i, t_{i+1}]$ it is absolutely continuous and satisfies the equation (1_i) almost everywhere.

Definition 2. The element $\sigma_m \in A_m$ is said to be admissible if the corresponding solution $\{x_i(t), t \in [\tau_i(t_i), t_{i+1}], i = 1,\dots,m\}$ satisfies the conditions

$$q^k(t_1,\dots,t_{m+1},x_{10},x_1(t_2),\dots,s_{m0},x_m(t_{m+1})) = 0, \quad k = 1,\dots,l.$$

The set of admissible elements will be denoted by A_m^0.

Definition 3. The element $\widetilde{\sigma}_m = (\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\dots,\widetilde{x}_{m0},\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_m, \widetilde{u}_1\dots,\widetilde{u}_m) \in A_m^0$ is said to be locally optimal if there exist a number $\delta > 0$ and a compact set $K_i \subset 0_i, i = 1,\dots,m$, such that for arbitrary elements $\sigma_m \in A_m^0$ satisfying

$$\sum_{i=1}^{m+1} |\widetilde{t}_i - t_i| + \sum_{i=1}^{m}\{|\widetilde{x}_{i0} - x_{i0}| + \|\widetilde{\varphi} - \varphi_i\| + \|\widetilde{f}_i - f_i\|_{K_i}\} \le \delta$$

the inequality

$$\begin{aligned}&q^0(\widetilde{t}_1,\dots,\widetilde{t}_{m+1},\widetilde{x}_{10},\widetilde{x}_1(\widetilde{t}_2),\dots,\widetilde{x}_{m0},\widetilde{x}_m(\widetilde{t}_{m+1})) \le \\ &\le q_0(t_1,\dots,t_{m+1},x_{10},x_1(t_2),\dots,x_{m0},x_m(t_{m+1}))\end{aligned}$$

is fulfilled.

Here

$$\begin{aligned}&\big\|\widetilde{f}_i - f_i\big\|_{K_i} = \int_J H\big(t;f_i,K_i\big)\,dt, H\big(t;f_i,K_i\big) = \\ &= \sup\Big\{\big|\widetilde{f}_i\big(t,x_i,y_i\big) - f_i\big(t,x_i,y_i\big)\big| + \Big|\frac{\partial \widetilde{f}_i(\cdot)}{\partial x_i} - \\ &-\frac{\partial f_i(\cdot)}{\partial x_i}\Big| + \Big|\frac{\partial \widetilde{f}_i(\cdot)}{\partial y_i} - \frac{\partial f_i(\cdot)}{\partial y_i}\Big|\big(x_i,y_i\big) \in K_i^2\Big\}; \\ &\widetilde{f}_i\big(t,x_i,y_i\big) = f_i\big(t,x_i,y_i,\widetilde{u}_i(t),\widetilde{u}_i(\theta_i(t))\big), f_i(t,x_i,y_i) = \\ &= f_i(t,x_i,y_i,u_i(t),u_i(\theta_i(t))\big),\ \widetilde{x}_i = x_i(t;\widetilde{\sigma}_i),\ x_i(t) = x_i(t;\sigma_i).\end{aligned}$$

The problem of optimal control consists in finding a locally optimal element.

Theorem 1. *(Necessary conditions of optimality). Let* $\widetilde{\sigma}_m \in A_m^0$ *be a locally optimal element;* $a \le \widetilde{t}_1 < \cdots < \widetilde{t}_{m+1} \le b$, $\widetilde{t}_i < \gamma_i = \gamma_i(\widetilde{t}_i) < \tau_{i+1}(\widetilde{t}_{i+1})$ $i = 1, \ldots, m_{-1}$, $\widetilde{t}_m < \gamma_m = \gamma_m(\widetilde{t}_m) \neq \widetilde{t}_{m+1}$*; let the functions* $(\widetilde{u}_i(t), \widetilde{u}_i(\theta_i(t)))$ $i = 1, \ldots, m$, *be continuous at the points* $\widetilde{t}_i, \gamma_i, \widetilde{t}_{i+1}$, *respectively,* $i = 1, \ldots, m$, *and the functions* $(\widetilde{\varphi}_i(t), \widetilde{\varphi}_i(\tau_i(t)))$, $i = 1, \ldots, m$ *be continuous at the points* $\widetilde{t}_i$, $i = 1, \ldots, m$, *respectively. Then there exist a non-zero vector* $n = (n_0, \ldots, n_l)$, $n_0 \le 0$, *and a solution* $\psi_i(t)$, $t \in [\widetilde{t}_i, \gamma_i(\widetilde{t}_{i+1})]$ *of the conjugate equation*

$$\begin{gathered}\dot{\psi}_i(t) = -\psi_i(t)\frac{\partial \widetilde{f}_i(t)}{\partial x_i} - \psi_i(\gamma(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t) - \\ -\chi_i(t)\psi_{i+1}(\gamma_{i+1}(t))\frac{\partial \widetilde{f}_{i+1}(\gamma_{i+1}(t))}{\partial y_{i+1}}\frac{\partial \widetilde{g}_{i+1}(t)}{\partial x_i}\dot{\gamma}_{i+1}(t) \\ t \in [\widetilde{t}_i, \widetilde{t}_{i+1}], \quad \psi_i(t) = 0, \quad t \in (\widetilde{t}_{i+1}, \gamma_i(\widetilde{t}_{i+1})], \\ i = m, \ldots, 1,\end{gathered} \tag{3_i}$$

such that the following conditions are fulfilled:

1) *the integral maximum principle*

$$\int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\gamma_i(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\widetilde{\varphi}_i(t)dt \ge$$

$$\int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} \psi_i(\gamma_i(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\varphi_i(t)dt, \ \ \forall \varphi_i \in \Delta_i, \quad i = 1, \ldots, m, \tag{4}$$

$$\begin{gathered}\int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t)\widetilde{f}_i(t)dt \ge \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \psi_i(t)f_i(t, \widetilde{x}_i(t), \widetilde{x}_i(\tau_i(t)u_i(t), u_i(\theta_i(t)))dt, \\ \forall \varphi_i \in \Delta_i, \quad i = 1, \ldots, m.\end{gathered} \tag{5}$$

2) *transversality conditions*

$$\begin{gathered}\pi\frac{\partial Q}{\partial t_i} = -a_i\psi_{i-1}(\widetilde{t}_i)\widetilde{f}_{i-1}(\widetilde{t}_i) + b_i\{\psi_i(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i) + \psi_i(\gamma_i) \\ [\widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{x}_{i0}) - \widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{\varphi}_i(\widetilde{t}_i) + \widetilde{g}_i(\widetilde{t}_i))]\dot{\gamma}_i(\widetilde{t}_i)\}, \\ i = 1, \ldots, m+1,\end{gathered} \tag{6}$$

$$\pi\frac{\partial Q}{\partial x_{i0}} = -\psi_i(\widetilde{t}_i), \quad \pi\frac{\partial Q}{\partial x_i} = \psi_i(\widetilde{t}_{i+1}), \quad i = 1, \ldots, m. \tag{7}$$

Here

$$\widetilde{f}_i(t) = \widetilde{f}_i(t, \widetilde{x}_i(t), \widetilde{x}_i(\tau_i(t))), \quad \frac{\partial \widetilde{f}_i(t)}{\partial x_i} = \frac{\partial}{\partial x_i} f_i(t, \widetilde{x}_i(t), \widetilde{x}_i(t)),$$
$$\widetilde{g}_i(t) = g_i(t, \widetilde{x}_{i-1}(t)), \quad \frac{\partial \widetilde{g}_{i+1}(t)}{\partial x_i} = \frac{\partial}{\partial x_i} g_{i+1}(t, \widetilde{x}_i(t));$$

$\chi_i(t)$ *is the characteristic function of the segment* $[\tau_i(\widetilde{t}_{i+1}), \widetilde{t}_{i+1})]$*;* $g_{m+1} = 0$*, i.e., the last summand in the right-hand side of the equation* (3_m) *equals zero;* $\gamma_i(t)$ *is the function inverse to* $\tau_i(t)$*;* $a_1 = 0$*,* $a_2 = \cdots = a_{m+1} = 1$*,* $b_1 = \cdots = b_m = 1$*,* $b_{m+1} = 0$*; tilde over* Q *denotes that the gradient is calculated at the point* $(\widetilde{t}_1, \ldots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \widetilde{x}_1(\widetilde{t}_2), \ldots, \widetilde{x}_{m0}, \widetilde{x}_m(\widetilde{t}_{m+1}),$

Remark 1. If

$$\operatorname{rank}\left(\frac{\partial Q}{\partial t_1}, \ldots \frac{\partial Q}{\partial t_{m+1}}, \frac{\partial Q}{\partial x_{10}}, \ldots \frac{\partial Q}{\partial x_{m0}}, \frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial Q}{\partial x_m}\right) = 1 + l,$$

then

$$\sum_{i=1}^{m} \max\{|\psi_i(t)||t \in [\widetilde{t}_i, \widetilde{t}_{i+1}]|\} \neq 0.$$

Remark 2. From the integral maximum principle 1) one can obtain in a standard way the pointwise maximum principle with respect to the functions $\widetilde{\varphi}_i(t)$, $i = 1, \ldots, m$:

$$\psi_i(\gamma_i(t)) \frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i} \dot{\gamma}_i(t) \widetilde{\varphi}_i(t) \geq \psi_i(\gamma_i(t)) \frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i} \dot{\gamma}_i(t) \varphi_i,$$
$$\forall \varphi_i \in N_i, \quad \text{a.e. on} \quad [\tau_i(\widetilde{t}_i), \widetilde{t}_i], \quad i = 1, \ldots, m;$$

with respect to the controls $\widetilde{u}_i(t)$, $i = 1, \ldots, m$,

$$\sum_{p=1}^{p_i} \dot{\theta}_i^{p-1}(t) \psi_i(\theta_i^{p-1}(t)) \widetilde{f}_i(\theta_i^{p-1}(t)) + \sum_{p=1}^{\nu_i - p_i} \cdot \dot{q}_i^p(t) \psi_i(q_i^p(t)) \widetilde{f}_i(q^p(t)) \geq$$
$$\geq \sum_{p=1}^{p_i} \dot{\theta}_i^{p-1}(t) \psi_i(\theta_i^{p-1}(t)) f_i(\theta^{p-1}(t), \widetilde{x}_i(\theta_i^{p-1}(t)), \widetilde{x}_i(\tau_i(\theta_i^{p-1}(t))),$$
$$u_{i,p_i-p+1}, u_{i,p_i-p}) + \sum_{p=1}^{\nu_i - p_i} \dot{\rho}_i^p(t) \psi_i(\rho_i^p(t)) f_i(\rho_i^p(t), \widetilde{x}_i(\rho_i^p(t)), \widetilde{x}_i(\tau_i(\rho_i^p(t))),$$
$$u_{i,p_i+p}, u_{i,p_i+p-1}), \quad \forall (u_{10}, \ldots, u_{1,\nu_i}) \in u_i^{1+\nu_i}, \quad \text{a.e. on}$$
$$[\xi_{i,p_i}, \xi_{i,p_i+1}], \quad p_i = 1, \ldots, \nu_i, \quad i = 1, \ldots, m.$$

Here

$$\xi_{i,p_i} = \rho_i(\xi_{i,p_i-1}), \quad p_i = 1, \ldots, \nu_i, \quad \xi_{i,0} = \theta_i(\widetilde{t}_i), \quad \xi_{i,p_i+1} = \widetilde{t}_{i+1};$$

$\theta_i^0(t) = t$, $\theta_i^p(t) = \theta_i(\theta_i^{p-1}(t))$; $\rho_i(t)$ is the function inverse to $\theta_i(t)$, $\rho_i^1(t) = \rho_i(t)$; we assume that $\sum_{p=1}^{0} \alpha_p = 0$.

Theorem 2. *Let $\widetilde{\sigma}_m \in A_m^0$ be a locally optimal element; $a < \widetilde{t}_1 < \cdots < \widetilde{t}_{m+1} < b$, $\widetilde{t}_i < \gamma_i < \tau_{i+1}\widetilde{t}_{i+1}$, $i = 1,\dots,m-1$, $\widetilde{t}_m < \gamma_m \neq \widetilde{t}_{m+1}$; let the functions $(\widetilde{u}_i(t), \widetilde{u}_i(\theta_i(t)))$, $i = 1,\dots,m$, be left- (right-) continuous at the points $\widetilde{t}_i, \gamma_i, \widetilde{t}_{i+1}$, $i = 1,\dots,m$, respectively. Then there exist a non-zero vector $\pi(\mp) = (\pi_0(\mp),\dots,\pi_q(\mp))$ and a solution $\psi_i(\mp)(t)$, $t \in [\widetilde{t}_i, \gamma_i(\widetilde{t}_{i+1})]$ of the conjugate equation (3_i), $i = 1,\dots,m$, such that the conditions in which we have to substitute π and $\psi_i(t)$ instead of $\pi(\mp)$ and $\psi_i^{(\mp)}$, respectively, $i = 1,\dots,m$, are fulfilled. Moreover, the equality (6) is replaced by the inequality*

$$\pi^- \frac{\partial\widetilde{\theta}}{\partial t_i} \geq -a_i\psi_{i-1}^-(\widetilde{t}_i)\widetilde{f}_{i-1}(\widetilde{t}_i-) + b_i\{\psi_i^-(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i-) + \psi_i^-(\gamma_i)[\widetilde{f}_i(\gamma_i-, \widetilde{x}_i(\gamma_i), \widetilde{x}_{10}) - \widetilde{f}_i(\gamma_i-, \widetilde{x}_i(\gamma_i), \varphi_i(\widetilde{t}_i-) + \widetilde{g}_i(\widetilde{t}_i))]\dot{\gamma}_i(\widetilde{t}_i)\}, \quad i = 1,\dots,m+1$$

$$\Big(\pi^+ \frac{\partial\widetilde{\theta}}{\partial t_i} \leq -a_i\psi_{i-1}^+(t)\widetilde{f}_{i-1}(\widetilde{t}_i+) + b_i\{\psi_i^+(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i+)\psi_i^+(\gamma_i)[\widetilde{f}_i(\gamma_i+, \widetilde{x}_i(\gamma_i), \widetilde{x}_{10}) - \widetilde{f}_i(\gamma_i-, \widetilde{x}_i(\gamma_i), \widetilde{\varphi}_i(\widetilde{t}_i+) + g_i(\widetilde{t}_i))]\dot{\gamma}_i(\widetilde{t}_i)\}\Big),$$

where

$$\widetilde{f}_i(\widetilde{t}_i^{(\mp)}) = \widetilde{f}(\widetilde{t}_i^{(\mp)}, \widetilde{x}_i(\widetilde{t}_i), \widetilde{x}_i(\widetilde{\tau}_i(\widetilde{t}_i^{(\mp)})).$$

Consider now for (1_i) and (2_i), $i = 1,\dots,m$, where $g_2 = \cdots = g_m = 0$, the problem with the boundary conditions

$$q_1(t_1, x_{10}) = 0, q_i(t_i, x_{i-1}(t_i)) - x_{i0} = 0, \quad i = r,\dots,m, \\ q_{m+1}(t_{m+1}, x_m(t_{m+1})) = 0, \tag{8}$$

and with the functional

$$q_0(t_{m+1}, x_m(t_{m+1})) \to \min. \tag{9}$$

The functions

$$q_1 : J \times 0_1 \to R^{l_1}, q_i : J \times 0_i \to R^{n_i}, \quad i = 2,\dots,m, \\ q_{m+1} : J \times 0_m \to R^{l_2}, \quad q^0 : J \times 0_m \to R^1$$

are assumed to be continuously differentiable in all arguments.

The function Q for the problem under consideration is of the form

$$Q = (q_0, q_1, q_2 - x_{20}, \dots, q_m - x_{m0}, q_{m+1}). \tag{10}$$

Taking into account (10), from Theorem 1 there follows

Theorem 3. *Let σ_m be a locally optimal element of the problem* (1_i), (2_i), $i=1,\dots,m$, (8), (9) *and let the conditions of Theorem 1 be fulfilled. Then there exist a non-zero vector* $\chi=(\chi_0,\chi_1,\dots,\chi_{m+1})$, $\chi_0\le 0$, $\chi_1\in R^{l_1}$, $\chi_i\in R^{n_i}$, $i=1,\dots,m$, $\chi_{m+1}\in R^{l_2}$ *and a solution* $\psi_i(t)$, $t\in[\widetilde{t}_i,\gamma_i(\widetilde{t}_{i+1})]$, *of the conjugate equation* (3_i) $i=1,\dots,m$, *where the last term in the right-hand side equals zero, such that the conditions* (4), (5) *are fulfilled, while the conditions of transversality take the form*

$$\chi_i\frac{\partial\widetilde{q}_i}{\partial t_i}=-a_i\psi_{i-1}(\widetilde{t}_i)\widetilde{f}_{i-1}(\widetilde{t}_i)+b_i\{\psi_i(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i)+$$

$$+\psi_i(\gamma_i)[\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{x}_{i0})-\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{\varphi}_i(\widetilde{t}_i))]\dot{\gamma}_i(\widetilde{t}_i)\}-c_i\chi_0\frac{\partial\widetilde{q}_i^0}{\partial t_{m+1}},$$

$$i=1,\dots,m+1,\chi_1\frac{\partial\widetilde{q}_1}{\partial x_{10}}=\psi_1(\widetilde{t}_1),\quad \psi_i(\widetilde{t}_i)=\chi_i,\quad i=2,\dots,m,$$

$$\chi_i\frac{\partial q_i}{\partial x_{i-1}}=\psi_{i-1}(\widetilde{t}_i),\quad i=2,\dots,m,$$

$$\chi_0\frac{\partial\widetilde{q}_0}{\partial x_m}+\chi_{m+1}\frac{\partial\widetilde{q}_{m+1}}{\partial x_m}=\psi_m(\widetilde{t}_{m+1}).$$

.

Here

$$c_1=\dots=c_m=0,\quad c_{m+1=1},\quad \widetilde{x}_{i0}=q_i(\widetilde{t}_i,\widetilde{x}_{i-1}(\widetilde{t}_i)),\quad i=2,\dots,m.$$

3. Proof of Theorem 1.

The necessary conditions of optimality are proved by the scheme given in [1, 2, 3]. When applying this scheme, the principal moment is the construction of a continuous and differentiable mapping which plays an important role in deriving the necessary conditions of optimality. To this end we present below and prove (see 3.1 and 3.2) the appropriate theorems.

3.1. Continuous Dependence and Differentiability of the Solution. Let $O\subset R^n$ be an open set: $E(J\times O^2;R^n)$ be a space of n-dimensional functions $f;J\times 0^2\to R^n$ satisfying the following conditions:

3) for a fixed $t\in J$ the function $f(t,x,y)$ is continuously differentiable with respect to $(x,y)\in O^2$;

4) for a fixed $(x,y)\in O^2$ the functions f,f_x,f_y are measurable with respect to t; for an arbitrary compactum $K\subset O$ and an arbitrary function f there exists a function $m_{f,k}(t)\in L_1(J,R^1_+)$, $R^1_+=[0,\infty)$ such that

$$|f(t,x,y)|+|f_x(\dot{})|+|f_y(\dot{})|\le m_{f,K}(t)\quad\forall(t,x,y)\in J\times K^2.$$

In the space $E(J\times O^2;R^n)$, let us introduce by means of the bases of neighborhoods of zero two locally convex separable topologies [1, 2, 3],

$\{V_s(K;\delta) \subset E(J \times 0^2; R^n)$ a compactum K and a number $\delta > 0$ are arbitrary$\}$, $s = 1, 2$, where

$$V_1(K;\delta) = \{\delta f \in E(J \times O^2, R^n) | h(\delta f; K) \le \delta\}, \tag{11}$$

$$V_2(K;\delta) = \{\delta f \in E(J \times O^2, R^n) | \int_J H_0(t;\delta f, K)dt \le \delta\}, \tag{12}$$

$$h(\delta f; K) = \sup\Big\{ \int_{t'}^{t''} \delta f(t,x,y)dt | (t',t'',x,y) \in J^2 \times K^2 \Big\},$$

$$H_0(t, \delta f; K) = \sup\Big\{ |f(t,x,y)| + |f_x(\cdot)| + |f_y(\cdot)| \big| (x,y) \in K^2 \Big\}.$$

Consider the sets

$$B_i = J^{1+i} \times \prod_{p=1}^{i} O_p \times \prod_{p=1}^{i} \Delta(J_{i1}; O_i) \times \prod_{p=1}^{i} E(J \times O_i^2; R^{n_i}), \quad i = 1, \dots, m,$$

with the elements $\mu_i = (t_1, \dots, t_{i+1}, x_{10}, \dots, x_{i0}, \varphi_1, \dots, \varphi_i, f_1, \dots, f_i)$, $i = 1, \dots, m$ respectively. In what follows we will assume that the topologies T_1, T_2 are prescribed in the spaces $E(J \times O_i^2; R^{n_i})$ and $i = 1, \dots, m$ (see (11) and (12)).

To every element $\mu_m \in B_m$ there corresponds the differential equation of variable structure:

$$\dot{x}_i(t) = f_i(t, x_i(t), x_i(\tau(t))), \quad t \in [t_i, t_{i+1}], \tag{13_i}$$

$$x_i(t) = \varphi_i(t) + g_i(t, x_{i-1}(t)), \quad t \in [\tau_i, (t_i), t_{i+1}),$$

$$x_i(t_i) = x_{i0}, \quad i = 1, \dots, m. \tag{14_i}$$

Definition 4. A set of functions $\{x_i(t) = x_i(t;\mu_i), t \in [\tau_i(t_i), t_{i+1}], i = 1, \dots, m\}$ is said to be a solution of the equation of variable structure corresponding to the element $\mu_m \in B_m$, if the function $x_i(t) \in 0_i$ on the segment $[\tau_i(t_i), t_i]$ satisfies the condition (14_i) and on the interval $[t_i, t_{i+1}]$ is absolutely continuous and satisfies the equation (13_i) a.e.

Theorem 4. *Let $\{\widetilde{x}_i(t), t \in [J_i(\widetilde{t}_i), \widetilde{t}_{i+1}], i = 1, \dots, m\}$, $a < \widetilde{t}_1 < \dots < \widetilde{t}_{m+1} < b$, be a solution corresponding to the element $\widetilde{\mu}_m = (\widetilde{t}_1, \dots, \widetilde{t}_{m+1}, \widetilde{x}_{10}, \dots, \widetilde{x}_{m0}, \widetilde{\varphi}_1, \dots, \widetilde{\varphi}_m, \widetilde{f}_1, \dots, \widetilde{f}_m) \in B_m$ and let $K_{i1} \subset 0_i$ be a compactum containing some neighborhood of the set*

$$K_{i0} = cl[\{\widetilde{x}_i(t) | t \in [\widetilde{t}_i, \widetilde{t}_{i+1}]\} \cup \{\widetilde{\varphi}_i(t) | \in J_{i1}\}].$$

Then for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that to every element

$$\mu_m \in V(\widetilde{\mu}_m; K_{11}, \dots, K_{m_1}, \delta, c_0) = \prod_{i=1}^{m+1} V(\widetilde{t}_i; \delta) \times \prod_{i=1}^{m} V(\widetilde{\varphi}_i; \delta) \times$$

$$\times \prod_{i=1}^{m} V(\widetilde{\varphi}_i;\delta) \times \prod_{i=1}^{m} V_1(\widetilde{f}_i;K_{i1}\delta) \cap V_2(\widetilde{f}_i;K_{i1},c_0)),$$

there corresponds a solution $\{x_i(t;\mu_i), t \in [J_i(t_i), t_{i+1}], i = 1,\dots,m\}$. *Moreover, the function* $x_i(t;\mu_i)$ *is defined on* $[J_i(t_i), \widetilde{t}_{i+1} + \delta] \subset (J_i(a), b)$, *and satisfies almost everywhere on* $[t_i, \widetilde{t}_i + \delta]$ *the equation* (13_i).

If

$$\mu_m^s \in V(\widetilde{\mu}_m; K_{11},\dots,K_{m_1},\delta,c_0), \quad s = 1,2,$$

then

$$|x_i(t;\mu_i^1) - x_i(t;\mu_i^2)| \le \varepsilon, \quad t \in [\max(t_i^1,t_i^2), \widetilde{t}_{i+1} + \delta], \\ i = 1,\dots,m. \tag{15}$$

Here $V(\widetilde{t}_i;\delta)$, $V(\widetilde{x}_{i0};\delta)$, $V(\widetilde{\varphi}_i;\delta)$, are the δ-neighborhoods of the points $\widetilde{t}_i$,$\widetilde{x}_{i0}$, $\widetilde{\varphi}_i$, in the spaces R^1, R^n, respectively; $\Delta(J_{i1};R^{n_i})$; $V_1(\widetilde{f}_i;K_i,\delta) = \widetilde{f}_i + V_1(K_{i1};\delta)$, $V_2(\widetilde{f}_i;K_{i1}c_0) = \widetilde{f}_i + V_2(K_{i1};c_0)$, $V_1(K_{i0};\delta) \subset E(J \times 0_i^2;R^{n_i}$, $V_2(K_{i1};c_0) \subset E(J \times 0_i^2;R^{n_i}$; $c_0 > 0$ is a fixed number.

Theorem 4 can be proved by the method given in [6] (see also [7]) and is used in proving the continuity of the mapping (see 3.3).

Remark 3. There exists a number $\widetilde{\delta} \in [0,\delta]$ (see (12)) such that

$$V_2(\widetilde{f}_i;K_{i1},\widetilde{\delta}) \subset V_1(\widetilde{f}_i;K_i,\widetilde{\delta}) \cap V_2(\widetilde{f}_i;K_{i1},c_0), \quad i = 1,\dots,m.$$

Consequently, the inequality (15) is the more so valid for

$$\mu_m^s \in \prod_{i=1}^{m+1} V(\widetilde{t}_i;\widetilde{\delta}) \times \prod_{i=1}^{m} V(\widetilde{\varphi}_i;\widetilde{\delta}) \times \prod_{i=1}^{m} V_2(\widetilde{f}_i;K_{i1},\widetilde{\delta}).$$

This fact is used in proving the openness of the set D_0 (see 3.2).

Introduce the set

$$V_i = \{\delta\mu_i = (\delta t_1,\dots,\delta t_{i+1},\delta x_{10},\dots,\delta x_{i0},\delta\varphi_1,\dots,\delta\varphi_i,\delta f_1,\dots,\delta f_i) \in \\ \in B_i - \widetilde{\mu}_i \big| \, |\delta t_i| \le c_i, \\ \delta x_{i0} \le c_1, \|\delta\varphi_i\| \le c_i, \delta f_p = \sum_{j=1}^{s} \alpha_j \varepsilon f_p^j, p = 1,\dots,i, \\ |\alpha_j| \le c_i\}, \quad i = 1,\dots,m,$$

$\delta f_p^j \in E(J \times 0_p^2;R^{n_p})$, $p = 1,\dots,i$, $j = 1,\dots,s$, are fixed points and s, $c_i > 0$ are fixed numbers.

From Theorem 4 we have

Theorem 5. *There exist numbers $\varepsilon_0 > 0$, $\delta_0 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu_m) \in [0, \varepsilon_0] \times V_m$, to the element $\widetilde{\mu}_m + \varepsilon\delta\mu_m$ there corresponds solution*

$$\{x_i(t; \widetilde{\mu}_{i+\varepsilon\delta\mu_i}), t \in [\tau_i(t_i), t_{i+1}], \quad i = 1, \dots, m\}, \quad t_i = \widetilde{t}_i + \varepsilon\delta t_i.$$

Moreover, $x_i(t; \widetilde{\mu}_i + \varepsilon\delta\mu_i)$ is defined on $[\tau_i(t_i), \widetilde{t}_{i+1} + \delta_0]$.

Remark 4. Due to the uniqueness, $x_i(t; \widetilde{\mu}_i)$ on the interval $[\tau_i(\widetilde{t}_i), \widetilde{t}_{i+1} + \delta_0]$ is a continuation of $\widetilde{x}_i(t)$. Therefore the function $\widetilde{x}_i(t)$ in the sequel is assumed to be defined on the whole interval $[\tau_i(\widetilde{t}_i), \widetilde{t}_{i+1} + \delta_0]$.

Using the numbers δ_0 and ε_0 (see Theorem 5), we introduce the notation

$$\Delta x_i(t; \varepsilon\delta\mu_i) = x_i(t; \mu_i + \varepsilon\delta\mu_i) - \widetilde{x}_i(t), t \in [\max(t_i, \widetilde{t}_i), \widetilde{t}_{i+1} + \delta_0], \varepsilon \in [0, \varepsilon_0].$$

Theorem 6. *Let $a < \widetilde{t}_1 < \dots < \widetilde{t}_{m+1} < b$, $\widetilde{t}_i < \gamma_i = \gamma_i(\widetilde{t}_i) < \tau_{i+1}(\widetilde{t}_{i+1})$, $i + 1, \dots, m - 1$, $\widetilde{t}_m < \gamma_m = \gamma_m(\widetilde{t}_m) \neq \widetilde{t}_{m+1}$; let the functions $\widetilde{f}_i(t, x_i, y_i)$, $i = 1, \dots, m$, be continuous respectively at the points $(\widetilde{t}_i, \widetilde{x}_{i0}, \widetilde{\varphi}_i(\tau_i(\widetilde{t}_i)))$, $(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{x}_{i0})$, $(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{\varphi}_i(\widetilde{t}_i))$, $i = 1, \dots, m$, and the functions $\widetilde{\varphi}_i(t)$, $i = 1, \dots, m$, be continuous respectively at the points $\widetilde{t}_i$, $i = 1, \dots, m$. Then there exist numbers $\varepsilon_1 \in [0, \varepsilon_0]$, $\delta_1 \in [0, \delta_0]$ such that the following formula is valid:*

$$\begin{aligned}\Delta x_i(t; \varepsilon\delta\mu_i) = \varepsilon\delta x_i(t; \delta\mu_i) + o_i(t; \varepsilon\delta\mu_i), \quad \forall (t, \varepsilon, \delta\mu_i) \in [\tau_{i+1}(\widetilde{t}_{i+1}),\\ \widetilde{t}_{i+1} + \delta_1] \times [0, \varepsilon_1] \times V_i, \quad i = 1, \dots, m,\end{aligned} \tag{16}$$

where

$$\begin{aligned}\delta x_i(t; \delta\mu_i) = Y_i(\widetilde{t}_i; t)\Big[\delta x_{i0} - \widetilde{f}_i(\widetilde{t}_i)\delta t_i\Big] + \int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} Y_i(\gamma_i(s); t)\frac{\partial \widetilde{f}_i(\gamma_i(s))}{\partial y_i}\\ \Big[\delta\varphi_i(s) + \frac{\partial \widetilde{g}_i(s)}{\partial x_{i-1}}\delta x_{i-1}(s)\Big]\dot{\gamma}_i(s) ds + \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} Y_i(s; t)\delta\widetilde{f}_i ds - Y_i(\gamma_i; t)\\ \big[\widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{x}_{i0}) - \widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{\varphi}_i, (\widetilde{t}_i) + \widetilde{g}_i(\widetilde{t}_i))\big]\dot{\gamma}_i(\widetilde{t}_i)\delta t_i =\\ = \delta x_i^1(t; \delta\mu_i) - \delta x_i^2(t; \delta\mu_i), \quad i = 1, \dots, m,\end{aligned} \tag{17}$$

$Y_i(s; t)$ is a matrix function satisfying the equation

$$\frac{\partial Y_i(s; t)}{\partial s} = -Y_i(s; t)\frac{\partial \widetilde{f}_i(s)}{\partial x_i} - Y_i(\gamma_i(s); t)\frac{\partial \widetilde{f}_i(\gamma_i(s))}{\partial y_i}\dot{\gamma}_i(s), \quad s \in [\widetilde{t}_i, \widetilde{t}_{i+1}]$$

and also the condition

$$Y_i(t; t) = E, \quad Y_i(s; t) = 0, \quad s > t;$$

next,

$$\lim_{\varepsilon \to 0} o_i(t; \varepsilon\delta\mu_i)/\varepsilon = 0, \quad \textit{uniformly with respect to} \quad (t, \delta\mu_i) \in [\tau_{i+1}(\widetilde{t}_{i+1}),$$

$\widetilde{t}_{i+1}+\delta_1]\times V_i;\delta\widetilde{f}_i(t)=\delta f_i(t,\widetilde{x}_i(t),\widetilde{x}_i(\tau_i(t))),\quad \tau_{m+1}(\widetilde{t}_{m+1})=\widetilde{t}_{m+1}-\delta_1.$

The proof of Theorem 6 is conducted in a way described in [2, 7]. In the same way one can prove more general

Theorem 7. *Let $a<\widetilde{t}_1<\cdots<\widetilde{t}_{m+1}<b$, $\widetilde{t}_i<\gamma_i<\tau_{i+1}(\widetilde{t}_{i+1})$, $i=1,\dots,m-1$, $\widetilde{t}_m<\gamma_m\neq\widetilde{t}_{m+1}$, and let the conditions*

$$\lim_{(t,x_i)\to(t_i(\mp),\widetilde{x}_{i0})}\widetilde{f}_i(t,x_i,\widetilde{\varphi}_i(\tau_i(t)))=\widetilde{f}_i(\widetilde{t}_i(\mp)),\widetilde{x}_{i0},\widetilde{\varphi}(\tau_i(\widetilde{t}_i(\mp)))<\infty,$$
$$\lim_{(t,x_i,y_i)\to(\gamma_i(\mp),\widetilde{x}_i(\gamma),\widetilde{x}_{i0})}\widetilde{f}_i(t,x_i,y_i)=\widetilde{f}_i(\widetilde{\gamma}_i(\mp),\widetilde{x}_i(\gamma_i),\widetilde{x}_{i0})<\infty,$$

be fulfilled. Then there exist numbers $\varepsilon_1\in[0,\varepsilon_0]$, $\delta_1\in[0,\delta_0]$, such that the formula (16) is valid for an arbitrary point $(t,\varepsilon,\delta\mu_i)\in[\tau_{i+1}(\widetilde{t}_{i+1}),\widetilde{t}_{i+1}+\delta_1]\times[0,\varepsilon_1]\times V_i^{(\mp)}$, while in the formula (17) before δt_i there take place respectively the expressions

$$Y(\widetilde{t}_i;t)\widetilde{f}_i(\widetilde{t}_i(\mp))+Y(\gamma_i;t)\big[\widetilde{f}_i(\gamma_i,(\mp),\widetilde{x}_i(\gamma_i),\widetilde{x}_{i0})-$$
$$\widetilde{f}_i(\gamma_i(\mp),\widetilde{x}_i(\gamma_i),\widetilde{\varphi}_i(\widetilde{t}_i(\mp))+g(\widetilde{t}_i))\big]\dot{\gamma}_i(\widetilde{t}_i).$$

Here

$$V_i^-\{\delta\mu_i\in V_i|\ \ \delta t_K\le 0,\ \ K=1,\dots,i+1\},$$
$$V_i^+\{\delta\mu_i\in V_i|\ \ \delta t_K\ge 0,\ \ K=1,\dots,i+1\}.$$

Theorem 8 ([The Cauchy formula [8]]). *Let $A(t)$, $B(t)$, $t\in J_1=[s_1,s_2]$ be summable matrix functions of dimension $n\times n$; let $F(t)$, $t\in J_1$ be an n-dimensional summable vector function and $\tau(t)$ satisfy the same conditions as $\tau_i(t)$ do; let $\varphi(t)$, $t\in[\tau(s_1),s_2]$ be a piecewise continuous function. Then the solution of the equation*

$$\dot{x}(t)=A(t)x(t)+B(t)x(\tau(t))+F(t),\quad t\in J_1,$$
$$x(t)=\varphi(t),\ t\in[\tau(s_1),s_1),x(s_1)=x_0\in R^n,$$

can be represented as

$$x(t)=Y(s_1;t)x_0+\int_{\tau(s_1)}^{s_1}Y(\gamma(s);t)B(\gamma(s))\dot{\gamma}(s)\varphi(s)ds+\int_{s_1}^{t}Y(s;t)F(s)ds,$$
$$t\in J_1,$$

where $Y(s;t)$ is a matrix function satisfying the equation

$$\frac{\partial Y(s;t)}{\partial s}=-Y(s;t)A(s)-Y(\gamma(s);t)B(\gamma(s))\dot{\gamma}(s),\quad s\in[s_1,t]$$

and also the condition

$$Y(t;t) = E, \quad Y(s;t) = 0, \quad s > t;$$

$\gamma(s)$ is the function inverse to $\tau(s)$.

On the basis of Theorem 8 we can conclude that the function $\delta x_i^1(t;\delta\mu_i)$ (see (17)) satisfies the equation

$$\delta\dot{x}_i^1(t) = \frac{\partial\widetilde{f}_i(t)}{\partial x_i}\delta x_i^1(t) + \frac{\partial\widetilde{f}_i(t)}{\partial y_i}\delta x_i^1(\tau_i(t)) + \delta\widetilde{f}_i(t),$$
$$t \in [\widetilde{t}_i, \widetilde{t}_{i+1}] \tag{18}$$

with the initial condition

$$\delta x_i^1(t) = \delta\varphi_i(t) + \frac{\partial\widetilde{g}_i(t)}{\partial x_{i-1}}\delta x_{i-1}(t), \quad t \in [\tau_i(\widetilde{t}_i), \widetilde{t}_i),$$
$$\delta x_i^1(\widetilde{t}_i) = \delta x_{i0} - \widetilde{f}_i(\widetilde{t}_1)\delta t_i; \tag{19}$$

while the function $\delta x_i^2(t;\delta\mu_i)$ (see (17)) satisfies the equation

$$\delta\dot{x}_i^2(t) = \frac{\partial\widetilde{f}_i(t)}{\partial x_i}\delta x_i^2(t) + \frac{\partial\widetilde{f}_i(t)}{\partial y_i}\delta x_i^2(\tau_i(t)), \quad t \in [\gamma_i, \widetilde{t}_{i+1}], \tag{20}$$

with the initial condition

$$\delta x_i^2(t) = 0, \quad t < \gamma_i \;\; \delta x_i^2(\gamma_i) = [\widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{x}_{i0}) -$$
$$-\widetilde{f}_i(\gamma_i, \widetilde{x}_i(\gamma_i), \widetilde{\varphi}_i(\widetilde{t}_i))]\dot{\gamma}_i(\widetilde{t}_i)\delta t_i. \tag{21}$$

For the sake of brevity we denote the function, $\delta x_i(t;\delta\mu_i)$, $\delta x_i^1(t;\delta\mu_i)$ and $\delta x_i^2(t;\delta\mu_i)$, respectively by $\delta x_i(t)$, $\delta x_i^1(t)$ and $\delta x_i^2(t)$.

Theorem 9. *The following formula is valid:*

$$\sum_{i=1}^{m}\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i(\widetilde{t}_{i+1}) = \sum_{i=1}^{m}(Y_i(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i) - Y_i(\gamma_i)\delta x_i^2(\gamma_i) + \beta_i), \tag{22}$$

where

$$\beta_i = \int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} Y_i(\gamma_i(t))\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\delta\varphi_i(t)dt +$$
$$+\int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} Y_i(t)\delta\widetilde{f}_i(t)dt; \tag{23}$$

$Y_i(t)$ is a solution of the matrix equation

$$\dot{Y}_i(t) = -Y_i(t)\frac{\partial \widetilde{f}_i(t)}{\partial x_i} - Y(\gamma_i(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t) -$$
$$-\chi_i(t)Y_{i+1}(\gamma_{i+1}(t))\frac{\partial \widetilde{g}_{i+1}(t)}{\partial x_i}\dot{\gamma}_{i+1}(t),$$
$$t \in [\widetilde{t}_i \widetilde{t}_{i+1}] \tag{24}$$

$$Y_i(\widetilde{t}_{i+1}) = \frac{\partial \widetilde{Q}}{\partial x_i}, \quad Y_i(t) = 0, \quad t \in (\widetilde{t}_{i+1}, \gamma(\widetilde{t}_{i+1})] \;\; i = m, \dots, 1. \tag{25}$$

Proof. Obviously (see (25)),

$$\frac{\partial \widetilde{Q}}{\partial x_i}\delta x_i^1(\widetilde{t}_{i+1}) = Y_i(\widetilde{t}_{i+1})\delta x_i^1(\widetilde{t}_{i+1}) - Y_i(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i) + Y_i(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i) =$$
$$= \int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \frac{d}{dt}(Y_i(t)\delta x_i^1(t))dt + Y_i(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i). \tag{26}$$

From the fact that $\delta x_i^1(t)$ satisfies the equation (18), we obtain

$$\int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \frac{d}{dt}(Y_i(t)\delta x_i^1(t))dt = \int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \Big[\dot{Y}_i(t)\delta x_i^1(t) + Y_i(t)\Big(\frac{\partial \widetilde{f}_i(t)}{\partial x_i}\delta x_i^1(t) +$$
$$+\frac{\partial \widetilde{F}_i(t)}{\partial y_i}\delta x_i^1(\tau_i(t)) + \delta \widetilde{f}_i(t)\Big)\Big]dt. \tag{27}$$

Now, taking into account (19) and (25), we transform the integral

$$\int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} Y_i(t)\frac{\partial \widetilde{f}_i(t)}{\partial y_i}\delta x_i^1(\tau_i(t))dt = \int_{\tau_i(\widetilde{t}_i)}^{\tau_i(\widetilde{t}_{i+1})} \frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i},$$

$$\dot{\gamma}_i(t)\delta x_i^1(t)dt = \int_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} Y_i(\gamma_i(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\Big[\delta\varphi_i(t) + \frac{\partial \widetilde{g}_i(t)}{\partial x_{i-1}}\delta x_{i-1}(t)\Big]\dot{\gamma}_i(t)dt +$$
$$+\int_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} Y_i(\gamma_i(t))\frac{\partial \widetilde{f}_i(\gamma_i(t))}{\partial y_i}\delta x_i^1(t)\dot{\gamma}(t)dt. \tag{28}$$

Owing to (27) and (28), from (26) we obtain

$$\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i^1(\widetilde{t}_{i+1}) = \beta_i + \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \Big[\dot{Y}_i(t) + Y_i(t)\frac{\partial\widetilde{f}_i(t)}{\partial x_i} +$$

$$+Y_i(\gamma_i(t))\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\Big]\delta x_i^1(t)dt +$$

$$\int\limits_{\tau_i(\widetilde{t}_i)}^{\tau_i(\widetilde{t}_i} Y_i(\gamma_i(t)) + \frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\frac{\partial\widetilde{g}_i(t)}{\partial x_{i-1}}\dot{\gamma}_i(t)\delta x_{i-1}(t)dt +$$

$$+Y(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i). \tag{29}$$

Since $\delta x_i^2(t)$ satisfies equation (20) and also the condition (21), we have analogously

$$\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i^2(\widetilde{t}_{i+1}) = \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \Big[\dot{Y}_i(t) + Y_i(t)\frac{\partial\widetilde{f}_i(t)}{\partial x_i} + Y_i(\gamma_i(t))\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\Big] \times$$

$$\times\delta x_i^2(t)dt + Y_i(\gamma_i)\delta x_i^2(\gamma_i). \tag{30}$$

Thus

$$\sum_{i=1}^m \frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i(\widetilde{t}_{i+1}) = \sum_{i=1}^m \Big[\frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i^1(\widetilde{t}_{i+1}) - \frac{\partial\widetilde{Q}}{\partial x_i}\delta x_i^2(\widetilde{t}_{i+1})\Big] =$$

$$= \sum_{i=1}^m (Y_i(\widetilde{t}_i)\delta x_i^1(\widetilde{t}_i) - Y_i(\gamma_i)\delta x_i^2(\gamma_i) + \beta_i +$$

$$+ \int\limits_{\widetilde{t}_i}^{\widetilde{t}_{i+1}} \Big(\dot{Y}_i(t) + Y_i(t)\frac{\partial\widetilde{f}_i(t)}{\partial x_i} + Y_i(\gamma_i(t))\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\Big)\delta x_i(t)dt +$$

$$+ \int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} Y_i(\gamma_i(t))\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\frac{\partial\widetilde{g}_i(t)}{\partial x_{i-1}}\dot{\gamma}_i(t)\delta x_{i-1}(t)dt. \tag{31}$$

We can easily see that

$$\sum_{i=1}^m \int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i} Y_i(\gamma_i(t))\frac{\partial\widetilde{f}(\gamma_i(t))}{\partial y_i}\frac{\partial\widetilde{g}_i(t)}{\partial x_{i-1}}\dot{\gamma}_i(t)\delta x_{i-1}(t)dt =$$

$$= \sum_{i=1}^{m} \int_{\widetilde{t}_i}^{\widetilde{t}_i} \chi_i(t) Y_{i+1}(\gamma_{i+1}(t)) \frac{\partial \widetilde{f}_{i+1}(\gamma_{i+1}(t))}{\partial y_{i+1}} \frac{\partial \widetilde{g}_{i+1}(t)}{\partial x_i} \dot{\gamma}_{i+1}(t) \delta x_i(t) dt. \quad (32)$$

Because of (24) and (32), the equality (31) provides the formula (22). ■

3.2. Calculation of the differential of the mapping. Consider the space

$$E_{\mu_m} = R^{1+m} \times \prod_{i=1}^{m} R^{n_i} \times \prod_{i=1}^{m} \Delta(J_{i1}; R^{n_i}) \times \prod_{i=1}^{m} E(J \times 0_i^2; R^{n_i})$$

of the points μ_m. Topologies T_1, T_2 specify in E_{μ_m} two locally convex separable topologies; denote them respectively by Θ_1, Θ_2.

Denote by D_0 the set of elements $\mu_m \in B_m$ to which there corresponds a solution of equation of variable structure (14_i), (15_i), $i = 1, \dots, m$.

Define on a set D_0 the mapping

$$S : D_0 \to \prod_{i=1}^{m} R^{n_i}, \quad (33)$$

by the formula

$$S(\mu_m) = (S_1(\mu_1), \cdots, S_m(\mu_m)), \quad .S_i(\mu_i) = x_i(t_{i+1}; \mu_i), \quad i = 1, \dots, m.$$

Let $\widetilde{\mu}_m \in D_0$ passing through the point $\widetilde{\mu}_m$ and $L_{\widetilde{\mu}_m}$ be a finite-dimensional manifold

$$L_{\widetilde{\mu}_m} = \Big\{ \mu_m \in E_{\mu_m} | \mu_m = \widetilde{\mu}_m + \delta\mu_m,$$

$$\delta\mu_m = \sum_{j=1}^{S} \alpha_j \delta\mu_m^j, \ \alpha_j \in R^1, \ \delta\mu_m^j \in E_{\mu_m} - \widetilde{\mu}_m, \Big\};$$

$$W_m = \Big\{ \delta\mu = \sum_{j=1}^{S} \alpha_j \delta\mu_m^j | \ |\alpha_j| \le c_2 \Big\}$$

be a bounded neighborhood of zero in $L_{\widetilde{\mu}_m} - \widetilde{\mu}_m$; let $c_2 > 0$ $\widetilde{\mu}_m$ be a fixed number.

The finite openness of the set D_0 implies the existence of a number $\varepsilon_2 \in [0, \varepsilon_1]$, such that for arbitrary $(\varepsilon, \delta\mu_m) \in [0, \varepsilon_2] \times \Omega_m$ the point $\widetilde{\mu}_m + \varepsilon\delta\mu_m \in D_0 \cap L_{\widetilde{\mu}_m}$. ε_2 is assumed to be so small that $\varepsilon|\delta t_i| \le \delta_2$, $i = 2, \dots, m+1$ for $(\varepsilon, \delta\mu_m) \in [0, \varepsilon_2] \times W_m$.

Theorem 10. *Let the conditions of Theorem* 6 *be fulfilled. Moreover, the functions $\widetilde{F}_i(t, x_i, y_i)$, $i = 1, \dots, m$, are assumed to be continuous respectively at the point $(\widetilde{t}_i, \widetilde{x}_i(\widetilde{t}), \widetilde{x}_i(\tau_i(\widetilde{t}_i)))$, $i = 1, \dots, m$. Then the differential of the mapping* (33) *at the point $\widetilde{\mu}_m$ has the form*

$$dS_{\widetilde{\mu}_m}(\delta\mu_m) = \big(dS_{1\mu_1}(\delta\mu_1), \dots, dS_{m\mu_m}(\delta\mu_m)\big),$$

where

$$dS_{i\widetilde{\mu}_i}(\delta\mu_i) = \delta x_i(\widetilde{t}_{i+1};\delta\mu_i) + \widetilde{f}_i(\widetilde{t}_{i+1})\delta t_{i+1}, \quad i = 1,\dots,m, \tag{34}$$
$$\delta\mu_i \in W_i = \{\delta\mu_i |\ (\delta\mu_i, 0,\dots,0) \in W_m\}, \quad i = 1,\dots,m-1.$$

Proof. It is easily seen that the set V_i can be chosen in such a way that $W_i \subset V_i$. Therefore for $(\varepsilon, \delta\mu_i) \in [0,\varepsilon_2] \times W_i$ we have

$$S_i(\widetilde{\mu}_i + \varepsilon\delta\mu_i) - S_i(\widetilde{\mu}_i = x_i(\widetilde{t}_{i+1} + \varepsilon\delta t_{i+1}; \widetilde{\mu}_i + \varepsilon\delta\mu_i) - \widetilde{x}_i(\widetilde{t}_{i+1}) =$$
$$= \Delta x_i(\widetilde{t}_{i+1} + \varepsilon\delta t_{i+1};$$
$$\widetilde{\mu}_i + \varepsilon\delta\mu_i) + \widetilde{x}_i(\widetilde{t}_{i+1} + \varepsilon\delta t_i) - \widetilde{x}_i(\widetilde{t}_{i+1}) = \varepsilon\delta x_i(\widetilde{t}_i + \varepsilon\delta t_i;\ \delta\mu_i) +$$
$$+ \int\limits_{\widetilde{t}_{i+1}}^{\widetilde{t}_{i+1}+\varepsilon\delta t_{i+1}} \widetilde{f}_i(t)dt + 0_i(\widetilde{t}_{i+1}; \varepsilon\delta\mu_i). \tag{35}$$

By assumption, $\widetilde{f}_i(t,x,y)$ is continuous at the point $(\widetilde{t}_i, \widetilde{x}_i(\widetilde{t}_i),\ \widetilde{x}_i(\tau_i(\widetilde{t}_i)))$, which implies the continuity of the function $\widetilde{f}_i(t)$ at the point $\widetilde{t}_{i+1}$. Applying now the theorem of the mean, we get

$$\int\limits_{\widetilde{t}_{i+1}}^{\widetilde{t}_{i+1}+\varepsilon\delta t_i} = \varepsilon\widetilde{f}_i(\widetilde{t}_{i+1})\delta t_i + 0_i(\varepsilon\delta\mu_i). \tag{36}$$

Further,

$$\lim_{\varepsilon\to 0}[\delta x_i(\widetilde{t}_{i+1} + \varepsilon\delta t_{i+1}; \delta\mu_i) - \delta x_i(\widetilde{t}_{i+1}; \delta\mu_i)] = 0,$$
$$\text{uniformly with respect to} \quad \delta\mu_i \in W_i. \tag{37}$$

Using (36) and (37), from (35) we have

$$S_i(\widetilde{\mu}_i + \varepsilon\delta\mu_i) - S_i(\widetilde{\mu}_i) = \varepsilon[\delta x_i(\widetilde{t}_i) + \widetilde{f}_i(\widetilde{t}_i)\delta t_i] + 0_i(\varepsilon\delta\mu_i),$$

where

$$0_i(\varepsilon\delta_i) = 0_i(\widetilde{t}_{i+1}; \varepsilon\delta\mu_i) + 0_i(\varepsilon\delta_i) + \varepsilon[\delta x_i(\widetilde{t}_i + \varepsilon\delta t_i;\ \delta\mu_i) - \delta x_i(\widetilde{t}_i)]$$
$$\text{and} \quad \lim_{\varepsilon\to 0} \overline{0}_i(\varepsilon\delta\mu_i)/\varepsilon = 0, \quad \text{uniformly with respect to} \quad \delta\mu_i \in \Omega_i.$$

Thus, the formula (34) is valid. ■

Let us consider now the mapping

$$P^1 : D_0 \to R^{1+l} \tag{38}$$

defined by the formula $P^1 = QL$, $L(\mu_m) = (t_1,\dots,t_{m+1}, x_{10},\dots,x_{m0}, x_1(t_1),\dots,x_m(t_{m+1}))$.

Theorem 11. *Let the conditions of Theorem 9 be fulfilled. Then the differential of the mapping* (38) *at the point* $\widetilde{\mu}_m$ *is of the form*

$$dP^1_{\widetilde{\mu}_m}(\delta\mu_m)=\sum_{i=1}^{m}\bigg\{\frac{\partial\widetilde{Q}}{\partial t_i}+a_iY_{i-1}(\widetilde{t}_i)\widetilde{f}_{i-1}-b_i\big[Y_i(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i)+$$

$$+Y_i(\gamma_i)\big(\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\ \widetilde{x}_{i0})-\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{\varphi}_i(\widetilde{t}_i)+\widetilde{g}_i(\widetilde{t}_i))\big)\dot{\gamma}_i(\widetilde{t}_i)\big]\bigg\}\delta t_i+$$

$$+\sum_{i=1}^{m}\bigg\{\Big[\frac{\partial\widetilde{Q}}{\partial x_{i0}}+Y_i(\widetilde{t}_i)\Big]\delta x_{i0}+\beta_i\bigg\},\tag{39}$$

where $a_1=0$, $a_2=\cdots=a_{m+1}$, $b_1=\cdots=b_m=1$, $b_{m+1}=0$.

Proof. It is easy to see that

$$dP^1_{\widetilde{\mu}_m}(\delta\mu_m)=\sum_{i=1}^{m+1}\bigg(\frac{\partial\widetilde{Q}}{\partial t_i}\delta t_i+\frac{\partial\widetilde{Q}}{\partial x_{i0}}\delta t_i+\frac{\partial\widetilde{Q}}{\partial x_i}dS_{i\widetilde{\mu}_i}(\delta\mu_i)\bigg)+$$

$$+\frac{\partial\widetilde{Q}}{\partial t_{m+i}}\delta t_{m+1}.\tag{40}$$

Taking into account (34), (22), (19) and (21), we obtain

$$\sum_{i=1}^{m+1}\frac{\partial\widetilde{Q}}{\partial t_i}\delta t_i+\sum_{i=1}^{m}\frac{\partial\widetilde{Q}}{\partial x_i}dS_{i\widetilde{\mu}_i}(\delta\mu_i)=\sum_{i=1}^{m+1}\bigg\{\frac{\partial\widetilde{Q}}{\partial t_i}+a_iY_{i-1}(\widetilde{t}_i)\widetilde{f}_{i-1}(\widetilde{t}_i)-$$

$$-b_i\Big[Y_i(\widetilde{t}_i)\widetilde{f}_i(\widetilde{t}_i)+Y_i(\gamma_i)\big(\widetilde{F}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{x}_{i0})-$$

$$-\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{\varphi}_i(\widetilde{t}_i)+\widetilde{g}_i(t_i))\big)\times\dot{\gamma}_i(\widetilde{t}_i)\Big]\bigg\}\delta t_i+\sum_{i=1}^{m}\Big[Y_i(\widetilde{t}_i)\delta x_{i0}+\beta_i\Big].\tag{41}$$

Owing to (41), from (40) we get the formula (39). ■

3.3. Necessary Conditions of Criticality. Deriving the necessary condition of criticality. Consider the space $E_z=R^1\times E_{\mu_m}$. Let

$$D=\{z=(s,\mu_m)|s\in R^1\}.$$

The finite openness of the set D_0 implies the finite openness of D. Define on the set D_0 the mapping

$$P:D\to R^{1+l}\tag{42}$$

by using the formula $P(z)=P^1(\mu_m)+(s,0,\dots,0)$.

The topology Θ_1 defines in E_2 a locally convex separable topology which we denote by Θ.

In E_2 we prescribe a filter $\phi_{\widetilde{z}}$, $\widetilde{z} = (0, \widetilde{\mu}_0)$, with elements

$$\Omega_{\widetilde{r}} = R^1_+ \cap V_0 \times \prod_{i=1}^{m+1} V_{\widetilde{t}_i} \times \prod_{i=1}^{m} V_{\widetilde{x}_{i0}} \times \prod_{i=1}^{m} \Omega_{\widetilde{\varphi}_i} \times \prod_{i=1}^{m} \Omega_{\widetilde{f}_i},$$

where V_0, $V_{\widetilde{t}_i}$, $V_{\widetilde{x}_{i0}}$ are arbitrary neighborhoods of the points $0 \in R^1$, $\widetilde{t}_i \in (a,b)$, $\widetilde{x}_{i0} \in 0_i$; $\Omega_{\widetilde{\varphi}_i} = \Delta_i \cap V_{\widetilde{\varphi}_i}$, $V_{\widetilde{\varphi}_i} \subset \Delta(J_{i1}; R^n)$ is an arbitrary neighborhood of the point $\widetilde{\varphi}_i$; $\Omega_{\widetilde{f}_i}$ is an arbitrary element of the filter $\phi_{\widetilde{f}_i}$.

Elements of the filter $\phi_{\widetilde{f}_i}$ are

$$\Omega_{\widetilde{f}_i} = f_i \cap V_{\widetilde{f}_i},$$

where $V_{\widetilde{f}_i} \subset E(J \times O_i^2; R^{n_i})$ is an arbitrary neighborhood of the point $\widetilde{f}_i = \widetilde{f}_i(t, x_i, y_i) = f_i(t, x_i; y_i, \widetilde{u}_i(t), \widetilde{u}_i(\theta_i(t)))$ in the topology T_2,

$$F_i = \{f_i(t, x_i, y_i, u_i(t),\ u_i(\theta_i(t)))|u_i \in \Omega_i\}.$$

Being a direct product of convex filters by quasi-convex filters $\phi_{\widetilde{f}_i}$ [1, 2, 3], the filter $\phi_{\widetilde{z}}$ is Θ-quasi-convex. The proof of quasi-convexity of $\phi_{\widetilde{f}_i}$ is given in [9].

The criticality of the mapping P onto $\phi_{\widetilde{z}}$ follows in a standard way from the local optimality of $\widetilde{\mu}_m$ [1, 2, 3].

From Remark 3 and Theorem 4 there follows the existence of an element $W_{\widetilde{z}}$, such that $c_0 w_{\widetilde{z}} \subset D$, and the mapping (42) onto $c_0 w_{\widetilde{z}}$ is continuous in the topology Θ.

The assumptions of Theorem 1 ensure the fulfillment of the conditions of Theorem 10 which in its turn ensures differentiability of the mapping (42) at the point $\widetilde{z}$:

$$dP_{\widetilde{z}}(\delta z) = dP_{\widetilde{\mu}_m}(\delta \mu_m) + (\delta s, 0, \ldots, 0), \quad \delta z = (\delta s, \delta \mu_m).$$

Thus all the premises for the necessary condition of criticality are fulfilled [1, 2, 3]. Consequently, there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_l)$ and the element $\widetilde{W}_z \in \phi_{\widetilde{z}}$ such that

$$\pi dP_{\widetilde{z}}(\delta z) \leq 0, \quad \forall \delta z \in K(c_0 \widetilde{w}_{\widetilde{z}} - \widetilde{z}), \tag{43}$$

$K(M)$ denotes a convex cone stretched on the set M.

Below from the necessary condition of criticality (43) we will derive the necessary conditions of optimality.

The condition $\delta z \in K(c_0 \widetilde{w}_{\widetilde{z}} - \widetilde{z})$ is equivalent to the condition $\delta s \in R^1_+$, $\delta t_i \in R^1$, $\delta t_{i0} \in R^{\pi_i}$, $\delta \varphi_i \in K(\widetilde{w}_{\widetilde{\varphi}_i} - \widetilde{\varphi}_i)$, $\delta f_i \in K(c_0 \widetilde{w}_{\widetilde{F}_i} - \widetilde{F}_i)$.

Using the expression (39), letting $\delta f_i = \delta \varphi_i = 0$, $i = 1, \ldots, m$, $\delta s = 0$ in (43) and taking into account that δt_i, $i = 1, \ldots, m+1$ $\delta x_i = 0$, $i = 1, \ldots, m$, may take arbitrary values, we obtain

$$\pi\frac{\partial\widetilde{Q}}{\partial t_i}=-a_i\pi Y_{i-1}(\widetilde{t}_i)\widetilde{f}_{i-1}(\widetilde{t}_i)+b_i\Big[\pi Y_i(\widetilde{t}_i)+\pi Y_i(\gamma_i)\big(\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{x}_{i0})-$$

$$-\widetilde{f}_i(\gamma_i,\widetilde{x}_i(\gamma_i),\widetilde{\varphi}_i(\widetilde{t}_i))\big)\dot{\gamma}_i(\widetilde{t}_i)\Big],\quad i=1,\dots,m+1,\tag{44}$$

$$\pi\frac{\partial\widetilde{Q}}{\partial t_i}=-\pi Y_i(\widetilde{t}_i),\quad i=1,\dots,m.\tag{45}$$

Letting $\delta t_i=0$, $i=1,\dots,m+1$, $\delta x_{i0}=\delta f_i=0$, $i=1,\dots,m$, $\delta s=0$ in (43) and taking into consideration that $\widetilde{W}_{\widetilde{\varphi}_i}-\widetilde{\varphi}_i$, $i=1,\dots,m$, contains a non-zero element, we get

$$\int\limits_{\tau_i(\widetilde{t}_i)}^{\widetilde{t}_i}\pi Y_i\big(\gamma_i(t)\big)\frac{\partial\widetilde{f}_i(\gamma_i(t))}{\partial y_i}\dot{\gamma}_i(t)\delta\varphi_i(t)dt\le 0,$$

$$\forall\delta\varphi_i\in K\big(\widetilde{W}_{\widetilde{\varphi}_i}-\widetilde{\varphi}_i\big),\quad i=1,\dots,m.\tag{46}$$

For $\delta t_i=0$, $i=1,\dots,m+1$, $\delta x_{i0}=\delta\varphi_i=0$, $i=1,\dots,m$, $\delta s=0$ we have

$$\int\limits_{t_i}^{\widetilde{t}_{i+1}}\pi Y_i(t)\delta f_i\big(t,\widetilde{x}(t),\widetilde{x}(\tau_i(t))\big)dt\le 0,\ \ \forall\delta f_i\in K\big(c_0\widetilde{W}_{\widetilde{F}_i}-\widetilde{F}_i\big),$$

$$i=1,\dots,m.\tag{47}$$

If $\delta t_i=0$, $i=1,\dots,m+1$, $\delta x_{i0}=\delta\varphi_i=\delta f_i=0$, $i=1,\dots,m$, we obtain that π_0.

Introduce the notation

$$\Psi_i(t)=\pi Y_i(t),\ t\in[\widetilde{t}_i,\gamma_i(\widetilde{t}_{i+1})],\quad i=1,\dots,m.\tag{48}$$

Obviously, $\Psi_i(t)$ on the interval $[\widetilde{t}_i,\widetilde{t}_{i+1}]$ satisfies equation (3_i) and also the condition

$$\Psi_i\big(\widetilde{t}_{i+1}\big)=\pi\frac{\partial\widetilde{Q}}{\partial x_i},\ \Psi_i(t)=0,\ t\in\Big(\widetilde{t}_{i+1},\gamma_i(\widetilde{t}_{i+1})\Big].\tag{49}$$

Conditions (44), (45) and (49) on account of (48) provide the condition of transversality (2).

From the convexity and boundedness of the set N_i follows the inclusion

$$\Delta_i-\widetilde{\varphi}_i\subset K(\widetilde{W}_{\widetilde{\varphi}_i}-\widetilde{\varphi}_i).\tag{50}$$

Condition (46), due to (48), (50) and the fact that $\delta\varphi_i=\varphi_i-\widetilde{\varphi}_i$, results with respect to $\widetilde{\varphi}_i$ in the maximum principle (4).

Prove now the condition (5). The mapping

$$\delta f_i \to \int_{t_i}^{\widetilde{t}_{i+1}} Y_i(t)\delta f_i\big(t,\widetilde{x}_i(t),\widetilde{x}_i(\tau_i(t))\big)dt, \quad \delta f_i \in \mathrm{V}_2\big(K_{i1};c_0\big), \tag{51}$$

is continuous in the topology T_1 (see [9]).

The continuity of the mapping (51) makes it possible to conclude that inequality (47) is valid for an arbitrary

$$\delta f_i \in K([\widetilde{W}_{\widetilde{F}_i}]^1_{M_i} - \widetilde{f}_i), \quad M_i = V_2(K_{i1};c_0),$$

where $[\widetilde{W}_{\widetilde{F}_i}]^1_{M_i}$ is a closure with respect to M_i of the set $\widetilde{W}_{\widetilde{f}_i} \cap M_i$ in the topology T_i.

Using now the proven in [9] inclusion

$$F_i - \widetilde{f}_i \subset K([\widetilde{W}_{\widetilde{F}_i}]^1_{M_i} - \widetilde{f}_i),$$

we can say that the inequality (47) is valid for arbitrary $\delta f_i \in f_i - \widetilde{F}_i$.

From (47), due to (48) as well as

$$\delta f_i = f_i\big(t,x_i,y_i,u_i(t),\ u_i(\theta_i(t))\big) - \widetilde{f}_i\big(t,x_i,y_i,\widetilde{u}_i(t),\widetilde{u}_i(\theta_i(t))\big) \in F_i - \widetilde{f}_i$$

we obtain with respect to the controls the maximum principle (5).

Thus Theorem 1 is proved completely. ■

Finally we note that Theorem 2 is proved analogously to Theorem 1. In this case for calculation of the differential we use Theorem 7, while in the filter $\phi_{\widetilde{z}}$ we replace respectively the neighborhoods $V_{\widetilde{t}_i}$, $i = 1,\dots,m+1$, by $R^-_{\widetilde{t}_i} \cap V_{\widetilde{t}_i}(R^+_{\widetilde{t}_i})$, where $R^-_{\widetilde{t}_i} = (-\infty,\widetilde{t}_i]$, $R^-_{\widetilde{t}_i} = [\widetilde{t}_i,\infty)$.

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