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SOLUTION OF THE SCHWARZ DIFFERENTIAL EQUATION

Abstract. A circular polygon of a general form with a finite number of vertices and arbitrary angles at these vertices is given. A single-valued analytic function mapping conformally a half-plane onto the given circular polygon is constructed in a general form. The function is proved to be a general solution of the Schwarz equation. First we construct functional series uniformly and rapidly convergent near all singular points and then fundamental local matrices which are connected by analytic continuation. The constructed analytic function satisfies nonlinear boundary conditions. In a general form, we compose and investigate all higher transcendental equations connecting geometric characteristics of circular polygons with unknown parameters of the Schwarz equation. Possible intervals of variation of unknown accessory parameters are established.

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Key words and Phrases. Analytic function, differential equations, conformal mapping, circular polygons, fundamental matrices.

რეზიუმე. მოცემულია ზოგადი სახის წრიული მრავალკუთხედი წვეროების სასრული რაოდენობით და ამ წვეროებთან ნებისმიერი კუთხეებით. აგებულია ზოგადი სახის ცალსახა ანალიზური ფუნქცია რომელიც ნახევრსიბრტყეს მოცემულ წრიულ მრავალკუთხედში გადასახავს. დამტკიცებულია, რომ ეს ფუნქცია შვარცის განტოლების ზოგადი ამონახსნია. ჩვენ ჯერ ვაგებთ ფუნქციონალურ მწკრივებს, რომლებიც თანაბრად და სწრაფად კრებადია ყოველი განსაკუთრებული წერტილის მიდამოში, შემდეგ კი ფუნდამენტურ ლოკალურ მატრიცებს, რომლებიც ანალიზური გაგრძელებით არიან ურთიერთდაკავშირებული. აგებული ანალიზური ფუნქცია აკმაყოფილებს არაწრფივ სასაზღვრო პირობებს. ზოგადი ფორმით ჩვენ ვადგენთ და ვიკვლევთ მაღალი რიგის ყველა ტრანსცენდენტურ განტოლებას, რომლებიც აკავშირებენ წრიული მრავალკუთხედების გეომეტრიულ მახასიათებლებს შვარცის განტოლების უცნობ პარამეტრებთან. დადგენილია უცნობი აქცესორული პარამეტრების ცვლილების შესაძლო ინტერვალები.

1. Introduction

Let on a complex plane w a simply connected domain $S(w)$ be given with the boundary l consisting of a finite number $m+1$ of circular arcs or linear segments; note that the latter are regarded as degenerated circular arcs. The vertices of circular polygons are denoted by $b_1, b_2, \ldots, b_{m+1}$, while the sizes of inward with respect to the domain $S(w)$ angles are denoted by $\pi\nu_1, \pi\nu_2, \ldots, \pi\nu_{m+1}$. The domain $S(w)$ may be assumed to be bounded. This always can be achieved by a suitable linear-fractional mapping.

Without restriction of generality, one can by means of a linear-fractional transformation combine one of the sides of circular polygons, say the side (b_m, b_{m+1}), with the a segment of abscissa axis, the origin coinciding with the vertex b_m. For $\nu_m \neq n$, $n = 0, 1, 2$, and the side (b_{m-1}, b_m) will likewise become a segment of a straight line forming with the abscissa axis the angle $\pi\nu_m$. This remark will be used in the sequel.

Find and investigate the function $w(\zeta)$ which conformally maps the half-plane $\Im(\zeta) > 0$ (or $\Im(\zeta) < 0$) of the plane $\zeta = t + i\tau$ onto the domain $S(w)$. Using the theorem on the correspondence of boundaries of the domains $\Im(\zeta) > 0$ and $S(w)$, we denote by a_k, $k = 1, 2, \ldots, m+1$, the points of the real axis of the plane $\zeta = t + i\tau$ (in this case $-\infty < a_1 < a_2 < \cdots < a_m < +\infty$) to which on the plane w there correspond the vertices of circular polygons b_k, $k = 1, 2, \ldots, m, m+1$. Suppose that the point $a_{m+1} = \infty$ is mapped into the point $w = b_{m+1}$. On every interval of the t-axis, the unknown function $w = w(\zeta)$ takes between neighboring points a_k, a_{k+1} the values which lie on the corresponding circular arc [5,6].

A not complete bibliography dealing with those problems can be found in [1–27].

The function $w = w(\zeta)$ is the solution of the Schwarz equation [5–7, 9–11]

$$w'''(\zeta)/w'(\zeta) - 1,5[w''(\zeta)/w'(\zeta)]^2 = R(\zeta), \tag{1.1}$$

$$R(\zeta) = \sum_{k=1}^{m} [0,5(1-\nu_k^2)/(\zeta - a_k)^2 + c_k/(\zeta - a_k)], \tag{1.2}$$

where c_k, $k = 1, 2, \ldots, m$ are unknown real accessory parameters which for the time being satisfy the conditions

$$\sum_{k=1}^{m} c_k = 0, \quad \sum_{k=1}^{m} [a_k c_k + 0,5(1-\nu_k^2)] = 0,5(1-\nu_{m+1}^2). \tag{1.3}$$

By b_k, b'_k, $k = 1, 2, \ldots, m+1$ we denote the complex coordinates of the vertices of a circular polygon at which two neighboring circumferences may intersect; but if the neighboring circumferences are tangent at the vertex $w = b_k$, then $b_k = b'_k$.

The function $w = w(\zeta)$ on the boundary l of $S(w)$ must satisfy the nonlinear boundary condition [19, 20]

$$iA(t)w(t)\overline{w(t)} + \overline{B(t)}w(t) - B(t)\overline{w(t)} + iD(t) = 0, \quad -\infty < t < +\infty, \tag{1.4}$$

$$B(t)\overline{B(t)} - A(t)D(t) = 1, \tag{1.5}$$

where $A(t)$, $B(t)$, $\overline{B(t)}$, $D(t)$ are given piecewise constant functions; $A(t)$, $D(t)$ are real, while $B(t)$ and $\overline{B(t)}$, $w(t)$ and $\overline{w(t)}$ are mutually complex conjugate.

It should be noted that (1.4) is the equation of the contour of the circular polygon.

It is known that every function $w(\zeta)$ conformally mapping $\Im(\zeta) > 0$ onto a circular polygon satisfies (1.1), and vice versa, every solution of (1.1) conformally maps the domain $\Im(\zeta) > 0$ on some circular polygon [10, p. 137]. Moreover, due to the boundary correspondence under conformal mapping, every solution of (1.1), $w = w(\zeta)$, will satisfy the boundary condition (1.4). Note hereat that when passing in (1.4) to complex conjugate values, the equation (1.4) remains unchanged.

If $w = w_1(\zeta)$ is a particular solution of (1.1), then the general solution of (1.1) is given by

$$w(\zeta) = [pw_1(\zeta) + q]/[rw_1(\zeta) + S], \quad ps - rq = 1, \tag{1.6}$$

where p, q, r, s are arbitrary, in general complex, parameters of integration of the equation (1.1) which are connected by the condition $ps - rq = 1$.

Equation (1.1) is invariant with respect to a linear-fractional transformation of the independent variable ζ and the dependent one w; given ζ, the coefficients of the linear-fractional transformation are real, but given w, they are complex. Therefore we can fix arbitrarily three of the parameters a_k, $k = 1, 2, \ldots, m, m+1$ one of which, $a_{m+1} = \infty$, is already fixed. It remains to fix the rest two parameters by taking, e.g., $a_1 = -m$, $a_m = m$.

After this it becomes evident that the equation (1.1) depends on $2(m-2)$ unknown parameters a_k, c_k, $k = 1, 2, \ldots, m$ and the number of singular points $\zeta = a_k$ equals $m + 1$.

The contour of the circular polygon l consists of arcs of $m + 1$ circumferences. For their definition, we need $3(m + 1)$ real parameters. As it will be seen, there are exactly $3(m + 1)$ parameters at our disposal. Indeed, the equation (1.1) depends both on $2(m - 2)$ unknown parameters a_k, c_k and on $m + 1$ known parameters ν_k, $k = 1, 2, \ldots, m + 1$. In defining the general solution of (1.1), there appear six more additional parameters of integration (see (1.6)). Thus we have $2(m - 2) + m + 1 + 6 = 3(m + 1)$ parameters [7].

If we assume that $w' = 1/u^2(\zeta)$, then the solution of (1.1) is reduced to that of the Fuchs class differential equation [5–13]

$$u''(\zeta) + 0,5R(\zeta)u(\zeta) = 0. \tag{1.7}$$

If we find linear independent partial $v_1(\zeta)$, $v_2(\zeta)$ solutions of (1.7), then the general solution of (1.1) can be obtained by the formula (1.6) assuming $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$.

Below we will consider the Fuchs class equation of the kind

$$v''(\zeta) + p(\zeta)v'(\zeta) + q(\zeta)v(\zeta) = 0, \tag{1.8}$$

where

$$p(\zeta) = \sum_{k=1}^{m} \beta_k/(\zeta - a_k), \quad q(\zeta) = \sum_{k=1}^{m}[\sigma_k/(\zeta - a_k)^2 + c_k/(\zeta - a_k)]; \tag{1.9}$$

β_k, σ_k are given constants and c_k are unknown $p'(s)$ accessory parameters.

Substituting

$$v(\zeta) = u(s)\exp\left[-\frac{1}{2}\int_0^s p(\zeta)ds\right], \tag{1.10}$$

the equation (1.8) is reduced to the equation (1.7), where

$$0,5R(\zeta) = q(\zeta) - 0,5(p'(s))^2 - 0,25(p(\zeta))^2. \tag{1.11}$$

One frequently uses equations of the type (1.8) in which $p(\zeta)$ and $q(\zeta)$ are of the form [4, 15]

$$\begin{gathered} p(\zeta) = \sum_{k=1}^{m}(1 - \nu_k)/(\zeta - a_k), \\ q(s) = \alpha'\alpha'' \prod_{k=1}^{m-2}(\zeta - \lambda_k)/\prod_{k=1}^{m}(\zeta - a_k), \end{gathered} \tag{1.12}$$

where

$$\sum_{k=1}^{m} \nu_k + \alpha' + \alpha'' = m - 1, \quad \alpha' - \alpha'' = \nu_{m+1}, \tag{1.13}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_{m-2}$ are accessory parameters.

If we consider a circular polygon with equal angles $\pi\nu_j = \pi$, $j = 1, 2, \ldots, m+1$ then $\alpha' = 0$, and hence in this case it is necessary to consider the limits $\lim(\alpha'\alpha''\lambda_k)$, $k = 1, 2, \ldots, m-2$ as $\alpha' \to 0$. Therefore it is better to write $q(\zeta)$ in the form [7]

$$q(\zeta) = \frac{\left[\alpha'\alpha''\zeta^{m-2} + \delta_1\zeta^{m-3} + \delta_2\zeta^{m-4} + \cdots + \delta_{m-3}\zeta + \delta_{m-2}\right]}{\prod_{k=1}^{m}(\zeta - a_k)}, \tag{1.14}$$

where δ_k, $k = 1, 2, \ldots, m-2$ are unknown accessory parameters.

The Fuchs class equations are solved by means of the power series, therefore we represent (1.14) as a sum of partial fractions,

$$q(\zeta) = \sum_{j=1}^{m} c_j/(\zeta - a_j), \tag{1.15}$$

where

$$\sum_{j=1}^{m} c_j = 0, \quad \sum_{j=1}^{m} c_j a_j = \alpha'\alpha'', \tag{1.16}$$

$$c_k = \frac{\left[\alpha'\alpha'' a_k^{m-2} + \delta_1 a_k^{m-3} + \cdots + \delta_{m-3} a_k + \delta_{m-2}\right]}{\prod\limits_{j=1, j\neq k}^{m} (a_k - a_j)}, \tag{1.17}$$

The equation (1.1) as well as the method of constructing $w(\zeta)$ for $m = 2$ have been obtained by H. A. Schwarz in 1873.

Equation (1.8) for $m = 3$ has been considered by K. Heun in 1889 and by Ch. Snow in 1952. But they have failed in connecting the constructed local solutions [3]. G.N. Goluzin [6] constructed $w(\zeta)$ for equilateral and equiangular circular polygons. V. Koppenfels and F. Stallmann constructed $w(\zeta)$ for some particular cases of circular polygons with the angles multiple of $\frac{\pi}{2}$ [10]. Approximate methods for finding the parameters a_k, c_k can be seen in [2].

P. Ya. Polubarinova-Kochina has obtained important results in constructing $w(\zeta)$ and in its application to the problems of the filtration theory when a finite number of new singular points, the so-called removable points, are added to the points $\zeta = a_k$.

General analytic solution of the equation (1.1) for any circular polygons with a finite number of vertices b_k $k = 1, 2, \ldots, m+1$ is given in [19–26]. In the same works, one can see the systems of equations for finding the parameters a_j, c_j, p, q, r, s, $j = 1, 2, \ldots, m$. The method making it possible to construct explicitly the solution of (1.1) for circular polygons with angles multiple of $\pi/2$ is described in [22].

Below we present our new not published yet results as well as the ones published earlier [19–26].

2. Application of Matrix Calculus to Determination of the Fundamental System of Solutions

Denote linearly independent local solutions of (1.8) near singular points $\zeta = a_k$, $k = 1, 2, \ldots, m+1$, by $v_{kj}(\zeta)$, $k = 1, 2$; $j = 1, \ldots, m+1$, while the solutions containing integration constants p, q, r, s satisfying $ps - rq = 1$

$$u_{1j}(\zeta) = p v_{1j}(\zeta) + q v_{2j}(\zeta), \quad u_{2j}(\zeta) = r v_{1j}(\zeta) + s v_{2j}(\zeta). \tag{2.1}$$

The ratios u_{1j}/u_{2j} are local solutions of (1.1) (see (1.6))

Linear independent local solutions of (1.8) are proved to be suitable only near the points $\zeta = a_k$, $k = 1, 2, \dots, m+1$.

The equation (1.8) can be written in the form of the system

$$\chi'(\zeta) = \chi(\zeta)\mathcal{P}(\zeta), \tag{2.2}$$

where

$$\chi(\zeta) = \begin{pmatrix} u_{1j}(\zeta), & u'_{1j}(\zeta) \\ u_{2j}(\zeta), & u'_{2j}(\zeta) \end{pmatrix}, \mathcal{P}(\zeta) = \begin{pmatrix} 0, & -q(\zeta) \\ 1, & -p(\zeta) \end{pmatrix}, \tag{2.3}$$

$$\chi'(\zeta) = \frac{d}{d\zeta}\chi(\zeta), u'_{kj}(\zeta) = \frac{d}{d\zeta}u_{kj}(\zeta). \tag{2.4}$$

and $u_1(\zeta)$, $u_2(\zeta)$ are linear independent solutions of (1.8).

Note that since the coefficients of (1.1) and (1.8) are real, it becomes obvious that if $w(\zeta)$ and $u_{kj}(\zeta)$, $k = 1, 2$, are solutions of (1.1) and (1.8), respectively then $\overline{w}(\overline{\zeta})$ and $\overline{u}_{kj}(\overline{\zeta})$ are also the solutions of (1.1) and (1.8) respectively.

In [26] we proved the basic

Theorem 2.1. *If $w(\zeta) = u_1(\zeta)/u_2(\zeta)$, where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of* (1.8)*, then the linear boundary condition* (1.4) *is equivalent to the conditions* [19, 20]

$$u_1(t) = \lambda[B(t)\overline{u}_1(t) - iD(t)\overline{u}_2(t)], \quad -\infty < t < +\infty, \tag{2.5}$$

$$u_2(t) = \lambda[iA(t)\overline{u}_1(t) + \overline{B}(t)\overline{u}_2(t)], \quad -\infty < t < +\infty, \tag{2.6}$$

where $\lambda = \lambda(t)$ takes on the intervals a_j, a_{j+1} constant values equal to $+1$ or -1; $u_k(\zeta)$, $\overline{u}_k(\zeta)$ are complex conjugate.

Proof. Assume $\lambda = \lambda(t)$. We rewrite (2.5) and (2.6) as

$$u_1(t) = \lambda(t)u_1^*(t), \quad u_2(t) = \lambda(t)u_2^*(t), \quad -\infty < t < +\infty, \tag{2.7}$$

where

$$u_1^*(t) = B(t)\overline{u}_1(t) - iD(t)\overline{u}_2(t), \tag{2.8}$$

$$u_2^*(t) = iA(t)\overline{u}_1(t) + \overline{B}(t)\overline{u}_2(t), \tag{2.9}$$

are linearly independent solutions of (1.8).

Substituting (2.7) in (1.8), we obtain

$$\lambda''(t)u_1^*(t) + \lambda'(t)[2(u_1^*(t))' + p(t)u_1^*(t)] = 0, \quad -\infty < t < +\infty, \tag{2.10}$$

$$\lambda''(t)u_2^*(t) + \lambda'(t)[2(u_2^*(t))' + p(t)u_2^*(t)] = 0, \quad -\infty < t < +\infty, \tag{2.11}$$

Multiplying (2.10) by $u_2^*(t)$ and (2.11) by $u_1^*(t)$ and then subtracting the first equality from the second one, we get

$$2\lambda'(t)\big[[u_1^*(t)]'u_2^*(t) - [u_2^*(t)]'u_1^*(t)\big] = 0, \tag{2.12}$$

The braces in (2.12) involve the Wronskian $w[u_1^*(t), u_2^*(t)] \neq 0$ for all ζ, with the exception of $\zeta = a_k$, $k = 1, 2, \ldots, m$. Hence (2.12) implies

$$\lambda(t) = \text{const}, \quad t \in (a_j, a_{j+1}), \quad j = 1, 2, \ldots, m. \tag{2.13}$$

From its side, (2.13) implies

$$\lambda'(t) = 0, \quad t \in (a_j, a_{j+1}), \quad j = 1, 2, \ldots, m. \tag{2.14}$$

If we calculate the Wronskian for (2.7) and take into account (2.14), then we obtain $\lambda^2 = 1$, and hence $\lambda = \pm 1$. ■

In §9, we will show which of the intervals (a_j, a_{j+1}), $j = 1, 2, \ldots, m$ requires $\lambda = 1$ and which one $\lambda = -1$.

As for the matrix $\chi(\zeta)$ which is defined by the (2.3), we can write the conditions (2.5) and (2.6) as:

$$\chi(t) = 6(t)\overline{\chi}(t), \quad -\infty < t < +\infty, \tag{2.15}$$

where

$$G(t) = \begin{pmatrix} B(t), & -iD(t) \\ iA(t), & \overline{B}(t) \end{pmatrix}, -\infty < t < +\infty, \tag{2.16}$$

is a given piecewise constant matrix, by (1.5) $\det G(t) = 1$, and $G(t)\overline{G(t)} = E$, where E is the unit matrix and $\overline{\chi}(t)$ is a matrix complex conjugate to the matrix $\chi(t)$.

For the intervals of the axis $\zeta = t$, the matrix $G(t)$ can be defined as

$$G(t) = G_j = \begin{pmatrix} B_j & -iD_j \\ iA_j & \overline{B}_j \end{pmatrix}, \quad a_j < t < a_{j+1}, \quad j = 1, 2, \ldots, m+1, \tag{2.17}$$

where $a_{j+1} = a_{m+2} = a_1$ when $j = m + 1$.

As it has been said above, without restriction of generality we may assume that $G_m = E$. Due to this fact, we can extend the matrix $\chi(\zeta)$ analytically through the interval (a_m, a_{m+1}) to the lower half-plane, or vice versa.

The matrix $\chi(\zeta)$ defined by (2.3) is a solution of (2.2). Since $\det \chi(\zeta) \neq 0$ for all ζ with the exception of the points $\zeta = a_k$, $k = 1, 2, \ldots, m+1$, we see that $\chi(\zeta)$ is likewise a fundamental matrix [8]. It is also known that if the matrix $\chi(\zeta)$ is a solution of (2.2), then the matrix $C \cdot \chi(\zeta)$ is likewise a solution of (2.2), where C is a nonsingular constant matrix.

Below we will construct locally linearly independent solutions of (1.8), $V_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ respectively for the points $\zeta = a_j$, $j = 1, 2, \ldots, m, m+1$, $\zeta = e_j = (a_j + a_{j+1})/2$, $j = 1, 2, \ldots, m-1$, where $k = 1, 2$, and then by means of these solutions we will construct for (2.2) the corresponding locally fundamental matrices:

$$\begin{array}{cc} \theta_j(\zeta) = \begin{pmatrix} V_{1j}(\zeta) & V'_{1j}(\zeta) \\ V_{2j}(\zeta) & V'_{2j}(\zeta) \end{pmatrix}, & H_j(\zeta) = \begin{pmatrix} \varphi_{1j}(\zeta) & \varphi'_{1j}(\zeta) \\ \varphi_{2j}(\zeta) & \varphi'_{2j}(\zeta) \end{pmatrix}, \\ j = 1, 2, 3, \ldots, m, m+1, & j = 1, 2, 3, \ldots, m-1. \end{array} \tag{2.18}$$

3. Local Solutions Near Singular Points, When the Difference of Characteristic Numbers is not an Integer

Equation (1.8) near $\zeta = a_j$ can be rewritten as

$$(\zeta - a_j)^2 V''(\zeta) + (\zeta - a_j) p_j(\zeta) V'(\zeta) + q_j(\zeta) V(\zeta) = 0, \tag{3.1}$$

where

$$p_j(\zeta) = \sum_{k=0}^{\infty} p_{kj} (\zeta - a_j)^k, \quad q_j(\zeta) = \sum_{k=0}^{\infty} q_{kj} (\zeta - a_j)^k. \tag{3.2}$$

For the point $\zeta = a_{m+1} = \infty$, by means of the transformation $\zeta = 1/x$ we can write the equation (1.8) as follows [1, 7, 13]:

$$x^2 V''(x) + x[2 - \sum_{k=0}^{\infty} p_k^{\infty} x^k] V'(x) + [\sum_{k=0}^{\infty} q_k^{\infty} x^k] V(x) = 0, \tag{3.3}$$

where

$$p(1/x) = x \sum_{k=0}^{\infty} p_k^{\infty} x^k, \quad q(1/x) x^2 \sum_{k=0}^{\infty} q_k^{\infty} x^k. \tag{3.4}$$

A solution of (3.1) respectively for the points $\zeta = a_i$, $\zeta = \infty$, $j = 1, 2, \ldots, m$, is sought in the form [1, 7, 8, 12, 13]

$$V_j(\zeta) = (\zeta - a_j)^{\alpha_j} \widetilde{V}_j(\zeta), \quad \widetilde{V}_j(\zeta) = \sum_{n=0}^{\infty} \gamma_{nj} (\zeta - a_j)^n, \tag{3.5}$$

$$V_{\infty}(\zeta) = \zeta^{-\alpha_{\infty}} \widetilde{V}_{\infty}(\zeta), \quad \widetilde{V}_{\infty}(\zeta) = \sum_{n=0}^{\infty} \gamma_{n\infty} (\zeta)^{-n}. \tag{3.6}$$

Theorem 3.1. *If near the point $t = a_j$ the equation (3.1) has a solution of the type (3.5), then after its substitution in (3.1) the following equality should identically be fulfilled:*

$$(\zeta - a_i)^{\alpha_j} \left[\sum_{k=0}^{\infty} M_{kj} (\zeta - a_j)^k \right] = 0. \tag{3.7}$$

From this equality we obtain an infinite recursion system of equations for *determination of* γ_{nj}, $n = 1, 2, \ldots$.

$$M_{0j}(\alpha_j) = \gamma_{0j} f_{0j}(\alpha_j), \quad f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + \alpha_j p_{0j} + q_{0j} = 0, \tag{3.8}$$

$$M_{1j}(\alpha_j) = \gamma_{1j}(\alpha_j) \cdot f_{0j}(\alpha_j + 1) + \gamma_{0j} f_{1j}(\alpha_j) = 0, \tag{3.9}$$

$$\begin{gathered} M_{2j}(\alpha_i) = \gamma_{2j}(\alpha_j) f_{0j}(\alpha_j + 2) + \\ \gamma_{1j}(\alpha_j) f_{1j}(\alpha_j + 1) + \gamma_{0j} f_{2j}(\alpha_j) = 0, \end{gathered} \tag{3.10}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned} M_{nj}(\alpha_j) = \gamma_{nj}(\alpha_j) f_{0j}(\alpha_j + n) + \gamma_{(n-1)j}(\alpha_i) f_{1j}(\alpha_j + n - 1) + \cdots + \\ + \gamma_{[n-(k-2)]j}(\alpha_j) f_{(k-2)j}(\alpha_j + n - k + 2) + \cdots + \\ + \gamma_{1j}(\alpha_j) f_{(n-1)j}(\alpha_j + 1) + \gamma_{0j} f_{nj}(\alpha_j) = 0, \end{aligned} \tag{3.11}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{kj}(\alpha_j) = \alpha_j p_{kj} + q_{kj} \tag{3.12}$$

Theorem 3.2. *If for the point* $\zeta = a_j$ *the determining equation* (3.8) *has the roots* α_{1j}, α_{2j} $(\alpha_{1j} > \alpha_{2j})$ *such that* $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$, *then for equation* (3.1) *we construct by formulas* (3.9)–(3.11) *two local linearly independent solutions of the type*

$$\begin{gathered} V_{kj}(\zeta) = (\zeta - a_j)^{\alpha_{kj}} \gamma_{0j} \widetilde{V}_{kj}(\zeta), \\ \widetilde{V}_{kj}(\zeta) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^{k} (\zeta - a_j)^n, \quad k = 1, 2. \end{gathered} \tag{3.13}$$

In complete analogy with the above theorem, we can formulate and prove the theorem for the point $\zeta = a_{m+1} = \infty$ [1, 7–13].

The convergence radius of the series $\widetilde{V}_{kj}(\zeta)$ is bounded by the distance from the point $\zeta = a_j$ to the nearest of the points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$ [1, 7,8].

The coefficient $\gamma_{0j} \neq 0$ will be defined below.

4. Construction of the Second Solution by Means of the Frobenius Method, When the Difference of Characteristic Numbers is Equal to an Integer

As it is known, when $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$, using the formulas (3.9)–(3.11), one can construct at the point $\zeta = a_j$ only one solution $V_{1j}(\zeta)$ corresponding to the root $\alpha_j = \alpha_{1j}$.

In such cases, there exist two methods for construction of the second solution $V_{2j}(\zeta)$: the Frobenius method and the method of lowering the order of the equation (1.8).

By the Frobenius method, $V_{2j}(\zeta)$ is sought as follows [8].

Consider the case where $\alpha_{1j} - \alpha_{2j} = 0$. In this case, for the point $\zeta = a_j$ we seek for the second solution of (3.1). First we differentiate (3.5) with respect to α_j and then calculate the limit $\alpha_j \to \alpha_{2j}$ and obtain $V_{2j}(\zeta)$. Thus we have

$$V_{2j}(\zeta) = V_{1j}(\zeta)\ln(\zeta - a_j) + (\zeta - a_j)^{\alpha_{2j}}\gamma_{0j} \times \\ \times \sum_{n=0}^{\infty} \left\{ \frac{d}{d\alpha_j}\gamma_{nj}^2(\alpha_j) \right\}_{\alpha_j=\alpha_{2j}} \times (\zeta - a_j)^n. \qquad (4.1)$$

Consequently, the following theorem is valid.

Theorem 4.1. *If for the point $\zeta = a_j$ the determining equation* (3.8) *has the roots such that $\alpha_{1j} - \alpha_{2j} = 0$ (at the point $w = b_j$, the two neighboring arcs are tangent, $\nu_j = 0$), then for the point $\zeta = a_j$ there exists the second solution $V_{2j}(\zeta)$ of the form* (4.1).

If for the point $\zeta = a_j$ the roots of (3.8) satisfy the condition $\alpha_{1j} - \alpha_{2j} = s$, $s \in \{1, 2\}$, then the second linearly independent solution of (3.1) is sought in the form [8]

$$V_j(\zeta, \alpha) = \gamma_{0j}(\zeta - a_j)\alpha_j \Big[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}(\alpha_j)(\zeta - a_j)^n\Big]. \qquad (4.2)$$

Substituting (4.2) in (3.1), we obtain for determination of $\gamma_n^2(\alpha_j)$, $n = 1, 2, \ldots$, a recursion system of equations. This system can also be obtained from (3.8)–(3.11), if instead of $\gamma_{0j}^2\ (\alpha_j - \alpha_{2j})$ we substitute $\gamma_{nj}^2\ (\alpha_j)$, $n = 1, 2, \ldots$. From this system we determine $\gamma_{nj}^2\ (\alpha_j)$, $n = 1, 2, \ldots$, and substitute them in (4.2). Then we differentiate (4.2) with respect to α_j and finally calculate the limits as $\alpha_j \to \alpha_{2j}$. As a result, we get the solution $V_{2j}(\zeta)$,

$$V_{2j}(\zeta) = \lim_{\alpha_j - \alpha_{2j}} \gamma_{0j}\Big\{(\zeta - a_j)^{\alpha_j}\left[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}(\alpha_j)(\zeta - a_j)^n\right] \times \\ \times \ln(\zeta - a_j) + (\zeta - a_j)^{\alpha_j}\Big[1 + \sum_{n=1}^{\infty} \frac{d}{d\alpha_j}[\gamma_{nj}^2(\alpha_j)](\zeta - a_j)^n\Big]\Big\} \qquad (4.3)$$

Reasoning as above, we have proved the following

Theorem 4.2. *If for the point $\zeta = a_j$ the equation* (3.8) *has the roots such that $\alpha_{1j} - \alpha_{2j} = s$ $s = \{1, 2\}$ (two neighboring circular arcs are tangent and $\nu_j = 1$ and $\nu_j = 2$), respectively then for the point $\zeta = a_j$ the second linearly independent solution of* (3.1) *is of the form* (4.3).

5. Conditions for the Absence of the Logarithmic Term in the Solution $V_{2j}(\zeta)$

The boundary l of the domain $s(w)$ may contain circular or rectilinear cuts of $s(w)$. For the cut end $w = b_j$, equation (3.8) possesses the roots such that $\alpha_{1j} - \alpha_{2j} = 2$. For the points $\zeta = a_j$, P. Ya. Polubarinova–Kochina has proved that solutions $V_{2j}(\zeta)$ contain no logarithmic terms. Moreover, for these points she has obtained the equation connecting the parameters a_j, c_j, ν of some circular polygons.

Below, using the method different from that used in [15], we derive for the end of the cut of the angle 2π an equation connecting parameters a_j, c_j, ν_j for any circular polygons and then prove that the second solution $V_{2j}(\zeta)$ constructed for this end should not contain a logarithmic term.

Denoting the first summand in formula (4.3) by $V^1_{2j}(\zeta)$, we have

$$V^1_{2j}(\zeta) = \gamma_{0j}(\zeta - a_j)^{\alpha_j} \times$$
$$\times\Big[\alpha_j - \alpha_{2j} + \sum_{k=1}^{\infty}\gamma^2_{nj}(\alpha_j)(\zeta - a_j)^n\Big]\ln(\zeta - a_j). \tag{5.1}$$

For determination of the coefficients $\gamma^2_{nj}(\alpha_j)$, we need the formulas (3.9)–(3.12) in which we replace γ_{0j} by γ_{0j} $(\alpha_j - \alpha_{2j})$. Having defined $\gamma_{nj}\gamma_{nj}(\alpha_j)$ α_j and passing to limit in γ^2_{nj} (α_j) as $\alpha_j \to \alpha_{2j}$, we obtain from (5.1) the equality

$$v^{1*}_{2j}(\zeta) = \lim_{\alpha_j \to \alpha_{2j}} V^1_{2j}(\zeta) = \gamma^2_{2j}(\alpha_{2j}) \cdot V_{1j}(\zeta)\ln(\zeta - a_j), \tag{5.2}$$

where $v_{1j}(\zeta)$ is the solution of (3.1) for $\alpha_j = \alpha_{1j}$.

Now we prove

Theorem 5.1. *A necessary and sufficient condition for the absence of a logarithmic term in the solution $v_{2j(\zeta)}$ constructed for the cut end is of the form*

$$\gamma^2_{2j}(\alpha_{2j}) = \frac{\gamma_{0j}}{2} \times$$
$$\times\{-f_{1j}(\alpha_{2j}) \cdot f_{1j}(\alpha_{2j} + 1)/f_{0j}(\alpha_{2j+1}) + f_{2j}(\alpha_{2j})\} = 0, \tag{5.3}$$

where $f_{kj}(\alpha)$, $k = 0, 1, 2$, are defined by (3.8) *and* (3.12).

Proof. Let us prove the sufficiency of (5.3). From (5.2) it is obvious that if (5.3) holds, then $v^{1*}_{2j}(\zeta) = 0$ which proves the sufficiency of the condition (5.3).

Let us prove now the necessity of the condition (5.3). As far as the equation (3.1) for the cut end $\zeta = a_j$ must have two locally independent solutions containing no logarithmic terms, we take this fact into account and construct the solution $v_{2j}(\zeta)$ by using the formulas (3.9)–(3.11) for, only the solutions of (3.1) constructed by (3.9)–(3.12) contain no logarithmic terms.

Really, all γ^2_{nj}, $n=1,3,4,\dots$, with the exception of $\gamma^2_{2j}(\alpha_{2j})$, are defined from the system (3.9)–(3.11). For definition of γ^2_{2j} we have equation (3.10) in which the first term $\gamma^2_{2j}(\alpha_j)f_{0j}(\alpha_j+2)=0$ for $\alpha_j=\alpha_{2j}$. Hence the sum of the last two summands in (3.10) must vanish,

$$\gamma^2_{1j}(\alpha_{2j})f_{1j}(\alpha_{2j}+1)+\gamma_{0j}f_{2j}(\alpha_{2j})=0; \tag{5.4}$$

moreover, the equation (5.4) coincides with (5.3) if we substitute in it $\gamma^2_{1j}(\alpha_{2j})$ defined by (3.9).

From (5.4), we have

$$q_{2j}+q^2_{1j}+q_{1j}p_{1j}=0, \tag{5.5}$$

where q_{2j}, q_{1j}, p_{1j} are defined from the corresponding coefficients of (3.2).

Finally, define $\gamma^2_{2j}(\alpha_{2j})$ uniquely. To this end, from (3.10) we define $\gamma_{2j}(\alpha_j)$ for $\alpha_j\neq\alpha_{2j}$. We have

$$\gamma_{2j}(\alpha_j)=-\frac{\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1)+\gamma_{0j}f_{2j}(\alpha_j)}{f_{0j}(\alpha_j+2)} \tag{5.6}$$

■

For $\alpha_j=\alpha_{2j}$, the numerator and the denominator in (5.6) vanish. Thus we have indeterminacy 0/0. If we develop it by means of the de L'Hospital rule, we will arrive at

$$\gamma^{2*}_{2j}(\alpha_{2j})=-0,5\gamma_{0j}[p_{1j}(p_{1j}+2q_{1j})+p_{2j}]. \tag{5.7}$$

Thus, by formulas (3.9)–(3.11), we define $v_{2j}(\zeta)$ uniquely and complete the proof of the necessity of the condition (5.3).

For the cut end $\zeta=a_j$, one can construct $v_{2j}(\zeta)$ by means of the Frobenius method under the condition (5.3). Indeed, if the condition (5.3) is fulfilled, then the first summand in (4.3) vanishes, while the second one takes the form

$$V_{2j}(\zeta)=(\zeta-a_j)^{\alpha_{2j}}\gamma_{0j}\left[1+\sum_{n=1}^{\infty}\gamma^{2*}_{nj}(\zeta-a_j)^n\right], \tag{5.8}$$

where all the coefficients γ^{2*}_{nj}, $n=1,2,\dots$, are defined by

$$\lim_{\alpha_j\to\alpha_{2j}}\frac{d}{d\alpha_j}[\gamma_{nj}(\alpha_j)]=\gamma^{2*}_{nj}\quad n=1,2,3,\dots. \tag{5.9}$$

Among them γ^{2*}_{2j} is defined by

$$\gamma^{2*}_{2j}=-0,5[p_{1j}(p_{1j}+2q_{1j})+p_{2j}], \tag{5.10}$$

which coincides with (5.7) since γ_{0j} in (5.8) is a factor standing out of brackets.

6. Searching for the Second Solution $v_{2j}(\zeta)$ by the Method of Lowering the Order of (1.8) when $\alpha_{1j} - \alpha_{2j} = s,\ s = 0, 1, 2.$

There naturally arises the question whether there is a more simple way of constructing $v_{2j}(\zeta)$ than that indicated by Frobenius. They may say that there is a second method, that is the method of lowering the order of equation (1.8) [7, 9, 10, 11, 12].

Using this method, one can get the well-known Liouville formula which in turn results in the following expression for $v_{2j}(\zeta)$:

$$v_{2j}(\zeta) = A_{0j} v_{1j}(\zeta) \ln(\zeta - a_j) + v_{2j}^2(\zeta), \tag{6.1}$$

where $v_{1j}(\zeta)$ is the solution corresponding to the root α_{1j}, A_{0j} is an unknown constant, and $v_{2j}^2(\zeta)$ for the case $\alpha_{1j} - \alpha_{2j} = 0$ takes the form

$$v_{2j}^2(\zeta) = (\zeta - a_j)^{\alpha_{2j}} \gamma_{0j} \sum_{n=1}^{\infty} h_{nj}(t - a_j)^n, \quad h_{1j} = 1. \tag{6.2}$$

For the cases $\alpha_{1j} - \alpha_{2j} = s$, $s = 1, 2$, the solution $v_{2j}^2(\zeta)$ is defined as follows:

$$v_{2j}^2(\zeta) = (\zeta - a_j)^{\alpha_{2j}} \gamma_{0j} \sum_{n=0}^{\infty} h_{nj}(\zeta - a_j)^n, \quad h_{0j} = 1, \tag{6.3}$$

where the coefficients h_{nj} $n = 1, 2, \dots,$ can be defined theoretically by the Liouville formula. Practically they cannot be defined in such a way.

Some well-known authors [9, 10, 12] recommend to substitute (6.1) in (3.1) and to obtain the recursion formulas which no longer has those defects we spoke about. Unfortunately, these statements are not true for $\alpha_{1j} - \alpha_{2j} = s$, $s = 1, 2$. Such an approach leaves again the coefficients h_{1j}, h_{2j} for $f_{0j}(\alpha_{2j} + s)$, where $f_{0j}(\alpha_{2j} + s) = 0$, $s = 1, 2$, undefined.

Indeed, the substitution of (6.1) in (3.1) results in

$$\begin{aligned}&(\zeta - a_j)^{\alpha_{1j} - \alpha_{2j}} A_j \{2\widetilde{v}_{1j}'(\zeta) + \widetilde{v}_{1j}(\zeta)(p_{1j}(\zeta) - 1)\} + \\ &+ \{(\widetilde{v}_{2j}^2(\zeta))'' + p_{1j}(\zeta)(\widetilde{v}_{2j}^2(\zeta))' + q_{1j}(\zeta)\widetilde{v}_{2j}^2(\zeta)\} = 0,\end{aligned} \tag{6.4}$$

where

$$v_{1j}(\zeta) = \gamma_{0j}(\zeta - a_j)^{\alpha_{1j}} \widetilde{v}_{1j}(\zeta), \quad \widetilde{v}_{1j}(\zeta) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^1 (\zeta - a_j)^n, \tag{6.5}$$

$$v_{1j}'(\zeta) = \gamma_{0j}(\zeta - a_j)^{\alpha_{1j} - 1} \widetilde{v}_{1j}^1(\zeta)$$

$$\widetilde{v}_{1j}^1(\zeta) = \alpha_{1j} + \sum_{n=1}^{\infty} \gamma_{nj}^1 (\alpha_{1j} + n)(\zeta - a_j)^n. \tag{6.6}$$

Formulas for $\widetilde{v}_{2j}^2(\zeta)$, $(\widetilde{v}_{2j}^2(\zeta))'$, $(\widetilde{v}_{2j}^2(\zeta))''$ are defined similarly.

After the substitution of $\widetilde{v}_{kj}(\zeta)$, $k=1,2$, in (6.4), we obtain

$$\sum_{k=0}^{\infty} Q_{kj}(\zeta - a_j)^n = 0, \tag{6.7}$$

The equation (6.7) implies

$$Q_{kj} = A_{0j} l_{(k-s)j} + M_{kj} = 0. \tag{6.8}$$

For $k=0$, we have

$$Q_{0j} = A_{0j} l_{(0-s)j} + M_{0j} = 0, \quad s = 0,1,2; \tag{6.9}$$

moreover,

$$l_{(k-s)j} = 0, \quad k - s < 0.$$

The coefficients M_{kj}, $k = 0,1,2,\dots$, can be defined by the formulas (3.8)–(3.11), while coefficients $l_{(k-s)j}$ are defined by

$$l_{0j} = 2\alpha_{1j} + p_{0j} - 1 = \alpha_{1j} - \alpha_{2j}, \tag{6.10}$$

$$l_{1j} = \gamma^1_{1j}[2(\alpha_{1j}+1) + p_{0j} - 1] + p_{1j}, \tag{6.11}$$

$$l_{2j} = \gamma^1_{2j}[2(\alpha_{1j}+2) + \alpha_{1j}(p_{0j}-1)] + \gamma^1_{1j}p_{1j} + p_{2j}, \tag{6.12}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{gathered} l_{nj} = \gamma^1_{nj}[2(\alpha_{1j}+n) + \alpha_{1j}(p_{0j}-1)] + \gamma^1_{(n-1)j}\alpha_{2j}p_{nj} + \cdots + \\ + \gamma^1_{2j}\alpha_{1j}p_{(n-2)j} + \gamma^1_{1j}\alpha_{1j}p_{(n-1)j} + p_{nj}, \end{gathered} \tag{6.13}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

According to (6.8), in order to define the parameter A_{0j} for the cases $s=1$ and $s=2$, respectively, we have the following equations:

$$A_{0j} + h_{1j}f_{0j}(\alpha_{2j}+1) + f_{1j}(\alpha_{2j}) = 0 \tag{6.14}$$

$$2A_{0j} + h_{2j}f_{0j}(\alpha_{2j}+1) + h_{1j}\cdot f_{1j}(\alpha_{2j}+1) + f_{2j}(\alpha_{2j}) = 0. \tag{6.15}$$

From (6.14) and (6.15) we can see that the recursion formulas (6.8) do not permit one to define $v_{2j}(\zeta)$ in the cases $\alpha_{1j} - \alpha_{2j} = s$, $s = 1,2$. Hence it remains to use the Frobenius method. But one can act differently: first calculate the coefficients h_{sj}, $s = 1,2$, by the Frobenius method and then the rest coefficients h_{nj}, $n \geq 3$, by the formula (6.8). The parameter A_{0j} can be defined as:

$$A_{0j} = -f_{1j}(\alpha_{1j}), \quad s = 1. \tag{6.16}$$

$$A_{0j} = -h_{1j}f_{0j}(\alpha_{2j}+1) - f_{2j}(\alpha_{2j}), \quad s = 2. \tag{6.17}$$

If we use the above-indicated method, then in the solution $v_{1j}(\zeta)$ instead of γ_{0j} we have to take $\gamma_{0j}A_{0j}$ and instead of $v_{2j}(\zeta)$ (formula (6.1)) the formula

$$v_{2j}(\zeta) = v_{1j}(\zeta)\ln(\zeta - a_j) + \gamma_{0j}v^2_{2j}(\zeta). \tag{6.18}$$

7. Local Matrices

For multi-valued functions $\exp[\alpha_{kj}\ln(\zeta - a_j)]$ encountered in local solutions, we select single-valued branches such as

$$\exp[\alpha_{kj}\ln(t-a_j)] > 0, \quad t > a_j;$$
$$\exp[\alpha_{kj}\ln(t-a_j)]^{\pm} = \exp[\pm i\pi\alpha_{kj}]\exp[\alpha_{kj}\ln(a_j - t)], \quad t < a_j;$$
$$\exp[-\alpha_{k\infty}\ln(-t)]\big]^{\pm} > 0, \quad -\infty < t < a_1;$$
$$\big[\exp[-\alpha_{k\infty}\ln t]\big]^{\pm} = \exp[\pm i\pi(-\alpha_{k\infty})]\exp[-\alpha_{k\infty}\ln t]. \quad a_m < t < +\infty.$$

Besides the matrix (2.18), we introduce the matrices

$$\theta_j^*(t) = \begin{pmatrix} v_{1j}^*(t), & v_{1j}^{'*}(t) \\ v_{2j}^*(t), & v_{2j}^{'*}(t) \end{pmatrix}, \quad a_{j-1} < t < a_j, \tag{7.1}$$

where

$$v_{kj}^*(t) = (a_j - t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}(t), \tag{7.2}$$
$$v_{kj}^{'*}(t) = -(a_j - t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}^{1*}(t) \tag{7.3}$$
$$v_{kj}'(t) = d[u_{kj}(t)]/dt,$$
$$\widetilde{v}_{kj}^{1*}(t) = \alpha_{kj} + \sum_{n=1}^{\infty}\gamma_{nj}^k(\alpha_{kj}+n)(t-a_j)^n,$$

Between the matrices $\theta_j(t)$ and $\theta_j^*(t)$, there is a (relation)

$$\theta_j^{\pm}(t) = \vartheta_j^{\pm}\theta_j^*(t), \quad a_{j-1} < t < a_j, \tag{7.4}$$
$$\theta_\infty^{\pm}(t) = \vartheta_\infty^{\pm}\theta_\infty^*(t), \quad a_m < t < \infty \tag{7.5}$$

Matrices $\vartheta_j^{\pm}$ for $\alpha_{1j} - \alpha_{2j} \neq s$, $s = 0, 1, 2$, are defined by

$$\vartheta_j^{\pm} = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}) & 0 \\ 0 & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix}. \tag{7.6}$$

For $\alpha_{1j} - \alpha_{2j} = s$, $s = 0, 1, 2$, they are defined by the equality

$$\vartheta_j^{\pm} = e^{\pm i\pi\alpha_{2j}}\begin{pmatrix} 1 & 0 \\ \pm\pi i & 1 \end{pmatrix}. \tag{7.7}$$

Matrices $\vartheta_j^{\pm}$ for the cut end $w = b_j$ are defined as follows: if the use is made of the equation (1.7), then the characteristic numbers can be defined as $\alpha_{1j} = 3/2$ and $\alpha_{2j} = -1/2$. To this case there correspond matrices $\vartheta_j^{\pm} = \mp iE$; however if we use the equation (1.8), then characteristic numbers are defined as $\alpha_{1j} = 2$, $\alpha_{2j} = 0$ with the corresponding matrices $\vartheta_j^{\pm} = E$.

The elements of the matrix $\theta_j^*(t)$ involving logarithmic terms are defined by the formulas

$$v_{2j}^*(t) = \gamma_{0j}\big\{(a_j - t)^{\alpha_{2j}}\big[(t-a_j)^s \widetilde{v}_{1j}(t)\ln(t-a_j) + \widetilde{v}_{2j}^2(t)\big]\big\}, \tag{7.8}$$

$$v_{2j}^{'*}(t) = -\gamma_{0j}(a_j - t)^{\alpha_{2j}-1} \times$$
$$\times\big\{\big[(a_j - t)^s e^{i\pi s}\widetilde{v}_{1j}^1(t)\ln(a_j - t) + \widetilde{v}_{1j}(t)\big] + \widetilde{v}_{2j}^2(t)\big\}, \tag{7.9}$$

In the local solutions $v_{kj}(\zeta)$ and $\varphi_{kj}(\zeta)$, there respectively appear constants γ_{0j} and φ_{0j} defined with the help of the Liouville formula

$$\gamma_{0j} = \Big\{\prod_{k=1,k\neq j}^{m} |a_j - a_k|^{\beta_k}\Big\}^{1/2}, \tag{7.10}$$

$$\varphi_{0j} = \Big\{\prod_{k=1}^{m} |e_j - a_k|^{\beta_k}\Big\}^{1/2} \tag{7.11}$$

8. Construction of the Fundamental Matrix

Construct the matrix

$$\chi(\zeta) = \begin{pmatrix} u_1(\zeta) & u_1'(\zeta) \\ u_2(\zeta) & u_2'(\zeta) \end{pmatrix}, \tag{8.1}$$

where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (1.8); moreover, $u_1'(\zeta) = du_1(\zeta)/d\zeta$ and $u_2'(\zeta) = du_2(\zeta)/d\zeta$.

Domain of convergence of the matrices $\theta_j(t)$, $H_j(t)$ always has a general part in which we can write the equalities

$$\theta_j^*(t) = T^* H_j(t), \quad H_j(t) = T_{0j}\theta_{j-1}(t), \quad a_{j-1} < t < a_j, \tag{8.2}$$

$$\theta_1^*(t) = T_{-\infty}\theta_\infty(t), \quad -\infty < t < a_1,$$
$$\theta_\infty^*(t) = T_\infty\theta_m(t), \quad a_m < t < +\infty, \tag{8.3}$$

where $T_j^*, T_{0j}, T_{-\infty}, T_\infty$ are the real constant matrices defined by equalities (8.2) and (8.3); in this case, we have to fix t in the domain where the two local matrices converge.

Define the matrix (8.1) along the axis t of the plane ζ:

$$\chi^\pm(t) = T\theta_m^\pm(t), \quad \theta_m^+(t) = \theta_m^-(t), \quad a_m < t < +\infty \tag{8.4}$$

$$\chi^\pm(t) = T\vartheta_m^\pm\vartheta_m^*(t), \quad a_{m-1} < t < a_m; \tag{8.5}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m\theta_{m-1}(t), \quad T_m = T_m^* \cdot T_{0m}, \quad a_{m-1} < t < a_m; \tag{8.6}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m\vartheta_{m-1}^\pm\theta_{m-1}^*(t), \quad a_{m-2} < t < a_{m-1}; \tag{8.7}$$

....

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \ldots T_1\vartheta_1^\pm\theta_1^*(t), \quad -\infty < t < a_1; \tag{8.8}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \ldots \vartheta_1^\pm T_{-\infty}\theta_\infty(t), \quad -\infty < t < a_1; \tag{8.9}$$

$$\chi^\pm(t) = T\vartheta_m^\pm T_m \ldots \vartheta_\infty^\pm T_\infty\vartheta_\infty^\pm(t), \quad a_m < t < \infty. \tag{8.10}$$

The upper signs $(\pm)$ in the matrices (8.4)–(8.10) denote the limiting values of the matrix $\chi(\zeta)$ from the upper and lower half-planes, respectively. The matrix T is defined by the equality

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{8.11}$$

Obviously, the matrices (8.4)–(8.10) are solutions of (2.2).

9. Solution of the Boundary Value Problem

Theorem 9.1. *The solution of the equation* (2.2) *satisfying the boundary condition* (2.15) *is given by formulas* (8.4)–(8.10).

Proof. We begin with the interval $(a_m, +\infty)$. We have

$$\begin{gathered} T\theta_m^+(t) = G_m T\theta_m^-(t), \quad \theta_m^+(t) = \theta_m^-(t), \\ G_m = E, \quad T = \overline{T}, \quad a_m < t < +\infty, \end{gathered} \tag{9.1}$$

For the interval (a_{m-1}, a_m), there takes place the equality

$$T\vartheta_m^+\theta_m^*(t) = G_{m-1}T\vartheta_m^-\theta_m^*(t), \quad a_{m-1} < t < a_m, \tag{9.2}$$

The equalities (9.1) and (9.2) result in the matrix equation

$$(\vartheta_m^+)^2 = TG_m^{-1}G_{m-1}T \tag{9.3}$$

It is seen from (9.3) that the matrices $(\vartheta_m^+)^2$ and $G_m^{-1}G_{m-1}$ are similar.

In a fashion analogous to the matrix equation (9.3), we find the corresponding matrix equations for the remaining points $\zeta = a_j$, $j = 1, 2, \ldots, m$, $m + 1$. We have

$$T\vartheta_m^+T_m\vartheta_{m-1}^+ = G_{m-2}T\vartheta_m^-T_m\vartheta_{m-1}^-, \tag{9.4}$$

$$T\vartheta_m^+T_m\vartheta_{m-1}^+T_{m-1}\vartheta_{m-2}^+ = G_{m-3}T\vartheta_m^-T_m\vartheta_{m-1}^-T_{m-1}\theta_{m-2}^-, \tag{9.5}$$

....

$$\begin{gathered} T\vartheta_m^+T_m\vartheta_{m-1}^+T_{m-1}\vartheta_{m-2}^+T_{m-2}\ldots T_1\vartheta_1^+ = \\ = G_{m+1}T\vartheta_m^-T_m\vartheta_{m-1}^-T_{m-1}\theta_{m-2}^-T_{m-2}\ldots T_1\vartheta_1^-, \end{gathered} \tag{9.6}$$

$$\begin{gathered} T\vartheta_m^+T_m\vartheta_{m-1}^+T_{m-1}\ldots T_{-\infty}\vartheta_\infty^+ = \\ = G_mT\vartheta_m^-T_m\vartheta_{m-1}^-T_{m-1}\ldots T_{-\infty}\vartheta_\infty^-. \end{gathered} \tag{9.7}$$

These equations can be written in terms of the equation (9.3), for example, the equation (9.4) can be written in the form

$$(\vartheta_{m-1}^+)^2 = T_m^{-1}(\vartheta_m^-)^{-1}T^{-1}G_{m-1}^{-1}G_{m-2}T\vartheta_m^-T_m.$$

As is said above, the matrices G_k can be defined first to within the factor $\lambda = \pm 1$, and then exactly. To define G_k exactly, we proceed from equation (3.8). Having defined χ_{kj}, it is necessary to construct the equation

$$\det(G_j^{-1}G_{j-1} - \lambda E) = 0. \tag{9.8}$$

Denote the roots of (9.8) by λ_{kj} and consider the equality

$$\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj} \tag{9.9}$$

The right-hand side of (9.9) is defined to within an integer summand. A suitable choice of $\lambda = \pm 1$ makes it always possible to fulfill the equation (9.9) and to define the matrices G_j, $j = 1, 2, \ldots, m, m+1$, exactly. But this operation should be done successively beginning, for example, with the matrix G_{m-1}.

It should be noted at this point that two neighboring circular arcs forming a cut with the end $w = b_j$ (in particular, segments of straight lines) belong to the same circumference. This implies that $G(t) = G_j$ for $\zeta > a_j$ and $G(t) = \lambda G_j$ for $\zeta < a_j$, where $\lambda = \pm 1$. If the use is made of the equation (1.7), then the equation (3.8) has the roots $3/2$ and $-1/2$, but if we use the equation (1.8), then the equation (3.8) has the roots 2 and 0. In the first case $\lambda = -1$, while in the second one $\lambda = 1$.

We rewrite the matrix equation (9.3) as follows:

$$T\vartheta_m^+ = G_{m-1} T\vartheta_m^- \tag{9.10}$$

From (9.10), we have

$$p\exp(i\pi\alpha_{1m}) = B_{m-1}p\exp(-i\pi\alpha_{1m}) - iD_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{9.11}$$
$$r\exp(i\pi\alpha_{1m}) = iA_{m-1}p\exp(-i\pi\alpha_{1m}) + \overline{B}_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{9.12}$$
$$q\exp(i\pi\alpha_{2m}) = B_{m-1}q\exp(-i\pi\alpha_{2m}) - iD_{m-1}s\exp(-i\pi\alpha_{2m}), \tag{9.13}$$
$$s\exp(i\pi\alpha_{2m}) = iA_{m-1}q\exp(-i\pi\alpha_{2m}) + \overline{B}_{m-1}s\exp(-i\pi\alpha_{2m}). \tag{9.14}$$

If we divide the corresponding parts of (9.11) and (9.12), (9.13) and (9.14), then we can see that the ratios p/r and q/s on the interval (a_{m-1}, a_m) satisfy the boundary condition (1.4):

$$\frac{p}{r} = \frac{B_{m-1}p/r - iD_{m-1}}{iA_{m-1}p/r + \overline{B}_{m-1}}, \quad \frac{q}{s} = \frac{B_{m-1}q/s - iD_{m-1}}{iA_{m-1}q/s + \overline{B}_{m-1}}. \tag{9.15}$$

The same boundary condition is satisfied by the coordinates of the points $w = b_m$, $w = b'_m$. Hence,

$$p/r = b_m, \quad q/s = b'_m. \tag{9.16}$$

On the plane w the origin of coordinates coincides with the point b_m, therefore $b_m = 0$, $b'_m = \infty$, and hence

$$p = 0, \quad s = 0. \tag{9.17}$$

If the determining equation (3.8) has for the point $\zeta = a_m$ the roots such that $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$, then we can define the matrix G_{m-1}:

$$G_{m-1} = \begin{pmatrix} B_{m-1} & 0 \\ 0 & \overline{B}_{m-1} \end{pmatrix} \tag{9.18}$$

Consider the matrix equation (9.4):

$$T_{*m}\vartheta^{+}_{m-1} = G_{m-2}\overline{T}_{*m}\vartheta^{-}_{m-1}, \quad T_{*m} = T\vartheta^{+}_{m}T_{m}. \tag{9.19}$$

Reasoning as above, from (9.19) we have the following system of equations:

$$p_{*m}/r_{*m} = b_{m-1}, \quad q_{*m}/s_{*m} = b'_{m-1}, \tag{9.20}$$

where p_{*m}, q_{*m}, r_{*m}, s_{*m} are the elements of the matrix T_{*m}.

The equalities (9.20) can be rewritten as

$$\frac{p_* p_m + q_* r_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{p_* p_m + q_* s_m}{r_* q_m + s_* s_m} = b'_{m-1}, \tag{9.21}$$

where p_*, q_*, r_*, s_* are the elements of the matrix $T_* = T\vartheta^{+}_{m}$.

Taking (9.16) into account, we can rewrite (9.21) as:

$$\frac{r_* p_m b_m + s_* r_m b'_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{r_* q_m b_m + s_* s_m b'_m}{r_* q_m + s_* s_m} = b'_{m-1}. \tag{9.22}$$

We rewrite (9.22) as

$$r_* p_m (b_m - b_{m-1}) + s_* r_m (b'_m - b_{m-1}) = 0, \tag{9.23}$$

$$r_* q_m (b_m - b'_{m-1}) + s_* s_m (b'_m - b'_{m-1}) = 0. \tag{9.24}$$

The condition of compatibility of the system of equations (9.23) and (9.24) with respect to r_* and s_* has the form

$$\frac{p_m s_m}{r_m q_m} = \frac{b'_m - b_{m-1}}{b_m - b_{m-1}} \cdot \frac{b_m - b'_{m-1}}{b'_m - b'_{m-1}}. \tag{9.25}$$

Exactly in the same way as above, from the matrix equation (9.5) we obtain a system of equations:

$$\begin{aligned} \frac{p_{*(m-1)} p_{m-1} + q_{*(m-1)} r_{m-1}}{r_{*(m-1)} p_{m-1} + s_{*(m-1)} r_{m-1}} &= b_{m-2}, \\ \frac{p_{*(m-1)} q_{m-1} + q_{*(m-1)} s_{m-1}}{r_{*(m-1)} q_{m-1} + s_{*(m-1)} s_{m-1}} &= b'_{m-2} \end{aligned} \tag{9.26}$$

Taking into consideration (9.20), after certain transformations we rewrite (9.26) as:

$$r_{*(m-1)} p_{m-1} (b_{m-1} - b_{m-2}) + s_{*(m-1)} r_{m-1} (b'_{m-1} - b_{m-2}) = 0, \tag{9.27}$$

$$r_{*(m-1)} q_{m-1} (b_{m-1} - b'_{m-2}) + s_{*(m-1)} s_{m-1} (b'_{m-1} - b'_{m-2}) = 0. \tag{9.28}$$

The condition of compatibility of the system of equations (9.27) and (9.28) with respect to $r_{*(m-1)}$ and $s_{*(m-1)}$ is of the form

$$\frac{p_{m-1}s_{m-1}}{r_{m-1}q_{m-1}} = \frac{b'_{m-1}-b_{m-2}}{b_{m-1}-b_{m-2}} \cdot \frac{b_{m-1}-b'_{m-2}}{b'_{m-1}-b'_{m-2}}. \tag{9.29}$$

Reasoning analogously we can successively consider all matrix equations (9.6) and (9.7).

The equations (9.25) and (9.29) represent invariant cross-ratios of four points belonging to the same circumference at which the latter intersects two neighboring circumferences.

From the matrix equations (9.3)–(9.7), we get all needed equations with respect to a_k, c_k as well as to the integration parameters p, q, r, s. For every point $\zeta = a_j$, the obtained system of two equations is homogeneous with respect to the elements of the matrix T_k. Its compatibility conditions, for example, for the points $\zeta = a_m$ and $\zeta = a_{m-1}$, are of the form (9.25) and (9.29). These equations have been obtained under the assumption $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$.

Consider the case where $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$.

Using the representation (8.4)–(8.10) for the interval (a_{j-1}, a_j), the unknown matrices $\chi^+(t)$, $\chi^-(t)$ must satisfy the boundary condition

$$\begin{aligned} &\begin{pmatrix} p_{*j} & q_{*j} \\ r_{*j} & s_{*j} \end{pmatrix} e^{i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} = \\ &= \begin{pmatrix} B_{j-1} & -iD_{j-1} \\ iA_{j-1} & \overline{B}_{j-1} \end{pmatrix} \begin{pmatrix} \overline{p}_{*j} & \overline{q}_{*j} \\ \overline{r}_{*j} & \overline{s}_{*j} \end{pmatrix} e^{-i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}, \end{aligned} \tag{9.30}$$

where p_{*j}, q_{*j}, r_{*j}, s_{*j} are defined by (8.4)–(8.10).

Reasoning in the same way as in deducing (9.11)–(9.14), we can see that the ratios

$$\frac{p_{*j} + \pi i q_{*j}}{r_{*j} + \pi i s_{*j}}, \quad \frac{q_{*j}}{s_{*j}} \tag{9.31}$$

satisfy the boundary condition (2.15). The same condition will likewise be satisfied by the coordinates of the point $w = b_j$ as well as by those of the points b_{j-1} or b'_{j-1}. Thus we obtain the following system of equations:

$$\frac{p_{*j} + \pi i q_{*j}}{r_{*j} + \pi i s_{*j}} = b_j, \quad \frac{q_{*j}}{s_{*j}} = b^*_j, \tag{9.32}$$

where b^*_j are equal either to b_{j-1} or b^*_{j-1}.

The system (9.32) is also homogeneous with respect to the elements of the corresponding matrices T_j whose compatibility conditions by this time does not provide the relations similar to (9.25)–(9.29).

As is said above, matrix equations similar to (9.3)–(9.7) can be obtained for all points, with the exception of the points $\zeta = a_k$. To these points there correspond the ends of the cuts $w = b_j$ for which $\nu_j = 2$. For such points we have conditions of the absence of logarithmic terms in the solutions $v_{2j}(\zeta)$, for example, the equation (5.5); the second equation will be given below.

From the matrix representations of $\chi^+(t)$ we first define $u_1^+(t)$, $u_2^+(t)$ and then compose the relation $w^+(t) = u_1^+(t)/u_2^+(t)$.

Suppose that the function $w^+(t)$ on the interval (a_k, a_{k+1}) is defined by

$$w^+(t) = [A_j^* v_{1j}^+(t) + B_j^* v_{2j}^+(t)]/[c_j^* v_{1j}(t) + D_j^* v_{2j}^+(t)], \qquad (9.33)$$

Using the formula (9.33) and calculating the limit as $\zeta \to a_j$, we get the equation

$$b_j = B_j^*/D_j^*. \qquad (9.34)$$

The corresponding equations for other points $\zeta = a_k$ can be obtained analogously.

Finally, for every point $t = a_j$ we obtain two real homogeneous equations with respect to p_j, q_j, r_j, s_j, for instance, the equations (9.11)–(9.14). From the conditions of compatibility of homogeneous equations for $\nu_j \neq 0, 1, 2$, we obtain invariant cross-ratios for four points of one and the same circle, for example, equations (9.25)–(9.29). In the case $\nu_j = 0, 1, 2$, the condition of compatibility of two equations provides certain condition rather than a cross-ratio.

Thus we can take from each system one equation and the compatibility condition, i.e, two equations for each point $\zeta = a_j$. The number of equations equals $2(m+1)$, and the number of unknown parameters a_k, c_k, p, q, r, s $(ps - rq = 1)$ will be $2m - 1$. Consequently, the number of equations is greater by three than the number of unknown parameters. This is connected with the fact that the bypass of all singular points a_k, $k = 1, 2, \ldots, m$, is equivalent to going around the point $\zeta = \infty$. This yields one matrix equation. Therefore these three equations are consequences of the remaining ones. This means that if we find all a_k, c_k and p, q, r, s and substitute them in the remaining system of equations, then they will identically be equal to zero. The appearance of three superfluous equations can be explained exactly in the same way as in the case of linear polygons. ■

Having found the system of equations for definition of a_k, c_k, p, q, r, s, we have to define the intervals of variation of the parameters c_k, $k = 1, 2, \ldots, m$, then to solve the system with respect to a_k, c_k, $k = 1, 2, \ldots, m$, and finally to specify p, q, r, s. Recall that p_j, q_j, r_j, s_j, $j = 1, 2, \ldots, m+1$, depend implicitly on the parameters a_k, c_k, $k = 1, 2, \ldots, m$.

Theorem 9.2. *If the contoure of the domain $s(w)$ of a circular polygon contains a cut with the end $w = b_j$ $\alpha_{1j} - \alpha_{2j} = 2$ for which, the then the second linearly independent solution* (3.1) *for the point $\zeta = a_j$ $v_{2j}(\zeta)$ does not contain the logarithmic term.*

Proof. Suppose the contrary. Let $v_{2j(\zeta)}$ contain a logarithmic term. For the point $\zeta = a_j$, we construct first a local fundamental matrix $\theta_j(\zeta)$ and then the matrices $\chi^+(t) = B_{0j}\theta_j^+(t)$, $\chi^-(t) = \overline{B}_{0j}\overline{\theta}_j(t)$, where B_{0j}, $\overline{B}_{0j}$ are the constants of the matrix constructed by (8.4)–(8.10). The matrices $\chi^+(t)$, $\chi^-(t)$ must satisfy the boundary conditions

$$B_{0j}\theta_j^+(t) = G_j\overline{B}_{0j}\theta_j^-(t), \quad \theta_j^+(t) = \theta_j^-(t), \quad t > a_j, \tag{9.35}$$

$$B_{0j}\vartheta_j^+(t) = G_j\overline{B}_{0j}\vartheta_j^-\theta_j^*(t), \quad t < a_j. \tag{9.36}$$

The equalities (9.35) and (9.36) imply that either

$$\text{either} \quad \text{and} \quad \vartheta_j^+ = \lambda\vartheta_j \quad \lambda = 1 \quad \text{or} \quad \lambda = -1. \tag{9.37}$$

When $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, and $\lambda = -1$ the equality (2.37) yields

$$i\begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}. \tag{9.38}$$

It follows from (9.38) that $\pi = 0$, which is not true. In the case $\alpha_{1j} = 2$, $\alpha_{2j} = 0$ and $\lambda = 1$, the equality (9.37) implies

$$\begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix}. \tag{9.39}$$

It again follows from (9.39) that $\pi = 0$, which is not true. Hence our supposition is invalid and the theorem is complete. ■

Theorem 9.2 has been proved in somewhat different way by P.Ya. Polubarinova-Kochina.

10. Representation of the Solutions $v_{kj}(\zeta)$, $j = 1, 2, \ldots, m+1$, by Means of Functional Series

It is known that the series $v_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \ldots, m, m+1$ converge near the points $\zeta = a_j$, $j = 1, 2, \ldots, m+1$, while the series $\varphi_{kj}(\zeta)$ converge near the points $\zeta = e_j = (a_j + a_{j+1})/2$. The radii of convergence of these series are bounded by the distance from the given point $t = a_j$ (or from the point $\zeta = e_j$) to the nearest points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$.

The constructed series $v_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ converge slowly thereby making numerical calculations more complicated. As n increases, the coefficients γ_{nj}^k sometimes increasen strongly, although their factory $(\zeta - a_j)^n$, on the contrary, strongly decrease as n increases. Electronic computers are unable to multiply γ_{nj}^k by $(t - a_j)^n$ despite the fact that these series converge. To remove this deficiency we suggest to represent these series as rapidly and uniformly convergent functional series.

Theorem 10.1. *If one considers the Fuchs class equation* (1.8), *with* $p(\zeta)$, $q(\zeta)$ *defined by* (1.9) (*or by* (1.12)), *and represent it near the points* $\zeta = a_j$ *and* $\zeta = \infty$ *in terms of the series* (3.2) *and* (3.4), *respectively, then the local solutions* $v_{kj}(\zeta)$, $j = 1, 2, \ldots, m+1$, *can be represented as rapidly and uniformly convergent functional series, the formulas* (3.9)–(3.11) *remaining valid.*

Proof. Consider the structure of the recursion formulas (3.9)–(3.11). The sum of the first subscripts for the expression $\gamma_{(k-n)j} \cdot f_{nj}(\alpha_j + k - n)$ is always equal to k, that is, to the exponent $(t - a_j)^k$. Consider instead of the series (3.5) the functional series

$$v_j(t) = (t - a_j)^{\alpha_j} \widetilde{v}_j(t - a_j), \quad \widetilde{v}_j(t - a_j) = \sum_{n=0}^{\infty} \gamma_{nj}(t - a_j), \quad (10.1)$$

where, owing to (3.9)–(3.11), γ_{nj} is defined in terms of $\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{(n-1)j}$, and the latters in terms of $f_{kj}(\alpha_j)$, where

$$f_{kj}(t - a_j, \alpha_j) = \alpha_j p_{kj}(t - a_j) + q_{kj}(t - a_j), \quad (10.2)$$

$$p_{nj}(t - a_j) = \sum_{k=1, k=j}^{m} (-1)^{n-1}(1 - \nu_k)\Big(\frac{t - a_j}{a_j - a_k}\Big)^n,$$

$$p_{0j} = 1 - \nu_j, \quad (10.3)$$

$$q_{nj}(t - a_j) = \sum_{k=1, k \neq j}^{m} (-1)^{n-2} \times$$

$$\times \{\sigma_k(n-1) + c_k(a_j - a_k)\}\Big(\frac{t - a_j}{a_j - a_k}\Big)^n \quad (10.4)$$

$$n = 2, 3, \ldots,$$

$$q_{0j} = \sigma_j, \quad q_{1j} = c_j \quad (10.5)$$

$$\Big|\frac{t - a_j}{a_j - a_k}\Big| < 1 \quad k \neq j, \quad (10.6)$$

$$|t - a_j| < M_{in}\{|a_j - a_{j-1}|,\ |a_j - a_{j+1}|\}. \quad (10.7)$$

It is seen from (10.6) that the functional series (10.1) converges uniformly near the point $\zeta = a_j$ and rapidly in comparison with the series (3.5).

The functional series for the point $\zeta = a_{m+1} = \infty$ can be constructed analogously.

In all the above formulas instead of the solution $v_{kj}(\zeta)$ we will represent the functional series (10.1).

Obviously, the functional series for regular points $t = e_j$, $e_j = (a_j + a_{j+1})/2$, $j = 1, 2, \ldots, m-1$, converge likewise uniformly and rapidly. ■

11. Determination of Intervals of Variation of Accessory Parameters

We have proved in [26] that $v_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m+1$, are entire functions of the accessory parameters, c_k, $k = 1, 2, \dots, m$ and in [23] we determined possible intervals of variation of these parameters.

Consider two cases: 1. A circular polygon with the angles $\nu_j = 1$, $j = 1, 2, \dots, m+1$, is given. We pass to that consisting of one circle. In this case, equation (1.1) takes the form

$$w(\zeta) = (A\zeta + B)/(C\zeta + D), \tag{11.1}$$

where A, B, C, D are unknown integration constants of (1.1).

Substitution of (11.1) in (1.1) results in the identity

$$R(\zeta) = \sum_{k=1}^{m} \frac{C_k}{\zeta - a_j} = 0. \tag{11.2}$$

From (11.2) it follows that

$$C_k = 0, \quad k = 1, 2, \dots, m.$$

2. On the plane w, a linear polygon is given. The accessory parameters vanish for this case and the solution of (1.1) is given by the Christofel–Schwarz's formula

$$w(\zeta) = M \int_0^{\zeta} \prod_{j=1}^{m} (\zeta - a_j)^{\nu_j - 1} d\zeta + N. \tag{11.3}$$

Substituting (11.3) in (1.1), we get

$$C_j^* = -(\nu_j - 1) \sum \sum_{k=1, k\neq j}^{m} (\nu_k - 1)/(a_j - a_k) \tag{11.4}$$

It follows from this reasoning that

$$\text{either} \quad c_j^* \leq c_j \leq 0 \quad \text{or} \quad c_j^* \geq c_j \geq 0. \tag{11.5}$$

To the equation (1.8), there corresponds the following Schwarz's equation:

$$\frac{w''(\zeta)}{w'(\zeta)} - \frac{3}{2}\Big(\frac{w''(\zeta)}{w'(\zeta)}\Big)^2 = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2, \tag{11.6}$$

where $p(\zeta)$ and $q(\zeta)$ are defined by (1.9) or (1.14).

For the equation (11.6) we consider the same two cases as above.

1. For this case, we have

$$\alpha'' = 0, \quad c_k = 0. \tag{11.7}$$

Thus we have obtained with respect to δ_k, $k = 1, 2, \ldots, m-3$, the following homogeneous system:

$$\delta_1 a_k^{m-2} + \cdots + \delta_{m-3} a_k^{m-3} + \delta_{m-2} = 0, \quad k = 1, 2, \ldots, m. \tag{11.8}$$

The equation (11.8) implies

$$\delta_k = 0, \quad k = 1, 2, \ldots, m-3. \tag{11.9}$$

2. In this case, we arrive at

$$\alpha' \neq 0, \quad \alpha'' \neq 0, \quad c_k = 0. \tag{11.10}$$

It follows from (11.10) that we get

$$\alpha' \alpha'' a_k^{m-2} + \delta_1 a_k^{m-3} + \cdots + \delta_{m-3} a_k + \delta_{m-2} = 0 \tag{11.11}$$

The system which this time is inhomogeneous with respect to δ_k, $k = 1, 2, \ldots, m-2$ (11.11) is solved with respect to δ_k, $k = 1, 2, \ldots, m-3$, hence in this case too one can determine possible intervals of variation of the accessory parameters.

12. Conclusion

Having known $w(\zeta)$ along the whole real axis t of the plane ζ, one can find $w = w(\zeta)$ for all $\Im(\zeta) > 0$ by the well-known formula [10, p. 152, formula (12.5.10)]

$$w(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} w^+(x) \frac{\tau dx}{(x-t)^2 + \tau^2}. \tag{12.1}$$

Along the whole real axis $w = w^+(t)$ is defined by (8.1):

$$w^+(t) = u_1^+(t)/u_2^+(t), \quad -\infty < t < +\infty, \tag{12.2}$$

where $u_1^+(t)$, and $u_2^+(t)$ as linearly independent solutions of (1.8), are defined uniquely by (8.4)–(8.10).

As is seen from the above-said, an algorithm for the construction of the single-valued analytic functions $w = w(\zeta)$ is given in a general form. These functions represent general solutions of (1.1) and map conformally the half-plane $\zeta = t + i\tau$ onto circular polygons with finite number of vertices and any angles at those vertices. At those vertices the system of equations is composed which connects geometrical characteristics of circular polygons with unknown parameters of the Schwarz's equation. Rapidly and uniformly convergent functional series are constructed.

Possible intervals of variation of the accessory parameters are defined. Consequently, the solution of (1.1) and the construction of $w = w(\zeta)$ are reduced, with regard for the boundary conditions (1.4), to the solution of a system of higher transcendental equations with respect to the parameters a_k, c_k, $k = 1, 2, \ldots, m$.

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