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# HOW PARABOLIC FREE BOUNDARIES APPROXIMATE HYPERBOLIC FRONTS

Abstract. Some recent results concerning existence and qualitative behaviour of the boundaries of the suppurts of solutions of the Cauchy problem for nonlinear first–order hyperbolic and second-order parabolic scalar conservation laws are discussed. Among other properties, it is shown that, under appropriate assumptions, parabolic interfaces converge to hyperbolic ones in the vanishing viscosity limit.

რეზიუმე. ნაშრომში მიმოხილულია ზოგიერთი ბოლოდროინდელი შედეგი პირველი და მეორე რიგის არაწრფივი სკალარული კონსერვატიული განტოლებებისათვის კოშის ამოცანის ამონახსნების არსებობისა და მათი მზიდების საზღვრების თვისებრივი ყოფაქცევის შესახებ. სხვა თვისებებთან ერთად ნაჩვენებია, რომ გარკვეულ პირობებში სიბლანტის ნულისკენ მისწრაფებისას პარაბოლური ტალღური ფრონტი მიისწრაფის ჰიპერბოლურისაკენ.

We consider phenomena associated with the nonnegative solution of the nonlinear first-order hyperbolic Cauchy problem

$$u_t + (f(u))_x = 0 \qquad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ \tag{1}$$

with the initial condition

$$u = u_0 \qquad \text{on } \mathbb{R} \times \{0\}\ , \tag{2}$$

and the nonnegative solution of the nonlinear second-order parabolic Cauchy problem

$$u_t + (f(u))_x = \varepsilon(a(u))_{xx} \qquad \text{in } \mathbb{R} \times \mathbb{R}^+ \tag{3}$$

(in which $\varepsilon > 0$ is a real parameter) with the same initial condition.

About the coefficients in these equations and the initial data function, we assume the following.

**($\mathbf{H}_1$)** The function $f \in C([0,\infty)) \cap C^1(0,\infty)$ with $f'$ locally Hölder continuous on $(0,\infty)$ and $f(0) = 0$.

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**($H_2$)** The function $a \in C([0,\infty)) \cap C^2(0,\infty)$ with $a''$ locally Hölder continuous on $(0,\infty)$, $a'(s) > 0$ for $s > 0$ and $a(0) = 0$.

**($H_3$)** The function $u_0 \in L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|\,\mathrm{d}x \to 0 \quad \text{as } h \to 0 \;, \tag{4}$$

$u_0$ is nonnegative, and is nontrivial in the sense that

$$M := \operatorname{ess\,sup}\{u_0(x) \mid x \in \mathbb{R}\} > 0 \;. \tag{5}$$

The problem (1),(2) may be regarded as the limit as $\varepsilon \downarrow 0$ of the problem (3),(2). Indeed, under the assumptions ($H_1$) and ($H_3$), it can be shown that the problem (1),(2) admits a unique *entropy* solution. While, under the assumptions ($H_1$)–($H_3$), the problem (3),(2) has a unique weak solution for any $\varepsilon > 0$. Moreover, the solution of the parabolic problem converges to the entropy solution of the hyperbolic problem as $\varepsilon \downarrow 0$ in $C([0,T]; L^1_{loc}(\mathbb{R}))$ for every $0 < T < \infty$ (see [1,3–5]).

In this report, we present some new results (see [2]) concerning the relationship between fronts associated with the propagation of the support of the entropy solution of the hyperbolic problem (1),(2) and the corresponding free boundary in the solution of the parabolic problem (3),(2). More precisely, we will study how the interface denoting the upper boundary of the support of the solution of (3),(2), also known as the "right front", approximates the corresponding interface of the solution of (1),(2) in the vanishing viscosity limit $\varepsilon \downarrow 0$.

Let us denote by $u(x,t;\varepsilon)$ the unique weak solution of the problem (3),(2) if $\varepsilon > 0$, and by $u(x,t;0)$ the unique entropy solution of the problem (1),(2). Define the front

$$\zeta(t;\varepsilon) := \sup\{x \in \mathbb{R} \mid w(x,t;\varepsilon) > 0\} \;,$$

where

$$w(x,t;\varepsilon) := \int_x^\infty u(y,t;\varepsilon)\,\mathrm{d}y \quad \text{for any } t > 0 \tag{6}$$

and $\varepsilon \geq 0$. Set also

$$\zeta_0 := \sup\{x \in \mathbb{R} \mid w_0(x) > 0\} \;,$$

where

$$w_0(x) := \int_x^\infty u_0(y)\,\mathrm{d}y \;. \tag{7}$$

Our objective is to study the existence and the behavior of the front $\zeta(t;\varepsilon)$ ($\varepsilon \geq 0$), as well as the relationship between the fronts $\zeta(\cdot;\varepsilon)$ for $\varepsilon > 0$ and the front $\zeta(\cdot;0)$.

A crucial rôle in the analysis is played by the quantities

$$\sigma_\delta := \sup\{f(s)/s \mid 0 < s \le \delta\} \tag{8}$$

and

$$\sigma_0 := \lim_{\delta \downarrow 0} \sigma_\delta = \limsup_{s \downarrow 0} f(s)/s \ . \tag{9}$$

Obviously, by definition $\sigma_0 \le \sigma_M$.

In the singular case $\sigma_0 = \infty$ then, irrespective of the initial data $u_0$, there holds $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \ge 0$. In this case, both the hyperbolic problem and the parabolic problem display *infinite speed of propagation*. On the other hand, if $\sigma_0 < \infty$, then the hyperbolic problem (1),(2) displays *finite speed of propagation*, i.e., if $\zeta_0 < \infty$, then there holds $\zeta(t;0) < \infty$ for all $t > 0$. For the parabolic equation, finite speed of propagation holds only under the additional necessary and sufficient condition

**(H$_4$)** $$\int_0^\delta \frac{a'(s)}{\max\{s, -f(s)\}} \, \mathrm{d}s < \infty \qquad \text{for some } \delta > 0 \ ,$$

which ensures that the diffusion is "slow" enough.

This is the content of the following theorems.

**Theorem 1.** *Let the assumptions* (H$_1$) *and* (H$_3$) *hold.*

**(a)** *If $\sigma_0 = \infty$, then $\zeta(t;0) = \infty$ for all $t > 0$.*

**(b)** *If $\sigma_0 < \infty$, then, as an extended function, $\zeta(\cdot;0)$ is lower semi-continuous and continuous from the right on $[0,\infty)$ with $\zeta(0;0) = \zeta_0$, and*

$$\zeta(t;0) \le \zeta(t_0;0) + \sigma_M(t - t_0) \qquad \textit{for all } t > t_0 \ge 0 \ . \tag{10}$$

*Moreover if $\sigma_0 > -\infty$, then $\zeta(\cdot;0)$ is continuous on $[0,\infty)$ with*

$$\zeta(t;0) \ge \zeta(t_0;0) + \sigma_0(t - t_0) \qquad \textit{for all } t > t_0 \ge 0 \ . \tag{11}$$

**Theorem 2.** *Let $\varepsilon > 0$ and the assumptions* (H$_1$)–(H$_3$) *hold.*

**(a)** *If $\sigma_0 = \infty$ or the assumption* (H$_4$) *is negated, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$.*

**(b)** *If $\sigma_0 < \infty$ and* (H$_4$) *is satisfied, then, as an extended function, $\zeta(\cdot;\varepsilon)$ is lower semi-continuous and continuous from the right on $[0,\infty)$ with $\zeta(0;\varepsilon) = \zeta_0$, and*

$$\zeta(t;\varepsilon) \le \zeta(t_0;\varepsilon) + \sigma_M(t - t_0) + Q_M(t - t_0, \varepsilon) \qquad \textit{for all } t > t_0 \ge 0 \ , \tag{12}$$

*where*

$$Q_M(t,\varepsilon) := \inf_{\sigma > \sigma_M} \left\{ (\sigma - \sigma_M)t + \varepsilon \int_0^M \frac{a'(s)}{\sigma_s - f(s)} \, \mathrm{d}s \right\} \ .$$

*Moreover, if $\sigma_0 > -\infty$, then $\zeta(\cdot;\varepsilon)$ is continuous on $[0,\infty)$ with*

$$\zeta(t;\varepsilon) \geq \zeta(t_0;\varepsilon) + \sigma_0(t-t_0) \quad \textit{for all } t > t_0 \geq 0 \ . \tag{13}$$

In the general situation with $\sigma_0 > -\infty$ we also completely clarify the relation between the parabolic free boundaries and the hyperbolic front. If $\sigma_0 = \infty$, or, if $|\sigma_0| < \infty$ and $\zeta_0 = \infty$, then by Theorems 1 and 2, $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$. Whereas if $|\sigma_0| < \infty$, $\zeta_0 < \infty$ and $(H_4)$ is negated, then $\zeta(t;0) < \infty = \zeta(t;\varepsilon)$ for all $t > 0$ and $\varepsilon > 0$. Thus in both these cases, the relation is clear from Theorems 1 and 2. In the case where $|\sigma_0| < \infty$, $\zeta_0 < \infty$ and $(H_4)$ holds with $\zeta(t;\varepsilon) < \infty$ for all $t \geq 0$ and $\varepsilon \geq 0$, we can prove the following result concerning the convergence of the interfaces as $\varepsilon \downarrow 0$.

**Theorem 3.** *Suppose that assumptions* $(H_1)$–$(H_4)$ *hold,* $|\sigma_0| < \infty$ *and* $\zeta_0 < \infty$*. Then*

$$\zeta(\cdot;\varepsilon) \to \zeta(\cdot;0) \quad \textit{as } \varepsilon \downarrow 0$$

*in* $C([0,T]) \cap C^{0+\alpha}([\tau,T])$ *for all* $0 < \tau < T < \infty$ *and* $0 < \alpha < 1$.

We conclude that in case $\sigma_0$ is finite, the hyperbolic front is the vanishing viscosity limit of the free boundary of the parabolic problem when the latter displays finite speed of propagation.

With regard to the singular situation $\sigma_0 = -\infty$, we obtain the following weaker result.

**Theorem 4.** *Suppose that the assumptions* $(H_1)$–$(H_4)$ *hold and* $\sigma_0 = -\infty$*. Then*

$$\liminf_{\varepsilon\downarrow 0} \zeta(t;\varepsilon) \geq \zeta(t;0) \quad \textit{for all } t > 0 \ ,$$

*and, if there exists a* $t_0 \geq 0$ *such that* $\limsup_{\varepsilon\downarrow 0} \zeta(t_0;\varepsilon) < \infty$*, then there holds*

$$\limsup_{\varepsilon\downarrow 0} \zeta(t;\varepsilon) \leq \limsup_{s\uparrow t} \zeta(s;0) \quad \textit{for all } t > t_0 \ .$$

It is known (see [3]) that in the above case both problems (1), (2) and (3), (2) admit *instantaneous shrinking*, this is to say that one can have $\zeta_0 = \infty$ and $\zeta(t;\varepsilon) < \infty$ for all $t > 0$, and *deferred instantaneous shrinking*, where $\zeta(t;\varepsilon) = \infty$ for all $0 \leq t < \tau$ and $\zeta(t;\varepsilon) < \infty$ for all $t > \tau$ for some $0 < \tau < \infty$.

It is informative to provide a synopsis of the previous results for the prototype equations

$$u_t + \lambda(u^n)_x = 0 \tag{14}$$

and

$$u_t + \lambda(u^n)_x = \varepsilon(u^m)_{xx} \tag{15}$$

with $\lambda \in \{-1,0,1\}$, $n > 0$ and $m > 0$ real parameters.

This is made in the following theorem.

**Theorem 5.** *Let $\lambda \in \{-1, 0, 1\}$, $n > 0$ and $m > 0$ be real constants, and suppose that $u_0$ satisfies the assumption* ($H_3$).

**(i)** *If $n < 1$ and $\lambda = 1$, then there holds $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon \geq 0$.*

**(ii)** *If $n = 1$, there holds $\zeta(t;0) = \zeta_0 + \lambda t$ for all $t > 0$. Moreover:*

- **(a)** *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
- **(b)** *If $m > 1$, then $\zeta_0 + \lambda t \leq \zeta(t;\varepsilon) \leq \zeta_0 + \lambda t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where*

$$R(t,\varepsilon,M) = C\varepsilon^{1/2} t^{1/2} M^{(m-1)/2} \tag{16}$$

*for some constant $C$ which depends only on $m$.*

**(iii)** *If $n > 1$ and $\lambda = -1$, there holds $\zeta(t;0) = \zeta_0$ for all $t > 0$. Moreover:*

- **(a)** *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
- **(b)** *If $m > 1$, then $\zeta_0 \leq \zeta(t;\varepsilon) \leq \zeta_0 + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by (16) or by*

$$R(t,\varepsilon,M) = \begin{cases} C\varepsilon^{(n-1)/(2n-m-1)} t^{(n-m)/(2n-m-1)} & \text{for } m < n \\ C\varepsilon\{\ln(\varepsilon^{-1} t M^{n-1} + 1) + 1\} & \text{for } m = n \\ C\varepsilon M^{m-n} & \text{for } m > n \end{cases} \tag{17}$$

*for some constant $C$ which depends only on $m$ and $n$.*

**(iv)** *If $n > 1$ and $\lambda = 1$, then there holds $\zeta_0 \leq \zeta(t;0) \leq \zeta_0 + M^{n-1}t$ for all $t > 0$. Moreover:*

- **(a)** *If $m \leq 1$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
- **(b)** *If $m > 1$, then $\zeta_0 \leq \zeta(t;\varepsilon) \leq \zeta_0 + M^{n-1}t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by (16) or by*

$$R(t,\varepsilon,M) = C\varepsilon M^{m-n}\{\ln(\varepsilon^{-1} t M^{2n-m-1} + 1) + 1\} \tag{18}$$

*for some constant $C$ which depends only on $m$ and $n$.*

**(v)** *If $n < 1$ and $\lambda = -1$, then there holds $\zeta(t;0) \leq \zeta_0 - M^{n-1}t$ for all $t > 0$. Moreover:*

- **(a)** *If $m \leq n$, then $\zeta(t;\varepsilon) = \infty$ for all $t > 0$ and $\varepsilon > 0$.*
- **(b)** *If $m > n$, then $\zeta(t;\varepsilon) \leq \zeta_0 - M^{n-1}t + R(t,\varepsilon,M)$ for all $t > 0$ and $\varepsilon > 0$, where $R(t,\varepsilon,M)$ is given by*

$$R(t,\varepsilon,M) = \begin{cases} C\varepsilon^{(1-n)/(m+1-2n)} t^{(m-n)/(m+1-2n)} & \text{for } m < 1 \\ C\varepsilon^{1/2} t^{1/2} \ln^{1/2}(\varepsilon t^{-1} M^{2(1-n)} + 1) & \text{for } m = 1 \\ C\varepsilon^{1/2} t^{1/2} M^{(m-1)/2} & \text{for } m > 1 \end{cases} \tag{19}$$

*or by*

$$R(t,\varepsilon,M) = C\varepsilon M^{m-n}\{\ln(\varepsilon^{-1} t M^{2n-m-1} + 1) + 1\} \tag{20}$$

*for some constant $C$ which depends only on $m$ and $n$.*

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