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ON BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. A general theorem (principle of a priori boundedness) on solvability of the boundary value problem

$$\frac{dx(t)}{dt} = f(x)(t), \quad h(x) = 0$$

is established, where

$$f : C([a,b]; R^n) \to L([a,b]; R^n) \quad \text{and} \quad h : C([a,b]; R^n) \to R^n$$

are continuous operators. As an application, a two-point boundary value problem for the system of ordinary differential equations is considered.

რეზიუმე. დამტკიცებულია ზოგადი თეორემა (აპრიორული შემოსაზღვრულობის პრინციპი)

$$\frac{dx(t)}{dt} = f(x)(t), \quad h(x) = 0$$

სასაზღვრო ამოცანის ამოხსნადობის შესახებ, სადაც $f : C([a,b]; R^n) \to L([a,b]; R^n)$ და $h : C([a,b]; R^n) \to R^n$ უწყვეტი ოპერატორებია. ამ თეორემის საფუძველზე გამოკვლეულია ორწერტილოვანი სასაზღვრო ამოცანა ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისათვის.

1. Statement of the Problem and Main Notation

Let n be a natural number, $I = [a,b]$ be a segment of the real axis, and let $f : C(I; R^n) \to L(I; R^n)$ and $h : C(I; R^n) \to R^n$ be continuous operators satisfying for every $\rho \in]0, +\infty[$ the conditions

$$\sup\big\{\|f(x)(\cdot)\| : x \in C(I; R^n), \|x\|_C \le \rho\big\} \in L(I; R),$$
$$\sup\big\{\|h(x)\| : x \in C(I; R^n), \|x\|_C \le \rho\big\} < +\infty.$$

Consider the functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

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with the boundary condition

$$h(x) = 0. \tag{2}$$

Under the solution of the equation (1) we mean an absolutely continuous vector function $x : I \to R^n$ which almost everywhere on I satisfies this equation, and under the solution of the problem (1), (2) we mean a solution of the equation (1) satisfying (2).

The theorem on the existence of a solution of the problem (1), (2) which will be proved below and be called the principle of a priori boundedness, generalizes Conti–Opial type theorems [2, 3, 7, 10–13] and supplements earlier known criteria for the solvability of boundary value problems for systems of ordinary differential and functional differential equations [1–14].

On the basis of the above-mentioned principle of a priori boundedness, we have obtained effective criteria for the solvability of the boundary value problem

$$\frac{dx(t)}{dt} = f_0(t, x(t)), \tag{3}$$

$$x(t_1(x)) = A(x)\, x(t_2(x)) + c_0, \tag{4}$$

where $f_0 : I \times R^n \to R^n$ is a vector function satisfying the local Carathéodory conditions, $c_0 \in R^n$ and $t_i : C(I; R^n) \to I$ $(i = 1, 2)$ and $A : C(I; R^n) \to R^n$ are continuous operators.

The use is made of the following notation:

$I = [a, b]$, $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$;

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in R$ $(i = 1, \dots, n)$ and the norm $\|x\| = \sum_{i=1}^n |x_i|$;

if $x = (x_i)_{i=1}^n$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n$;

$x \cdot y$ is the scalar product of the vectors x and $y \in R^n$;

$R^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with the components $x_{ik} \in R$ $(i, k = 1, \dots, n)$ and the norm $\|X\| = \sum_{i,k=1}^n |x_{ik}|$;

$C(I; R^n)$ is the space of continuous vector functions $x : I \to R^n$ with the norm $\|x\|_C = \max\{\|x(t)\| : t \in I\}$;

$L(I; R^n)$ is the space of summable vector functions $x : I \to R^n$ with the norm $\|x\|_L = \int_a^b \|x(t)\|\, dt$.

2. The Principle of a Priori Boundedness

To formulate our basic result, we will need the following

Definition 1. The pair (p, l) of continuous operators $p : C(I; R^n) \times C(I; R^n) \to L(I; R^n)$ and $l : C(I; R^n) \times C(I; R^n) \to R^n$ is said to be *consistent* if:

(i) for any fixed $x \in C(I; R^n)$ the operators $p(x, \cdot) : C(I; R^n) \to L(I; R^n)$ and $l(x, \cdot) : C(I; R^n) \to R^n$ are linear;

(ii) for any x and $y \in C(I;R^n)$ and for almost all $t \in I$ the inequalities

$$\|p(x,y)(t)\| \le \alpha(t,\|x\|_C)\,\|y\|_C, \quad \|l(x,y)\| \le \alpha_0(\|x\|_C)\,\|y\|_C$$

are fulfilled, where $\alpha_0 : R_+ \to R_+$ is nondecreasing and $\alpha : I \times R_+ \to R_+$ is summable in the first argument and nondecreasing in the second one;

(iii) there exists a positive number β such that for any $x \in C(I;R^n)$, $q \in C(I;R^n)$ and $c_0 \in R^n$ an arbitrary solution y of the boundary value problem

$$\frac{dy(t)}{dt} = p(x,y)(t) + q(t), \quad l(x,y) = c_0 \tag{5}$$

admits the estimate

$$\|y\|_C \le \beta(\|c_0\| + \|q\|_L). \tag{6}$$

Theorem 1. *Let there exist a positive number ρ and a consistent pair (p,l) of continuous operators $p : C(I;R^n) \times C(I;R^n) \to L(I;R^n)$ and $l : C(I;R^n) \times C(I;R^n) \to R^n$ such that for any $\lambda \in]0,1[$ an arbitrary solution of the problem*

$$\frac{dx(t)}{dt} = p(x,x)(t) + \lambda\big[f(x)(t) - p(x,x)(t)\big], \tag{7}$$

$$l(x,x) = \lambda[l(x,x) - h(x)] \tag{8}$$

admits the estimate

$$\|x\|_C \le \rho. \tag{9}$$

Then the problem (1), (2) *is solvable.*

Proof. Let α, α_0 and β be the functions and numbers appearing in Definition 1. Set

$$\gamma(t) = 2\rho\alpha(t,2\rho) + \sup\big\{\|f(x)(t)\| : x \in C(I;R^n),\ \|x\|_C \le 2\rho\big\},$$

$$\gamma_0 = 2\rho\alpha_0(2\rho) + \sup\big\{\|h(x)\| : x \in C(I;R^n),\ \|x\|_C \le 2\rho\big\},$$

$$\sigma(s) = \begin{cases} 1 & \text{for } \ 0 \le s \le \rho \\ 2 - s/\rho & \text{for } \ \rho < s < 2\rho \ , \\ 0 & \text{for } \ s \ge 2\rho \end{cases} \tag{10}$$

$$\begin{aligned} q(x)(t) &= \sigma(\|x\|_C)\big[f(x)(t) - p(x,x)(t)\big], \\ c_0(x) &= \sigma(\|x\|_C)\big[l(x,x) - h(x)\big]. \end{aligned} \tag{11}$$

Then $\gamma \in L(I;R)$, $\gamma_0 < +\infty$, and for every $x \in C(I;R^n)$ and almost all $t \in I$, the inequalities

$$\|q(x)(t)\| \le \gamma(t), \quad \|c_0(x)\| \le \gamma_0. \tag{12}$$

hold.

For an arbitrarily fixed $x \in C(I;R^n)$, let us consider the linear boundary value problem

$$\frac{dy(t)}{dt} = p(x,y)(t) + q(x)(t), \quad l(x,y) = c_0(x). \tag{13}$$

By virtue of the condition (iii) from Definition 1, the homogeneous problem

$$\frac{dy(t)}{dt} = p(x,y)(t), \quad l(x,y) = 0 \tag{13_0}$$

has only the trivial solution. However, by Theorem 1.1 in [9], the conditions (i) and (ii) from Definition 1 and the absence of nontrivial solutions of the problem (13_0) guarantee the unique solvability of the problem (13). On the other hand, by virtue of the conditions (ii) and (iii) from Definition 1 and the inequalities (12), the solution y of the problem (11) admits the estimate

$$\|y\|_C \le \rho_0, \quad \|y'(t)\| \le \gamma^*(t) \quad \text{for almost all} \quad t \in I, \tag{14}$$

where $\rho_0 = \beta(\gamma_0 + \|\gamma\|_L)$, $\gamma^*(t) = \alpha(t,\rho_0)\rho_0 + \gamma(t)$.

Let $u : C(I;R^n) \to C(I;R^n)$ be an operator which to every $x \in C(I;R^n)$ assigns the solution y of the problem (13). Due to Corollary 1.6 from [9], the operator u is continuous. On the other hand, by (14) we have

$$\|u(x)\|_C \le \rho_0, \quad \|u(x)(t) - u(x)(s)\| \le \left| \int_s^t \gamma^*(\xi)\, d\xi \right| \quad \text{for} \quad s \ \text{ and } \ t \in I.$$

Consequently, the operator u continuously maps the ball $C_{\rho_0} = \{x \in C(I;R^n) : \|x\|_C \le \rho_0\}$ into its own compact subset. Therefore, owing to Schauder's principle, there exists $x \in C_{\rho_0}$ such that $u(x)(t) = x(t)$ for $t \in I$. By the equalities (11), x is obviously a solution of the problem (7), (8), where

$$\lambda = \sigma(\|x\|_C). \tag{15}$$

Let us show that x admits the estimate (9). Suppose the contrary. Then either

$$\rho < \|x\|_C \le 2\rho, \tag{16}$$

or

$$\|x\|_C > 2\rho. \tag{17}$$

If we assume that the inequality (16) is fulfilled, then because of (10) and (15) we have $\lambda \in]0,1[$. However, by the conditions of the theorem, in this case we have the estimate (9) which contradicts (16). Suppose now that (17) is fulfilled. Then by (10) and (15), we have $\lambda = 0$. Hence x is a solution of the problem (13_0). But this is impossible because (13_0) has only the trivial solution. The above-obtained contradiction proves the validity of the estimate (9).

By virtue of (9), (10) and (15), it is clear that $\lambda = 1$ and hence x is a solution of the problem (1), (2). ■

Following [10], we introduce

Definition 2. Let $p : C(I;R^n) \times C(I;R^n) \to L(I;R^n)$ and $l : C(I;R^n) \times C(I;R^n) \to R^n$ be arbitrary, while $p_0 : C(I;R^n) \to L(I;R^n)$ and $l_0 : C(I;R^n) \to R^n$ be linear operators. We say that the pair (p_0, l_0) belongs to the set $\mathcal{E}^n_{p,l}$ if there exists a sequence $x_k \in C(I;R^n)$ $(k = 1, 2, \dots)$ such that for every $y \in C(I;R^n)$ the following conditions are fulfilled:

$$\lim_{k\to\infty} \int_0^t p(x_k, y)(s)ds = \int_0^t p_0(y)(s)ds \quad \text{uniformly on} \quad I,$$

$$\lim_{k\to\infty} l(x_k, y) = l_0(y).$$

Definition 3. We say that the pair (p, l) of continuous operators $p : C(I;R^n) \times C(I;R^n) \to L(I;R^n)$ and $l : C(I;R^n) \times C(I;R^n) \to R^n$ belongs to the *Opial class* O^n_0 if:

(i) for any fixed $x \in C(I;R^n)$ the operators $p(x, \cdot) : C(I;R^n) \to L(I;R^n)$ and $l(x, \cdot) : C(I;R^n) \to R^n$ are linear;

(ii$'$) for any x and $y \in C(I;R^n)$ and for almost all $t \in I$, the inequalities

$$\|p(x, y)(t)\| \le \alpha(t)\|y\|_C, \quad \|l(x, y)\| \le \alpha_0\|y\|_C$$

are fulfilled, where $\alpha : I \to R_+$ is summable and $\alpha_0 \in R_+$;

(iii$'$) for every $(p_0, l_0) \in \mathcal{E}^n_{pl}$ the problem

$$\frac{dy(t)}{dt} = p_0(y)(t), \quad l_0(y) = 0 \tag{18}$$

has only the trivial solution.

By Lemma 2.2 from [10], if $(p, l) \in O^n_0$, then the pair (p, l) is consistent. Therefore from Theorem 1 we have

Corollary 1. *Let there exist a positive number* ρ *and a pair of operators* $(p, l) \in O^n_0$ *such that for every* $\lambda \in]0, 1[$ *an arbitrary solution of the problem* (7), (8) *admits the estimate* (9). *Then the problem* (1), (2) *is solvable.*

Definition 4. The linear operator $p_0 : C(I;R^n) \to L(I;R^n)$ is said to be *strongly bounded* if there exists a summable function $\alpha : I \to R_+$ such that for every $y \in C(I;R^n)$, the inequality $\|p_0(y)(t)\| \le \alpha(t)\|y\|_C$ is fulfilled almost everywhere on I.

Let $p(x,y)(t) \equiv p_0(y)(t)$ and $l(x,y) \equiv l_0(y)$, where $p_0 : C(I;R^n) \to L(I;R^n)$ is a strongly bounded linear operator and $l_0 : C(I;R^n) \to R^n$ is a bounded linear operator. Then by Definition 3, for the condition $(p,l) \in O_0^n$ to be fulfilled, it is necessary and sufficient that the problem (18) have only the trivial solution. Therefore from Corollary 1 follows

Corollary 2. *Let there exist a positive number ρ, a linear strongly bounded operator $p_0 : C(I;R^n) \to L(I;R^n)$ and a linear bounded operator $l_0 : C(I;R^n) \to R^n$ such that the problem* (18) *has only the trivial solution and for every $\lambda \in]0,1[$ an arbitrary solution of the problem*

$$\frac{dx(t)}{dt} = p_0(x)(t) + \lambda\big[f(x)(t) - p_0(x)(t)\big], \quad l_0(x) = \lambda[l_0(x) - h(x)]$$

admits the estimate (9). *Then the problem* (1), (2) *is solvable.*

3. Theorem on the Solvability of the Problem (3), (4)

As is mentioned in Section 1, we investigate the problem (3), (4) under the assumptions that the vector function $f_0 : I \times R^n \to R^n$ satisfies the local Carathéodory conditions, and the operators $t_i : C(I;R^n) \to I$ $(i = 1,2)$ and $A : C(I;R^n) \to R^{n\times n}$ are continuous.

Assume

$$I_0 = \big\{t_1(x) : x \in C(I;R^n)\big\},$$
$$\|A(x)\|_0 = \max\big\{\|A(x)y\| : y \in R^n,\ \|y\| = 1\big\}.$$

The following theorem holds.

Theorem 2. *Let there exist summable functions $g_1 : I \to R$, $g_2 : I \to R_+$ and a number $\delta \in]0,1[$ such that*

$$f_0(t,x) \cdot \operatorname{sgn}[(t-t_0)x] \le \\ \le g_1(t)\|x\| + g_2(t) \quad \textit{for} \quad t \in I,\ t_0 \in I_0,\ x \in R^n \tag{19}$$

and

$$\exp\Bigg[\int\limits_{t_1(x)}^{t_2(x)} g_1(t)\,dt \cdot \operatorname{sgn}(t_2(x) - t_1(x))\Bigg]\,\|A(x)\|_0 \le \\ \le \delta \quad \textit{for} \quad x \in C(I;R^n). \tag{20}$$

Then the problem (3), (4) *is solvable.*

Proof. For every x and $y \in C(I;R^n)$, we suppose $f(x)(t) = f_0(t,x(t))$, $h(x) = x(t_1(x)) - A(x)x(t_2(x)) - c_0$,

$$p(x,y)(t) = \big[g_1(t)\operatorname{sgn}(t - t_1(x))\big]y(t), \quad l(x,y) = y(t_1(x)).$$

Obviously, the operators $p : C(I;R^n) \times C(I;R^n) \to L(I;R^n)$ and $l : C(I;R^n) \times C(I;R^n) \to R^n$ are continuous and the pair (p,l) is consistent.

By Theorem 1, to prove Theorem 2 it suffices to establish the uniform with respect to $\lambda \in]0,1[$ a priori boundedness of solutions of the problem

$$\frac{dx(t)}{dt} = (1-\lambda)\big[g_1(t)\operatorname{sgn}(t-t_1(x))\big]x(t) + \lambda f_0(t,x(t)),$$
$$x(t_1(x)) = \lambda\big[A(x)x(t_2(x)) + c_0\big].$$

Let x be an arbitrary solution of this problem for some $\lambda \in]0,1[$. Suppose $u(t) = \|x(t)\|$. Then by (19),

$$u'(t)\operatorname{sgn}(t-t_1(x)) \le g_1(t)u(t) + g_2(t) \quad \text{for} \quad t \in I. \tag{21}$$

On the other hand,

$$u(t_1(x)) \le \|A(x)\|_0 \, u(t_2(x)) + \|c_0\|. \tag{22}$$

The inequality (21) implies

$$u(t) \le \exp\bigg(\int\limits_{t_1(x)}^{t} g_1(s)\operatorname{sgn}(s-t_1(x))\,ds\bigg)u(t_1(x)) + \rho_1 \quad \text{for} \quad t \in I, \tag{23}$$

where $\rho_1 = \exp(\|g_1\|_L)\,\|g_2\|_L$. This, with regard for (20) and (22), yields

$$u(t_2(x)) \le \delta u(t_2(x)) + \|c_0\|\exp(\|g_1\|_L) + \rho_1$$

and, consequently,

$$u(t_2(x)) \le \rho_2, \tag{24}$$

where $\rho_2 = (1-\delta)^{-1}[\|c_0\|\exp(\|g_1\|_L) + \rho_1]$. However, as it is clear from (20),

$$\|A(x)\|_0 \le \delta\exp(\|g_1\|_L).$$

According to this inequality, from (22)–(24) there follows the estimate (9), where $\rho_0 = \delta\exp(2\|g_1\|_L)(\delta\rho_2 + \|c_0\|) + \rho_1$ is a positive constant, which does not depend on λ and x. ■

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References

1. N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, Introduction to the theory of functional differential equations. (Russian) *Nauka, Moscow*, 1991.

2. S. R. Bernfeld and V. Lakshmikantham, An introduction to nonlinear boundary value problems. *Academic Press, Inc., New York and London*, 1974.

3. R. Conti, Problèmes linéaires pour les équations différentielles ordinaires. *Math. Nachr.* 23(1961), 161–178.

4. Sh. Gelashvili and I. Kiguradze, On multi-point boundary value problems for systems of functional differential and difference equations. *Mem. Differential Equations Math. Phys.* 5(1995), 1–113.

5. P. Hartman, Ordinary differential equations. *John Wiley, New York*, 1964.

6. I. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

7. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. *J. Soviet Math.* 43(1988), No. 2, 2259–2339.

9. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* 47(1997), No. 2, 341–373.

8. I. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations I. (Russian) *Metsniereba, Tbilisi*, 1997.

10. I. Kiguradze and B. Půža, Conti–Opial type theorems for systems of functional differential equations. (Russian) *Differentsial'nye Uravneniya* 33(1997), No. 2, 185–194.

11. I. Kiguradze and B. Půža, On the solvability of boundary value problems for systems of nonlinear differential equations with deviating arguments. *Mem. Differential Equations Math. Phys.* 10(1997), 157–161.

12. I. Kiguradze and B. Půža, On the solvability of nonlinear boundary value problems for functional differential equations. *Georgian Math. J.* (*to appear*).

13. Z. Opial, Linear problems for systems of nonlinear differential equations. *J. Diff. Eqs.* 3(1967), 580–594.

14. N. I. Vasil'ev and Yu. A. Klokov, Foundations of the theory of boundary value problems for ordinary differential equations. (Russian) *Zinatne, Riga*, 1978.

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