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A SINGULAR EIGENVALUE PROBLEM FOR SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The Sturm-Liouville equation of the form

$$(p(t)x')' + \lambda q(t)x = 0 \quad (p(t) > 0, \;\; q(t) > 0), \tag{A}$$

is considered on an infinite interval $[a, +\infty[$ and the problem of finding the values of λ for which (A) has a principal solution $x_0(t;\lambda)$ satisfying $\alpha x_0(a;\lambda) - \beta p(a)x_0'(a;\lambda) = 0$, $\alpha^2 + \beta^2 > 0$, is studied: Assuming that (A) is strongly nonoscillatory in the sense of Nehari, a general theorem is proved asserting that, similarly to the regular eigenvalue problems on compact intervals, there exists a sequence $\{\lambda_n\}$ of eigenvalues such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$, and the eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly n zeros in (a, ∞).

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$$(p(t)x')' + \lambda q(t)x = 0 \quad (p(t) > 0, \;\; q(t) > 0), \tag{A}$$

განტოლება უსასრულო $[a, +\infty[$ ინტერვალზე და შესწავლილია ამოცანა λ-ს ისეთი მნიშვნელობების პოვნის შესახებ, რომელთათვისაც (A)-ს $\alpha x(a) - \beta p(a)x'(a)$ პირობებში, სადაც, $\alpha^2 + \beta^2 > 0$, აქვს მთავარი ამონახსნი $x_0(t,\lambda)$. (A) განტოლების ნეჰარის აზრით ძლიერად არარხევადობის პირობებში დამტკიცებულია ზოგადი თეორემა რომლის ძალითაც, კომპაქტურ ინტერვალებზე განხილული ამოცანების მსგავსად, არსებობს საკუთრივ მნიშვნელობათა ისეთი $\{\lambda_n\}$ მიმდევრობა, რომ $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $= \lim_{n\to\infty} \lambda_n = \infty$ და $\lambda = \lambda_n$-ის შესაბამის საკუთრივ $x_0(t;\lambda_n)$ ფუნქციას ზუსტად n ნული აქვს $(a, +\infty)$-ზე.

0. Introduction

We consider the Sturm-Liouville equation of the form

$$(p(t)x')' + \lambda q(t)x = 0, \quad t \geq a, \tag{A}$$

where $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a \geq 0$, and λ is a positive parameter. We assume that (A) is strongly nonoscillatory,

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that is, (A) is nonoscillatory for all $\lambda > 0$. In this case, it is known ([2, p. 355]) that there exists, for every $\lambda > 0$, a principal solution $x_0(t;\lambda)$ of (A) which is uniquely determined up to a constant factor by the requirement

$$\int^{\infty} \frac{dt}{p(t)(x_0(t;\lambda))^2} = \infty. \tag{0.1}$$

We are interested in the problem of finding those values of λ for which (A) has a principal solution $x_0(t;\lambda)$ satisfying the boundary condition at $t = a$:

$$\alpha x_0(a;\lambda) - \beta p(a) x_0'(a;\lambda) = 0, \tag{0.2}$$

where α and β are real constants such that $\alpha^2 + \beta^2 > 0$. This problem may well be called the singular eigenvalue problem for (A) since requiring $x_0(t;\lambda)$ to be a principal solution can be regarded as imposing on $x_0(t;\lambda)$ a boundary condition at $t = \infty$; see, e.g., Hartman [3]. A solution $x_0(t;\lambda)$ of this problem will be termed a principal eigenfunction and the corresponding value of λ a principal eigenvalue. The main purpose of this paper is to prove that, similarly to regular eigenvalue problems on compact intervals, there exists a sequence of principal eigenvalues $\{\lambda_n\}$ tending to infinity and the principal eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly n zeros in (a,∞).

Our goal is to prove the following theorem.

Theorem. *Assume that (A) is strongly nonoscillatory. Put*

$$\begin{aligned} &\gamma = \alpha && \text{for the case where } \int_a^{\infty} \frac{dt}{p(t)} = \infty; \\ &\gamma = \alpha + \beta\left(\int_a^{\infty} \frac{dt}{p(t)}\right)^{-1} && \text{for the case where } \int_a^{\infty} \frac{dt}{p(t)} < \infty. \end{aligned}$$

(i) *Let $\beta = 0$ or $\beta\gamma > 0$. Then there exists a sequence of principal eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ such that*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty, \tag{0.3}$$

and the principal eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly n zeros in (a,∞), $n = 0,1,2,\ldots$.

(ii) *Let $\gamma = 0$ or $\beta\gamma < 0$. Then there exists a sequence of principal eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ such that*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty, \tag{0.4}$$

and the principal eigenfunction $x_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ has exactly n zeros in (a,∞), $n = 1,2,\ldots$.

1. Preliminaries

We note that it suffices to give a proof for the special case where $p(t) \equiv 1$:

$$x'' + \lambda q(t)x = 0, \quad t \geq a, \tag{B}$$

since the general case of (A) can be reduced to (B) by means of a change of independent and/or dependent variables. It is known ([6]) that (B) is strongly nonoscillatory if and only if $q(t) \in L(a, \infty)$ and

$$\lim_{t\to\infty} t \int_t^\infty q(s)ds = 0. \tag{1.1}$$

This condition is all that is required for the equation (B).

1.1. Normalized Principal Solutions. Let $\lambda > 0$ be fixed arbitrarily and consider the fundamental set of solutions $\{x_1(t;\lambda), x_2(t;\lambda)\}$ of (B) determined by the initial conditions

$$x_1(a;\lambda) = 1, \quad x_1'(a;\lambda) = 0, \quad x_2(a;\lambda) = 0, \quad x_2'(a;\lambda) = 1.$$

Let $x_0(t;\lambda)$ be a principal solution of (B). Then there exist constants $c_1(\lambda)$ and $c_2(\lambda)$ depending on λ such that

$$x_0(t;\lambda) = c_1(\lambda)x_1(t;\lambda) + c_2(\lambda)x_2(t;\lambda), \quad t \geq a. \tag{1.3}$$

Since a principal solution is unique up to a constant factor, we may suppose that the coefficients $c_1(\lambda)$ and $c_2(\lambda)$ in (1.3) satisfy

$$c_1(\lambda)^2 + c_2(\lambda)^2 = 1. \tag{1.4}$$

We require in addition that $x_0(t;\lambda)$ be eventually positive. This requirement together with (1.4) determines a unique prinicpal solution for each $\lambda >\ 0$. The principal solution constructed in this manner is referred to as the normalized principal solution of (B) and is denoted by $X_0(t;\lambda)$.

1.2. Properties of the Normalized Principal Solutions. The following properties of $X_0(t;\lambda)$ are needed in the proof of the main theorem.

(I) $X_0(t;\lambda)$ is a continuous function of $(t,\lambda) \in [a,\infty) \times (0,\infty)$.
(II) If $\lambda > 0$ is sufficiently small, then $X_0(t;\lambda) > 0$ on $[a,\infty)$.
(III) For every $\lambda > 0$, $X_0(t;\lambda) = O(t^{1/2})$ and $X_0'(t;\lambda) = o(t^{-1/2})$ as $t \to \infty$; furthermore, $X_0(t;\lambda)/X_0'(t;\lambda) \to \infty$ as $t \to \infty$.
(IV) $X_0(a;\lambda)/X_0'(a;\lambda) \to \infty$ as $\lambda \to +0$.

To prove (I), we use the relations

$$\lim_{t\to\infty} t^{1/2}\int_t^\infty s^{1/2}q(s)ds = 0, \tag{1.5}$$

$$\lim_{t\to\infty} t^{-1/2}\int_a^t s^{3/2}q(s)ds = 0, \tag{1.6}$$

which are straightforward consequences of (1.1). Take any constant $\Lambda > 0$ and choose $T > a$ large enough so that for $t \geq T$

$$\Lambda t^{1/2}\int_t^\infty s^{1/2}q(s)ds \leq 1/3 \quad \text{and} \quad \Lambda t^{-1/2}\int_a^t s^{3/2}q(s)ds \leq 1/3. \tag{1.7}$$

We denote by E the set of the functions $x(t;\lambda) \in C([T,\infty)\times(0,\Lambda))$ such that

$$\|x\| = \sup\left\{t^{-1/2}|x(t;\lambda)| : \ (t,\lambda) \in [T,\infty)\times(0,\Lambda)\right\} < \infty. \tag{1.8}$$

Clearly E is a Banach space with the norm $\|\cdot\|$ given by (1.8). Consider the subset $X \subset E$ and the mapping $M : X \to E$ defined by

$$X = \left\{x \in E : \ 0 \leq x(t;\lambda) \leq t^{1/2}, \ (t,\lambda) \in [T,\infty)\times(0,\lambda)\right\} \tag{1.9}$$

and

$$(Mx)(t;\lambda) = \frac{1}{3}T^{1/2} + \lambda t\int_t^\infty q(s)x(s;\lambda)ds + \\ + \lambda\int_T^t sq(s)x(s;\lambda)ds, \quad (t,\lambda) \in [T,\infty)\times(0,\Lambda). \tag{1.10}$$

Using (1.7), we easily see that M is well defined, maps X into itself, and satisfies

$$\|Mx_1 - Mx_2\| \leq \frac{2}{3}\|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in X.$$

The contraction mapping principle then implies that there exists a unique element $x_0 \in X$ such that $x_0 = Mx_0$, which satisfies the integral equation

$$x_0(t;\lambda) = \frac{1}{3}T^{1/2} + \lambda t\int_t^\infty q(s)x_0(s;\lambda)ds + \\ + \lambda\int_T^t sq(s)x_0(s;\lambda)ds, \quad (t,\lambda) \in [T,\infty)\times(0,\Lambda). \tag{1.11}$$

Differentiation of (1.11) shows that $x_0'(t;\lambda)$ is continuous on $(t,\lambda) \in [T,\infty)\times (0,\Lambda)$ and that, for each fixed $\lambda \in (0,\Lambda)$, $x_0(t;\lambda)$ is a solution of (B) on $[T,\infty)$. There exist constants $k_1(\lambda)$ and $k_2(\lambda)$ depending on λ such that

$$\begin{aligned} x_0(t;\lambda) &= k_1(\lambda)x_1(t;\lambda) + k_2(\lambda)x_2(t;\lambda), \\ x_0'(t;\lambda) &= k_1(\lambda)x_1'(t;\lambda) + k_2(\lambda)x_2'(t;\lambda), \end{aligned}$$

from which, using the fact that $x_1(t;\lambda)x_2'(t;\lambda) - x_2(t;\lambda)x_1'(t;\lambda) \equiv 1$, we have

$$\begin{aligned} k_1(\lambda) &= x_0(t;\lambda)x_2'(t;\lambda) - x_2(t;\lambda)x_0'(t;\lambda), \\ k_2(\lambda) &= x_0'(t;\lambda)x_1(t;\lambda) - x_1'(t;\lambda)x_0(t;\lambda). \end{aligned}$$

This shows that $k_1(\lambda)$ and $k_2(\lambda)$ depend continuously on $\lambda \in (0,\Lambda)$. Define

$$\begin{aligned} X_0(t;\lambda) = {} & \frac{k_1(\lambda)}{(k_1(\lambda)^2 + k_2(\lambda)^2)^{1/2}}\, x_1(t;\lambda) + \\ & + \frac{k_2(\lambda)}{(k_1(\lambda)^2 + k_2(\lambda)^2)^{1/2}}\, x_2(t;\lambda), \quad (t,\lambda) \in [a,\infty)\times(0,\Lambda). \end{aligned}$$

Then $X_0(t;\lambda)$ is the normalized principal solution of (B) and is a continuous function of $(t;\lambda)$ in $[a,\infty)\times(0,\Lambda)$. Since $\Lambda > 0$ is arbitrary and since the normalized principal solution $X_0(t;\lambda)$ is unique, we conclude that $X_0(t;\lambda)$ is continuous in $(t,\lambda) \in [a,\infty)\times(0,\infty)$, establishing the property (I) of $X_0(t;\lambda)$.

The property (II) of $X_0(t;\lambda)$ is an existence result. Choose $\lambda_* > 0$ so small that for $t \geq a$

$$\begin{aligned} \lambda_*(t-a+1)^{1/2} \int_t^\infty (s-a+1)^{1/2} q(s)\,ds &\leq 1/3, \\ \lambda_*(t-a+1)^{-1/2} \int_a^t (s-a+1)^{3/2} q(s)\,ds &\leq 1/3; \end{aligned}$$

this is possible because of (1.5) and (1.6). Define E to be the Banach space of continuous functions $x(t;\lambda) \in C([a,\infty)\times(0,\lambda_*))$ satisfying

$$\|x\| = \sup\Big\{(t-a+1)^{-1/2}|x(t;\lambda)| : \ (t,\lambda) \in [a,\infty)\times(0,\lambda_*)\Big\} < \infty.$$

Consider the set $X \subset E$ and the mapping $M : X \to E$ defined by

$$X = \Big\{x \in E : \ 0 \leq x(t;\lambda) \leq (t-a+1)^{1/2}, \ t \geq a\Big\}$$

and

$$(Mx)(t;\lambda) = \frac{1}{3} + \lambda(t-a+1)\int_t^\infty q(s)x(s;\lambda)ds +$$
$$+ \lambda \int_a^t (s-a+1)q(s)x(s;\lambda)ds, \quad (t,\lambda) \in [a,\infty) \times (0,\lambda_*).$$

Then it can be shown as in the proof of (I) that M is a contraction mapping on X, so that there exists a function $x_0 \in X$ such that $x_0 = Mx_0$, which gives rise to a principal solution of (B) which is positive throughout $[a,\infty)$. The normalization procedure applied to x_0 ensures that the normalized principal solution $X_0(t;\lambda)$ of (B) is positive on $[a,\infty)$ provided that $0 < \lambda < \lambda_*$.

We now turn to the property (III) of $X_0(t;\lambda)$. It suffices to prove this property for the solution $x_0(t;\lambda)$ that was constructed in the above proof of (I) (see (1.11)). It is almost trivial to see that $x_0(t;\lambda) = O(t^{1/2})$ as $t \to \infty$. That $x_0'(t;\lambda) = o(t^{-1/2})$ as $t \to \infty$ follows from the inequality

$$x_0'(t;\lambda) = \lambda \int_t^\infty q(s)x_0(s;\lambda)ds \le t^{-1/2} \cdot \lambda t^{1/2} \int_t^\infty s^{1/2} q(s)ds.$$

Since $x_0(t;\lambda)$ is eventually increasing in t, we have

$$\frac{x_0(t;\lambda)}{x_0'(t;\lambda)} \ge \frac{x(b;\lambda)}{x_0'(t;\lambda)} = x(b;\lambda)\,\frac{1}{\lambda \int_t^\infty q(s)x_0(s;\lambda)ds}\ , \quad t \ge b,$$

for some large b, which shows that $x_0(t;\lambda)/x_0'(t;\lambda) \to \infty$ as $t \to \infty$.

The property (IV) of $X_0(t;\lambda)$ is an easy consequence of the following inequality for the solution $x_0(t;\lambda)$ that was constructed in the proof of (II):

$$\frac{x_0(a;\lambda)}{x_0'(a;\lambda)} = \frac{x_0(a;\lambda)}{\lambda \int_a^\infty q(s)x_0(s;\lambda)ds} \ge \frac{1/3}{\lambda \int_a^\infty (s-a+1)^{1/2} q(s)ds}.$$

The properties (I)–(IV) of the normalized principal solutions have thus been verified.

2. Proof of the theorem

We now prove the main theorem for the equation (B). Let $X_0(t;\lambda)$ be the normalized principal solution of (B). We perform the Prüfer transformation:

$$X_0(t;\lambda) = r(t;\lambda)\sin\theta(t;\lambda), \quad X_0'(t;\lambda) = r(t;\lambda)\cos\theta(t;\lambda), \qquad (2.1)$$

or equivalently

$$
\begin{aligned}
r(t;\lambda) &= \left(X_0(t;\lambda)^2 + X_0'(t;\lambda)^2\right)^{1/2}, \\
\theta(t;\lambda) &= \arctan \frac{X_0(t;\lambda)}{X_0'(t;\lambda)}\ ;
\end{aligned}
\tag{2.2}
$$

see e.g., Hartman [2; p. 332]. From the property (I) of $X_0(t;\lambda)$ it follows that $r(t;\lambda)$ and $\theta(t;\lambda)$ are continuous functions of $(t,\lambda) \in [a,\infty) \times (0,\infty)$; in particular, $\theta(t;\lambda)$ is a continuous function of λ for each fixed $t \geq a$. Furthermore, $r(t;\lambda)$ and $\theta(t;\lambda)$ are continuously differentiable with respect to t and $\theta(t;\lambda)$ satisfies the differential equation

$$
\theta'(t;\lambda) = \cos^2\theta(t;\lambda) + \lambda q(t)\sin^2\theta(t;\lambda), \quad t \geq a. \tag{2.3}
$$

From (2.3) it is easy to see that, for each fixed $\lambda > 0$, $\theta(t;\lambda)$ is a strictly increasing function of t and that, if $0 < \lambda < \lambda'$ and $\theta(t_0;\lambda) \leq \theta(t_0;\lambda')$ for some $t_0 \in [a,\infty)$, then $\theta(t;\lambda) < \theta(t;\lambda')$ for $t \in (t_0,\infty)$. Using (2.1), (2.2) and the property (III), we have

$$
\sin\theta(t;\lambda) = 1 \,/\, \left[1 + (X_0'(t;\lambda)/X_0(t;\lambda))^2\right]^{1/2} \to 1
$$

and

$$
\cos\theta(t;\lambda) = \left(X_0'(t;\lambda)/X_0(t;\lambda)\right) \,/\, \left[1 + (X_0'(t;\lambda)/X_0(t;\lambda))^2\right]^{1/2} \to 0
$$

as $t \to \infty$. This implies that $\lim_{t\to\infty} \theta(t;\lambda) \equiv \pi/2 \pmod{2\pi}$ and so we may assume with no loss of generality that

$$
\lim_{t\to\infty} \theta(t;\lambda) = \frac{\pi}{2}, \quad \lambda > 0. \tag{2.4}
$$

We claim that, for each fixed $t \geq a$, $\theta(t;\lambda)$ is strictly decreasing in $\lambda > 0$. To see this, take any λ_1, λ_2 with $\lambda_1 < \lambda_2$, and put

$$
W(t) = X_0(t;\lambda_1)X_0'(t;\lambda_2) - X_0(t;\lambda_2)X_0'(t;\lambda_1).
$$

We then see that

$$
W'(t) = (\lambda_1 - \lambda_2)q(t)X_0(t;\lambda_1)X_0(t;\lambda_2) < 0 \quad \text{for all large} \quad t.
$$

Since, by the property (III) of $X_0(t;\lambda)$,

$$
W(t) = t^{-1/2}X_0(t;\lambda_1)\cdot t^{1/2}X_0'(t;\lambda_2) - t^{-1/2}X_0(t;\lambda_2)\cdot t^{1/2}X_0'(t;\lambda_1) \to 0
$$

as $t \to \infty$, it follows that $W(t) > 0$, that is,

$$
X_0(t;\lambda_1)/X_0'(t;\lambda_1) > X_0(t;\lambda_2)/X_0'(t;\lambda_2) \quad \text{for all large} \quad t.
$$

In view of (2.4), this inequality implies

$$
\theta(t;\lambda_1) = \arctan\frac{X_0(t;\lambda_1)}{X_0'(t;\lambda_1)} > \arctan\frac{X_0(t;\lambda_2)}{X_0'(t;\lambda_2)} = \theta(t;\lambda_2) \tag{2.5}
$$

for all sufficiently large t. Actually, (2.5) holds for all $t \geq a$; in fact, if (2.5) were violated at some $t_1 \geq a$, then we would have $\theta(t;\lambda_1) < \theta(t;\lambda_2)$ for all $t \geq t_1$, a contradiction to (2.5).

We observe that

$$\lim_{\lambda \to +0} \theta(a;\lambda) = \frac{\pi}{2}, \quad \lim_{\lambda \to \infty} \theta(a;\lambda) = -\infty. \tag{2.6}$$

The first of (2.6) is an immediate consequence of the properties (II) and (IV) of the normalized principal solutions, while the second follows from the fact that the number of zeros of any nontrivial solution of (B) can be made as large as possible by taking λ sufficiently large.

Our final task is to examine the possibility of finding those values of λ for which the normalized solution $X_0(t;\lambda)$ satisfies the boundary condition

$$\alpha X_0(a;\lambda) - \beta X_0'(a;\lambda) = 0, \quad \alpha^2 + \beta^2 > 0. \tag{2.7}$$

Case 1: $\beta = 0$. The boundary condition (2.7) takes the form $X_0(a;\lambda) = 0$. This is equivalent to $\theta(a;\lambda) \equiv 0 \pmod{\pi}$. Noting that $\theta(a;\lambda)$ is continuous and strictly decreasing in λ, and using (2.6), we can choose, for every $n = 0, 1, 2, \ldots$, a unique value λ_n of λ such that $\theta(a;\lambda_n) = -n\pi$. Therefore, the principal solution $X_0(t;\lambda_n)$ corresponding to $\lambda = \lambda_n$ satisfies the boundary condition $X_0(a;\lambda_n) = 0$. It is clear that $X_0(t;\lambda_n)$ has exactly n zeros in (a,∞).

Case 2: $\alpha\beta > 0$. Then, the boundary condition (2.7) is translated into $\theta(a;\lambda) = \arctan \beta/\alpha$. Choose a unique $\gamma \in (0, \pi/2)$ such that $\tan\gamma = \beta/\alpha$. We then use (2.6) to make sure that, for every $n = 0, 1, 2, \ldots$, there exists a unique $\lambda = \lambda_n$ such that $\theta(a;\lambda_n) = \gamma - n\pi$. It follows that the principal solution $X_0(t;\lambda_n)$ satisfies the boundary condition in question and has exactly n zeros in (a,∞).

Case 3: $\alpha = 0$. The boundary condition then reduces to $X_0'(a;\lambda) = 0$, which is equivalent to $\theta(a;\lambda) \equiv \pi/2 \pmod{\pi}$. In this case, (2.6) guarantees, for every $n = 1, 2, \ldots$, the existence of the value of $\lambda = \lambda_n$ for which $\theta(a;\lambda_n) = \pi/2 - n\pi$. The corresponding solution $X_0(t;\lambda_n)$ then satisfies $X_0'(a;\lambda) = 0$ and has exactly n zeros in (a,∞).

Case 4: $\alpha\beta < 0$. In this case there is a unique $\delta \in (-\pi/2, 0)$ such that $\tan\delta = \beta/\alpha$. In view of (2.6) there exists, for each $n = 1, 2, \ldots$, a unique value of $\lambda = \lambda_n$ such that $\theta(a;\lambda_n) = \delta - (n-1)\pi$. Consider the normalized principal solution corresponding to $\lambda = \lambda_n$. Then, it satisfies the required boundary condition at $t = a$ and possesses exactly n zeros in (a,∞).

This completes the proof of the main theorem for the equation (B).

Example. Consider the Hermite differential equation

$$(e^{-t^2} x')' + \lambda e^{-t^2} x = 0. \tag{2.8}$$

As is well-known (see e.g. [1]), for $\lambda = 2n$, $n \in N \cup \{0\}$, (2.8) has a polynomial solution of degree n; these solutions, suitably normalized, define the Hermite polynomials $H_n(t)$ satisfying

$$H_n(-t) = (-1)^n H_n(t), \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!},$$
$$H'_{2n}(0) = 0, \quad H_{2n+1}(0) = 0.$$

We now restrict our attention to (2.8) on the half-axis $[0,\infty)$. It is easy to check that (2.8) is strongly nonoscillatory on $[0,\infty)$ and that all the Hermite polynomials are principal solutions of (2.8) for $\lambda = 2n$, $n \in N \cup \{0\}$. We now apply the main theorem to the principal eigenvalue problems for (2.8) conjoined with the boundary conditions $x_0(0;\lambda) = 0$ and $x_0'(0;\lambda) = 0$, respectively. The theorem then guarantees the existence of two sequences of positive numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\widetilde{\lambda}_n\}_{n=1}^{\infty}$ which grow monotonically to ∞ with n and have the property that, for $\lambda = \lambda_n$, (2.8) possesses a principal solution $x_0(t;\lambda_n)$ satisfying $x_0(0;\lambda_n) = 0$ and having exactly n zeros in $(0,\infty)$, $n = 0,1,2,\ldots$, and for $\lambda = \widetilde{\lambda}_n$, (2.8) possesses a principal sotution $x_0(t;\widetilde{\lambda}_n)$ satisfying $x_0'(0;\widetilde{\lambda}_n) = 0$ and having exactly n zeros in $(0,\infty)$, $n = 1,2,\ldots$. In view of the uniqueness of the sequence of principal eigenvalues in the main theorem, we conclude that (i) $\lambda_n = 2(2n+1)$ and the corresponding principal eigenfunction $x_0(t;\lambda_n)$ is a constant multiple of $H_{2n+1}(t)$, $n = 0,1,2,\ldots$; and (ii) $\widetilde{\lambda}_n = 4n$ and the corresponding principal eigenfunction $x_0(t;\widetilde{\lambda}_n)$ is a constant multiple of $H_{2n}(t)$, $n = 1,2,\ldots$.

Remark. For earlier studies of similar singular eigenvalue problems the reader is referred to the papers [4] and [5].

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